THE

ELEMENTS OF ALGEBRA.

BY

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TEACHERS' EDITION.

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PREFACE.

THIS edition is intended for teachers, and for them only. The publishers will make every effort to keep the book from pupils; and teachers are urged to exercise the utmost care not to lose their copies, or to leave them where pupils can have access to them.

It is hoped that young teachers will derive great advantage from studying the systematic arrangement of the algebraic work, for such attention has been paid to this as the limitation of the page would allow.

It is also expected that many teachers, who are pressed for time, will find great relief by not being obliged to work out every problem in the Algebra.

G. A. WENTWORTH.

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ALGEBRA.

EXERCISE I.

When $a=1,\ b=2,\ c=3,\ d=4,\ e=5,\ f=0$:

1. $9a+2b+3c-2f$
$=9+4+9-0$
$=22.$

2. $4e-3a-3b+5c$
$=20-3-6+15$
$=26.$

3. $8abc-bcd+9cde-def$
$=48-24+540-0$
$=564.$

4. $\dfrac{4ac}{b}+\dfrac{8bc}{d}-\dfrac{5cd}{e}$
$=6+12-12$
$=6.$

5. $7e+bcd-\dfrac{3bde}{2ac}$
$=35+24-20$
$=39.$

6. $abc^2+bcd^2-dea^2+f^3$
$=18+96-20+0$
$=94.$

7. $e^4+6e^2b^2+b^4-4e^3b-4eb^3$
$=625+600+16-1000-160$
$=81.$

8. $\dfrac{8a^2+3b^2}{a^2b^2}+\dfrac{4c^2+6b^2}{c^2-b^2}-\dfrac{c^2+d^2}{e^2}$
$=\dfrac{8+12}{4}+\dfrac{36+24}{5}-\dfrac{9+16}{25}$
$=5+12-1$
$=16.$

9. $\dfrac{d^c}{b^e}$
$=\dfrac{4^3}{2^5}$
$=2.$

10. $\dfrac{e^c+b^a}{c^b-b^c}$
$=\dfrac{5^3+2}{3^2-2^3}$
$=\dfrac{125+2}{9-8}$
$=127.$

11. $\dfrac{b^c+d^c}{b^2+d^2-bd}$
$=\dfrac{2^3+4^3}{2^2+4^2-8}$
$=\dfrac{8+64}{4+16-8}$
$=6.$

12. $\dfrac{e^c-d^c}{e^2+ed+d^2}$
$=\dfrac{5^3-4^3}{5^2+20+4^2}$
$=\dfrac{125-64}{25+20+16}$
$=1.$

13. $100+80\div 4$
$=100+20$
$=120.$

14. $75 - 25 \times 2$
$= 75 - 50$
$= 25.$

15. $25 + 5 \times 4 - 10 \div 5$
$= 25 + 20 - 2$
$= 43.$

16. $24 - 5 \times 4 \div 10 + 3$
$= 24 - 20 \div 10 + 3$
$= 24 - 2 + 3$
$= 25.$

17. $(24 - 5) \times (4 \div 10 + 3)$
$= 19 \times (\frac{2}{5} + 3)$
$= 19 \times \frac{17}{5}$
$= \frac{323}{5}$
$= 64\frac{3}{5}.$

18. $xy + 4a \times 2$
$= 15 + 16$
$= 31.$

19. $xy - 15b \div 5$
$= 15 - \frac{150}{5}$
$= 15 - 30$
$= -15.$

20. $3x + 7y \div 7 + a \times y$
$= 9 + 35 \div 7 + 2 \times 5$
$= 9 + 5 + 10$
$= 24.$

21. $6b - 8y \div 2y \times b - 2b$
$= 60 - 40 \div 100 - 20$
$= 60 - \frac{2}{5} - 20$
$= 39\frac{3}{5}.$

22. $(6b - 8y) \div 2y \times b + 2b$
$= (60 - 40) \div 10 \times 10 + 20$
$= 20 \div 100 + 20$
$= 20\frac{1}{5}.$

23. $(6b - 8y) \div (2y \times b) + 2b$
$= (60 - 40) \div (10 \times 10) + 20$
$= 20 \div 100 + 20$
$= 20\frac{1}{5}.$

24. $6b - (8y \div 2y) \times b - 2b$
$= 60 - (40 \div 10) \times 10 - 20$
$= 60 - 40 - 20$
$= 0.$

25. $6b \div (b - y) - 3x + bxy \div 10a$
$= 60 \div (10 - 5) - 9 + 150 \div 20$
$= 12 - 9 + 7\frac{1}{2}$
$= 10\frac{1}{2}.$

26. Express the sum of a and b.

$$a + b.$$

27. Express the double of x.

$$2x.$$

28. By how much is a greater than 5?

$$a - 5.$$

29. If x be a whole number, what is the next number above it?

$$x + 1.$$

30. Write five numbers in order of magnitude, so that x shall be the middle number.

$$x - 2, \quad x - 1, \quad x, \quad x + 1, \quad x + 2.$$

31. What is the sum of $x + x + x + \cdots\cdots$ written a times?

$$ax.$$

32. If the product be xy and the multiplier x, what is the multiplicand?

$$xy \div x = y.$$

33. A man who has a dollars spends b dollars; how many dollars has he left?

$$a - b.$$

34. A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many men does the regiment consist?

$$ab + c.$$

35. Write, the sum of x and y divided by c is equal to the product of a, b, and m, diminished by six times c, and increased by the quotient of a divided by the sum of x and y.

$$\frac{x+y}{c} = abm - 6c + \frac{a}{x+y}.$$

36. Write, six times the square of n, divided by m minus a, increased by five b into the expression c plus d minus a.

$$\frac{6n^2}{m-a} + 5b(c+d-a).$$

37. Write, four times the fourth power of a, diminished by five times the square of a into the square of b, and increased by three times the fourth power of b.

$$4a^4 - 5a^2b^2 + 3b^4.$$

Exercise II.

3. The greater of two numbers is six times the smaller, and their sum is 35. Required the numbers.

Let $x =$ smaller number.
Then $6x =$ larger number,
and $6x + x =$ sum of numbers.
But $35 =$ sum of numbers.
Therefore, $6x + x = 35$, $x = 5$, and $6x = 30$.

4. Thomas had 75 cents. After spending a part of his money, he found he had twice as much left as he had spent. How much had he spent?

Let $x =$ number of cents spent.
Then $75 - x =$ number of cents left.
But $2x =$ number of cents left.
Therefore, $75 - x = 2x$, $-3x = -75$, and $x = 25$.

5. A tree 75 feet high was broken, so that the part broken off was four times the length of the part left standing. Required the length of each part.

Let $x =$ number of feet left standing.
Then $4x =$ number of feet broken off,
and $x + 4x =$ number of feet in height.
But $75 =$ number of feet in height.
Therefore, $5x = 75$, $x = 15$, and $4x = 60$.

6. Four times the smaller of two numbers is three times the greater, and their sum is 63. Required the numbers.

Let $x =$ smaller number.
Then $63 - x =$ larger number,
and $4x = 4$ times smaller;
also, $3(63 - x) = 3$ times greater.
$\therefore 4x = 3(63 - x)$, $4x = 189 - 3x$, $7x = 189$, $x = 27$, and $63 - x = 36$.

7. A farmer sold a sheep, a cow, and a horse, for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?

Let $x =$ number of dollars received for sheep.
Then $7x =$ number of dollars received for cow,
and $28x =$ number of dollars received for horse,
and $x + 7x + 28x =$ number of dollars received for all.
But $216 =$ number of dollars received for all.
$\therefore x + 7x + 28x = 216$, $36x = 216$, $x = 6$, $7x = 42$, and $28x = 168$.

8. George bought some apples, pears, and oranges, for 91 cents. He paid twice as much for the pears as for the apples, and twice as much for the oranges as for the pears. How much money did he spend for each?

Let $x =$ number of cents paid for apples.
Then $2x =$ number of cents paid for pears,
and $4x =$ number of cents paid for oranges,
and $x + 2x + 4x =$ number of cents paid for all.
But $91 =$ number of cents paid for all.
$\therefore x + 2x + 4x = 91$, $7x = 91$, $x = 13$, $2x = 26$, and $4x = 52$.

9. A man bought a horse, wagon, and harness, for $350. He paid for the horse four times as much as for the harness, and for the wagon one-half as much as for the horse. What did he pay for each?

Let $x =$ number of dollars paid for harness.
Then $4x =$ number of dollars paid for horse,
and $2x =$ number of dollars paid for wagon,
and $x + 4x + 2x =$ number of dollars paid for all.
But $350 =$ number of dollars paid for all.

$\therefore x + 4x + 2x = 350,\ 7x = 350,\ x = 50,\ 4x = 200,$ and $2x = 100.$

10. Distribute $3 among Thomas, Richard, and Henry, so that Thomas and Richard shall each have twice as much as Henry.

Let $x =$ number of dollars Henry has.
Then $2x =$ number of dollars Thomas has,
and $2x =$ number of dollars Richard has,
and $x + 2x + 2x =$ number of dollars all have.
But $3 =$ number of dollars all have.

Therefore, $x + 2x + 2x = 3,\ 5x = 3,\ x = \frac{3}{5},$ and $2x = 1\frac{1}{5}.$

11. Three men, A, B, and C, pay $1000 taxes. B pays four times as much as A, and C an amount equal to the sum of what the other two pay. How much does each pay?

Let $x =$ number of dollars A pays.
Then $4x =$ number of dollars B pays,
and $5x =$ number of dollars C pays,
and $x + 4x + 5x =$ number of dollars all pay.
But $1000 =$ number of dollars all pay.

$\therefore x + 4x + 5x = 1000,\ 10x = 1000,\ x = 100,\ 4x = 400,$ and $5x = 500.$

Exercise III.

1. $+16 + (-11)$
$= 16 - 11$
$= 5.$

2. $-15 + (-25)$
$= -15 - 25$
$= -40.$

3. $+68 + (-79)$
$= 68 - 79$
$= -11.$

4. $-7 + (+4)$
$= -7 + 4$
$= -3.$

5. $+33 + (+18)$
$= 33 + 18$
$= 51.$

6. $+378 + (+709) + (-592)$
$= 378 + 709 - 592$
$= 495.$

7. A man has $5242 and owes $2758. How much is he worth?

$$\$5242 + (-\$2758) = \$5242 - \$2758 = \$2484.$$

8. The First Punic War began B.C. 264, and lasted 23 years. When did it end?

$$-264 + (+23) = -264 + 23 = -241;\ \textit{i.e.}\ 241\ \text{B.C.}$$

9. Augustus Cæsar was born B.C. 63, and lived 77 years. When did he die?

$$-63+(+77)=+14;\ i.e.\ 14\ \text{A.D.}$$

10. A man goes 65 steps forwards, then 37 steps backwards, then again 48 steps forwards. How many steps did he take in all? How many steps is he from where he started?

$$65+37+48=150. \qquad 65-37+48=76.$$

Exercise IV.

1. $5ab+(-5ab)$
$=5ab-5ab$
$=0.$

2. $8mx+(-2mx)$
$=8mx-2mx$
$=6mx.$

3. $-13mng+(-7mng)$
$=-13mng-7mng$
$=-20mng.$

4. $-5x^2+(+8x^2)$
$=-5x^2+8x^2$
$=3x^2.$

5. $25my^2+(-18my^2)$
$=25my^2-18my^2$
$=7my^2.$

6. $7ab+(-5ab)$
$=7ab-5ab$
$=2ab.$

7. $120my+(-95my)$
$=120my-95my$
$=25my.$

8. $-33ab^2+(11ab^2)$
$=-33ab^2+11ab^2$
$=-22ab^2.$

9. $-75xy+(+20xy)$
$=-75xy+20xy$
$=-55xy.$

10. $+15a^2x^2+(-a^2x^2)$
$=15a^2x^2-a^2x^2$
$=14a^2x^2.$

11. $-b^2m^3+(+7b^2m^3)$
$=-b^2m^3+7b^2m^3$
$=6b^2m^3.$

12. $5a+(-3b)+(+4a)+(-7b)$
$=5a-3b+4a-7b$
$=9a-10b.$

13. $4a^2c+(-10xyz)+(+6a^2c)+(-9xyz)+(-11a^2c)+(+20xyz)$
$=4a^2c-10xyz+6a^2c-9xyz-11a^2c+20xyz$
$=-a^2c+xyz.$

14. $3x^2y+(-4ab)+(-2mn)+(+5x^2y)+(-x^2y)+(-4x^2y)$
$=3x^2y-4ab-2mn+5x^2y-x^2y-4x^2y$
$=3x^2y-4ab-2mn.$

Exercise V.

1.
$$\begin{array}{r} 5a+3b+c \\ 3a+3b+3c \\ a+3b+5c \\ \hline 9a+9b+9c \end{array}$$

2.
$$\begin{array}{r} 7a-4b+c \\ 6a+3b-5c \\ -12a\ +4c \\ \hline a-b \end{array}$$

3.
$$\begin{array}{r}
a+b-c\\
-a+b+c\\
a-b+c\\
a+b-c\\
\hline
2a+2b
\end{array}$$

4.
$$\begin{array}{r}
a+2b+3c\\
2a-b-2c\\
-a+b-c\\
-a-b+c\\
\hline
a+b+c
\end{array}$$

5.
$$\begin{array}{r}
a-2b+3c-4d\\
-2a+3b-4c+5d\\
+3a-4b+5c-6d\\
-4a+5b-4c+7d\\
\hline
-2a+2b\qquad+2d
\end{array}$$

6.
$$\begin{array}{r}
x^3-4x^2+5x-3\\
2x^3-14x^2-14x+5\\
-x^3+9x^2+x+8\\
\hline
2x^3-9x^2-8x+10
\end{array}$$

7.
$$\begin{array}{r}
x^4-2x^3+3x^2\qquad\qquad\\
x^3+x^2+x\qquad\\
4x^4+5x^3\qquad\qquad\qquad\qquad\\
+2x^2+3x-4\\
-3x^2-2x-5\\
\hline
5x^4+4x^3+3x^2+2x-9
\end{array}$$

8.
$$\begin{array}{r}
a^3+3ab^2-3a^2b-b^3\qquad\\
2a^3-6ab^2+5a^2b\qquad-7b^2\\
a^3-ab^2\qquad+2b^3\qquad\\
\hline
4a^3-4ab^2+2a^2b+b^3-7b^2
\end{array}$$

9.
$$\begin{array}{r}
2ab+2a^2x-3ax^2\qquad\\
12ab-6a^2x+10ax^2\qquad\\
-8ab-5a^2x\qquad+ax^3\\
\hline
6ab-9a^2x+7ax^2+ax^3
\end{array}$$

10.
$$\begin{array}{r}
c^4-3c^3+2c^2-4c+7\\
2c^4+3c^3+2c^2+5c+6\\
-4c^4\qquad-4c^2\qquad-5\\
\hline
-c^4\qquad\qquad+c+8
\end{array}$$

11.
$$\begin{array}{r}
3x^2-xy+xz-3y^2+4yz-z^2\qquad\qquad\\
-5x^2-xy-xz\qquad+5yz\qquad\qquad\qquad\\
6x^2\qquad\qquad\qquad\qquad\qquad-6y-6z\\
4yz\qquad\qquad\qquad\\
-5yz+3z^2\qquad\qquad\\
-4x^2\qquad+y^2+3yz+3z^2\qquad\qquad\\
\hline
-2xy\qquad-2y^2+11yz+5z^2-6y-6z
\end{array}$$

12.
$$\begin{array}{r}
m^5-3m^4n-6m^3n^2\qquad\qquad\qquad\qquad\\
-5m^4n+m^3n^2+m^2n^3\qquad\qquad\qquad\\
+7m^3n^2+4m^2n^3-3mn^4\qquad\\
-2m^2n^3-3mn^4+4n^5\\
3m^5\qquad\qquad\qquad\qquad\qquad+2mn^4+2n^5\\
2m^5+7m^4n\qquad\qquad\qquad\qquad\qquad-n^5\\
\hline
6m^5-m^4n+2m^3n^2+3m^2n^3-4mn^4+5n^5
\end{array}$$

Exercise VI.

1. $+25-(+16)$
$=25-16$
$=9.$

2. $-50-(-25)$
$=-50+25$
$=-25.$

3. $-31-(+58)$
$=-31-58$
$=-89.$

4. $+107-(-93)$
$=107+93$
$=200.$

5. Rome was ruled by emperors from B.C. 30 to its fall, A.D. 476. How long did the empire last?

$$476-(-30)=476+30=506;\ i.e.\ 506 \text{ years.}$$

6. The continent of Europe lies between 36° and 71° north latitude, and between 12° west and 63° east longitude (from Paris). How many degrees does it extend in latitude, and how many in longitude?

$$71-(+36)=71-36=35.$$
$$12-(-63)=12+63=75.$$

Exercise VII.

1. $5x-(-4x)$
$=5x+4x$
$=9x.$

2. $-3ab-(+5ab)$
$=-3ab-5ab$
$=-8ab.$

3. $3ab^2-(+10ab^2)$
$=3ab^2-10ab^2$
$=-7ab^2.$

4. $15m^2x^2-(-7m^2x^2)$
$=15m^2x^2+7m^2x^2$
$=22m^2x^2.$

5. $-7ay-(-3ay)$
$=-7ay+3ay$
$=-4ay.$

6. $17ax^3-(-24ax^3)$
$=17ax^3+24ax^3$
$=41ax^3.$

7. $5a^2x-(-3a^2x)$
$=5a^2x+3a^2x$
$=8a^2x.$

8. $-4xy-(-5xy)$
$=-4xy+5xy$
$=xy.$

9. $8ax-(-3ay)$
$=8ax+3ay.$

10. $2ab^2y-(+aby)$
$=2ab^2y-aby.$

11. $9x^2+(5x^2)-(+8x^2)$
$=9x^2+5x^2-8x^2$
$=6x^2.$

12. $5x^2y-(-18x^2y)+(-10x^2y)$
$=5x^2y+18x^2y-10x^2y$
$=13x^2y.$

13. $17ax^3-(-ax^3)-(+24ax^3)$
$=17ax^3+ax^3-24ax^3$
$=-6ax^3.$

14. $-3ab+(+2mx)-(-4mx)$
$=-3ab+2mx+4mx$
$=-3ab+6mx.$

15. $3a-(+2b)-(-4c)$
$=3a-2b+4c.$

Exercise VIII.

1.
$$\begin{array}{r}6a-2b-\ c\\ 2a-2b-3c\\ \hline 4a\qquad\ +2c\end{array}$$

2.
$$\begin{array}{r}3a-2b+3c\\ 2a-8b-\ c\\ \hline a+6b+4c\end{array}$$

3. $$\begin{array}{r} 7x^2-8x-1\\ 5x^2-6x+3\\ \hline 2x^2-2x-4 \end{array}$$

4. $$\begin{array}{r} 4x^4-3x^3-2x^2-7x+9\\ x^4-2x^3-2x^2+7x-9\\ \hline 3x^4-x^3\qquad-14x+18 \end{array}$$

5. $$\begin{array}{r} 2x^2-2ax+3a^2\\ x^2-ax+a^2\\ \hline x^2-ax+2a^2 \end{array}$$

6. $$\begin{array}{r} x^2-3xy-y^2+yz-2z^2\qquad\\ x^2+2xy-3y^2\qquad-2z^2+5xz\\ \hline -5xy+2y^2+yz\qquad-5xz \end{array}$$

7. $$\begin{array}{r} a^3-3a^2b+3ab^2-b^3\\ -a^3+3a^2b-3ab^2+b^3\\ \hline 2a^3-6a^2b+6ab^2-2b^3 \end{array}$$

8. $$\begin{array}{r} x^2-5xy+xz-y^2+7yz+2z^2\\ x^2-xy-xz\qquad+2yz+3z^2\\ \hline -4xy+2xz-y^2+5yz-z^2 \end{array}$$

9. $$\begin{array}{r} 2ax^2+3abx-4b^2x+12b^3\qquad\qquad\\ ax^2-4abx-5b^2x\qquad\qquad+bx^2-x^3\\ \hline ax^2+7abx+b^2x+12b^3-bx^2+x^3 \end{array}$$

10. $$\begin{array}{r} 6x^3-7x^2y+4xy^2-2y^3-5x^2+xy-4y^2+2\\ 8x^3-7x^2y+xy^2-y^3+9x^2-xy+6y^2-4\\ \hline -2x^3\qquad+3xy^2-y^3-14x^2+2xy-10y^2+6 \end{array}$$

11. $$\begin{array}{r} a^4-b^4\qquad\qquad\qquad\qquad\\ +4a^3b-6a^2b^2+4ab^3\\ \hline a^4-b^4-4a^3b+6a^2b^2-4ab^3\\ 2a^4-2b^4-4a^3b+6a^2b^2+4ab^3\\ \hline -a^4+b^4\qquad\qquad\qquad-8ab^3 \end{array}$$

12. $$\begin{array}{r} x^3y^2-3x^2y^3+4xy^4-y^5\qquad\qquad\\ -4xy^4-4y^5-x^5+2x^4y\\ \hline x^3y^2-3x^2y^3+8xy^4+3y^5+x^5-2x^4y \end{array}$$

$$\begin{array}{r} x^3y^2-3x^2y^3+4xy^4-y^5\qquad\qquad\\ -4xy^4-4y^5-x^5+2x^4y\\ \hline x^3y^2-3x^2y^3\qquad-5y^5-x^5+2x^4y\\ x^3y^2-3x^2y^3+8xy^4+3y^5+x^5-2x^4y\\ \hline -8xy^4-8y^5-2x^5+4x^4y \end{array}$$

13. $$\begin{array}{r} a^2b^2-a^2bc-8ab^2c-a^2c^2+abc^2-6b^2c^2\\ +2a^2bc-5ab^2c\qquad+2abc^2-5b^2c^2\\ \hline a^2b^2-3a^2bc-3ab^2c-a^2c^2-abc^2-b^2c^2 \end{array}$$

14. $$\begin{array}{r} 12a+3b-5c-2d=69\\ 10a-b+4c-3d=45\\ \hline 2a+4b-9c+d=24 \end{array}$$

15. $b-a$.
$$\begin{array}{r} 2a^3-6a^2b+6ab^2-2b^3\\ a^3-7a^2b\qquad-3b^3\\ \hline a^3+a^2b+6ab^2+b^3 \end{array}$$

16. $$\begin{array}{r} 2x^2-y^2-2xy+z^2\\ x^2-y^2+2xy-z^2\\ \hline x^2\qquad-4xy+2z^2 \end{array}$$

17. $$\begin{array}{r} 12ac+8cd-9\\ -7ac-9cd+8\\ \hline 19ac+17cd-17 \end{array}$$

18.
$$\begin{array}{r} -\ \ 6a^2 + 2ab - 3c^2 \\ 4a^2 + 6ab - 4c^2 \\ \hline -10a^2 - 4ab + \ \ c^2 \end{array}$$

19.
$$\begin{array}{r} 9xy - 4x - 3y + 7 \\ 8xy - 2x + 3y + 6 \\ \hline xy - 2x - 6y + 1 \end{array}$$

20.
$$\begin{array}{r} -\ \ a^2bc - \ \ ab^2c + \ \ abc^2 - \ \ abc \\ a^2bc + \ \ ab^2c - \ \ abc^2 + \ \ abc \\ \hline -2a^2bc - 2ab^2c + 2abc^2 - 2abc \end{array}$$

21.
$$\begin{array}{l} 7x^2 - 2x + 4 \\ 2x^2 + 3x - 1 \\ \hline 5x^2 - 5x + 5 \end{array}$$

22.
$$\begin{array}{r} 3x^2 + 2xy - \ \ y^2 \\ -\ \ x^2 - 3xy + 3y^2 \\ \hline 4x^2 + 5xy - 4y^2 \\ 3x^2 + 4xy - 5y^2 \\ \hline x^2 + \ \ xy + \ \ y^2 \end{array}$$

23.
$$\begin{array}{r} ax^2 - by^2 \qquad\qquad\quad \\ + cx^2 - dy^2 \\ \hline ax^2 - by^2 - cx^2 + dy^2 \end{array}$$

24.
$$\begin{array}{r} ax + \ \ bx + \ \ by + cy \\ ax - \ \ bx - \ \ by + cy \\ \hline 2bx + 2by \end{array}$$

25.
$$\begin{array}{r} 5x^2 + 4x - 4y + 3y^2 \\ 5x^2 - 3x + 3y + \ \ y^2 \\ \hline 7x - 7y + 2y^2 \end{array}$$

26.
$$\begin{array}{r} a^2b^2 + 12abc - 9ax^2 \qquad\qquad\qquad\qquad\quad \\ 4ab^2 - 6acx + 3a^2x \\ \hline a^2b^2 + 12abc - 9ax^2 - 4ab^2 + 6acx - 3a^2x \end{array}$$

27.
$$\begin{array}{r} a^2 - 2ab + c^2 - 3b^2 \\ 2a^2 - 2ab \qquad + 3b^2 \\ \hline -a^2 \qquad\quad + c^2 - 6b^2 \end{array}$$

28.
$$\begin{array}{r} a^2 + \ \ b^2 + \ \ c^2 + \ \ d^2 \\ b^2 + \ \ c^2 + \ \ d^2 \\ a^2 + \ \ b^2 - \ \ c^2 - \ \ d^2 \\ a^2 - \ \ b^2 + \ \ c^2 + \ \ d^2 \\ \hline 3a^2 + 2b^2 + 2c^2 + 2d^2 \end{array}$$

$$\begin{array}{r} b^2 + \ \ c^2 + \ \ d^2 \\ a^2 + \ \ b^2 - \ \ c^2 - \ \ d^2 \\ a^2 - \ \ b^2 + \ \ c^2 + \ \ d^2 \\ -a^2 + \ \ b^2 + \ \ c^2 + \ \ d^2 \\ \hline a^2 + 2b^2 + 2c^2 + 2d^2 \end{array}$$

$$\begin{array}{l} 3a^2 + 2b^2 + 2c^2 + 2d^2 \\ \ \ \ a^2 + 2b^2 + 2c^2 + 2d^2 \\ \hline 2a^2 \end{array}$$

29.
$$\begin{array}{r} 2x^2 - 2y^2 - \ \ z^2 \\ 2x^2 + 3y^2 - \ \ z^2 \\ \hline -5y^2 \qquad \\ -x^2 - 2y^2 + 3z^2 \\ \hline x^2 - 3y^2 - 3z^2 \end{array}$$

30.
$$\begin{array}{r} -2a^3 + \ \ a^2c + 2ac^2 \\ +\ \ a^3 - \ \ a^2c - \ \ ac^2 \\ \hline -\ \ a^3 \qquad\qquad + \ \ ac^2 \end{array}$$

$$\begin{array}{r} a^3 - 2a^2c + 3ac^2 \\ -\ \ a^3 \qquad\qquad + \ \ ac^2 \\ \hline 2a^3 - 2a^2c + 2ac^2 \end{array}$$

Exercise IX.

1. $(a + b) + (b + c) - (a + c)$
$= a + b + b + c - a - c$
$= 2b.$

2. $(2a - b - c) - (a - 2b + c)$
$= 2a - b - c - a + 2b - c$
$= a + b - 2c.$

3. $(2x-y)-(2y-z)-(2z-x)$
$=2x-y-2y+z-2z+x$
$=3x-3y-z.$

4. $(a-x-y)-(b-x+y)+(c+2y)$
$=a-x-y-b+x-y+c+2y$
$=a-b+c.$

5. $(2x-y+3z)+(-x-y-4z)-(3x-2y-z)$
$=2x-y+3z-x-y-4z-3x+2y+z$
$=-2x.$

6. $(3a-b+7c)-(2a+3b)-(5b-4c)+(3c-a)$
$=3a-b+7c-2a-3b-5b+4c+3c-a$
$=-9b+14c.$

7. $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$
$=1-1+a+1-a+a^2-1+a-a^2+a^3$
$=a+a^3.$

8. $a-\{2b-(3c+2b)-a\}$
$=a-\{2b-3c-2b-a\}$
$=a-2b+3c+2b+a$
$=2a+3c.$

9. $2a-\{b-(a-2b)\}$
$=2a-\{b-a+2b\}$
$=2a-b+a-2b$
$=3a-3b.$

10. $3a-\{b+(2a-b)-(a-b)\}$
$=3a-\{b+2a-b-a+b\}$
$=3a-b-2a+b+a-b$
$=2a-b.$

11. $7a-[3a-\{4a-(5a-2a)\}]$
$=7a-[3a-\{4a-5a+2a\}]$
$=7a-[3a-4a+5a-2a]$
$=7a-3a+4a-5a+2a$
$=5a.$

12. $2x+(y-3z)-[(3x-2y)+z]+5x-(4y-3z)$
$=2x+y-3z-[3x-2y+z]+5x-4y+3z$
$=2x+y-3z-3x+2y-z+5x-4y+3z$
$=4x-y-z.$

13. $\{(3a-2b)+(4c-a)\}-\{a-(2b-3a)-c\}+\{a-(b-5c-a)\}$
$=\{3a-2b+4c-a\}-\{a-2b+3a-c\}+\{a-b+5c+a\}$
$=3a-2b+4c-a-a+2b-3a+c+a-b+5c+a$
$=-b+10c.$

14. $a-[2a+(3a-4a)]-5a-\{6a-[(7a+8a)-9a]\}$
$=a-[2a+3a-4a]-5a-\{6a-[7a+8a-9a]\}$
$=a-2a-3a+4a-5a-\{6a-7a-8a+9a\}$
$=a-2a-3a+4a-5a-6a+7a+8a-9a$
$=-5a.$

15. $2a-(3b+2c)-[5b-(6c-6b)+5c-\{2a-(c+2b)\}]$
$=2a-3b-2c-[5b-6c+6b+5c-\{2a-c-2b\}]$
$=2a-3b-2c-[5b-6c+6b+5c-2a+c+2b]$
$=2a-3b-2c-5b+6c-6b-5c+2a-c-2b$
$=4a-16b-2c.$

16. $a - [2b + \{3c - 3a - (a + b)\} + \{2a - (b + c)\}]$
$= a - [2b + \{3c - 3a - a - b\} + \{2a - b - c\}]$
$= a - [2b + 3c - 3a - a - b + 2a - b - c]$
$= a - 2b - 3c + 3a + a + b - 2a + b + c$
$= 3a - 2c.$

17. $16 - x - [7x - \{8x - (9x - \overline{3x - 6x})\}]$
$= 16 - x - [7x - \{8x - (9x - 3x + 6x)\}]$
$= 16 - x - [7x - \{8x - 9x + 3x - 6x\}]$
$= 16 - x - [7x - 8x + 9x - 3x + 6x]$
$= 16 - x - 7x + 8x - 9x + 3x - 6x$
$= 16 - 12x.$

18. $2a - [3b + (2b - c) - 4c + \{2a - (3b - \overline{c - 2b})\}]$
$= 2a - [3b + 2b - c - 4c + \{2a - (3b - c + 2b)\}]$
$= 2a - [3b + 2b - c - 4c + \{2a - 3b + c - 2b\}]$
$= 2a - [3b + 2b - c - 4c + 2a - 3b + c - 2b]$
$= 2a - 3b - 2b + c + 4c - 2a + 3b - c + 2b$
$= 4c.$

19. $a - [2b + \{3c - 3a - (a + b)\} + 2a - (b + 3c)]$
$= a - [2b + \{3c - 3a - a - b + 2a - b - 3c]$
$= a - [2b + 3c - 3a - a - b + 2a - b - 3c]$
$= a - 2b - 3c + 3a + a + b - 2a + b + 3c$
$= 3a.$

20. $a - [5b - \{a - (3c - 3b) + 2c - (a - 2b - c)\}]$
$= a - [5b - \{a - 3c + 3b + 2c - a + 2b + c\}]$
$= a - [5b - a + 3c - 3b - 2c + a - 2b - c]$
$= a - 5b + a - 3c + 3b + 2c - a + 2b + c$
$= a.$

Exercise X.

1. $2a - 3b - 4c + d + 3e - 2f$
$= (2a - 3b) - (4c - d) + (3e - 2f)$
$= (2a - 3b - 4c) + (d + 3e - 2f).$

2. $a - 2x + 4y - 3z - 2b + c$
$= (a - 2x) + (4y - 3z) - (2b - c)$
$= (a - 2x + 4y) - (3z + 2b - c)$

3. $a^5 + 3a^4 - 2a^3 - 4a^2 + a - 1$
$= (a^5 + 3a^4) - (2a^3 + 4a^2) + (a - 1)$
$= (a^5 + 3a^4 - 2a^3) - (4a^2 - a + 1).$

4. $-3a - 2b + 2c - 5d - e - 2f$
$= -(3a + 2b) + (2c - 5d) - (e + 2f)$
$= -(3a + 2b - 2c) - (5d + e + 2f).$

5. $ax - by - cz - bx + cy + az$
$= (ax - by) - (cz + bx) + (cy + az)$
$= (ax - by - cz) - (bx - cy - az).$

6. $2x^5 - 3x^4y + 4x^3y^2 - 5x^2y^3 + xy^4 - 2y^5$
$= (2x^5 - 3x^4y) + (4x^3y^2 - 5x^2y^3) + (xy^4 - 2y^5)$
$= (2x^5 - 3x^4y + 4x^3y^2) - (5x^2y^3 - xy^4 + 2y^5)$.

7.
(1.) $2a-3b-4c+d+3e-2f=(2a-3b-4c)+(d+[3e-2f])$.
(2.) $a-2x+4y-3z-2b+c=(a-2x+4y)-(3z+[2b-c])$.
(3.) $a^5+3a^4-2a^3-4a^2+a-1=(a^5+3a^4-2a^3)-(4a^2-[a-1])$.
(4.) $-3a-2b+2c-5d-e-2f=-(3a+2b-2c)-(5d+[e+2f])$
(5.) $ax-by-cz-bx+cy+az=(ax-by-cz)-(bx-[cy+az])$
(6.) $2x^5-3x^4y+4x^3y^2-5x^2y^3+xy^4-2y^5=(2x^5-3x^4y+4x^3y^2)$
$-(5x^2y^3-[xy^4-2y^5])$

8. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy$
$= (2a - 4b - 2c)x - (6a + 3c)y + 4bz$.

9. $ax - bx + 2ay + 3y + 4az - 3bz - 2z$
$= (a - b)x + (2a + 3)y + (4a - 3b - 2)z$.

10. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz$
$= (a - 4b - 2c)x - (a + 2b + 3c)y + (a + 4b + 5c)z$.

11. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz$
$= (12a - 15c)x + (12a + 4b + 6c)y - (12b - 3c)z$.

12. $2ax - 3by - 7cz - 2bx + 2cx + 8cz - 2cx - cy - cz$
$= (2a - 2b + 2c - 2c)x - (3b + c)y - (7c - 8c + c)z$
$= (2a - 2b)x - (3b + c)y$.

Exercise XI.

1. $-17 \times 8 = -136$.

2. -12.8×25
$= -12.8 \times 100 \div 4 = -320$.

3.
$$\begin{array}{r} 3.29 \\ \underline{5.49} \\ 2961 \\ 1316 \\ \underline{1645} \\ -18.0621 \end{array}$$

4. $-18 \times -5 = +90$.

5. $43 \times -6 = -258$.

6. $457 \times 100 = 45700$.

7. $(-358 - 417) \times -79$
$= -775 \times -79$
$= 61225$.

8. $(7.512 - \{-2.894\}) \times (-6.037 + \{13.963\})$
$= (7.512 + 2.894) \times (-6.037 + 13.963)$
$= 10.406 \times 7.926$
$= 82.477956$.

9. $13 \times 8 \times -7$
$= 104 \times -7$
$= -728.$

10.
$$\begin{array}{r} -38 \\ 9 \\ \hline -342 \\ -6 \\ \hline 2052 \end{array}$$

11.
$$\begin{array}{r} -20.9 \\ -1.1 \\ \hline 209 \\ 209 \\ \hline 22.99 \\ 8 \\ \hline 183.92 \end{array}$$

12. $-78.3 \times -0.57 = 44.631$;
$1.38 \times -27.9 = -38.502$;
$44.631 \times -38.502 = -1718.382762.$

13. $-2.906 \times -2.076 = 6.032856$;
$6.032586 \times -1.49 = -8.98895544$;
$-8.98895544 \times 0.89 = -8.0001703416.$

Exercise XII.

9. $6a \times -2a = -12a^2.$

10. $5mn \times 9m = 45m^2n.$

11. $3ax \times -4by = -12abxy.$

12. $-8cm \times dn = -8cdmn.$

13. $-7ab \times 2ac = -14a^2bc.$

14. $5m^2x \times 3mx^2 = 15m^3x^3.$

15. $5a^m \times -2a^n = -10a^{m+n}.$

16. $3a^2x^2 \times 7a^3x^4 = 21a^5x^6.$

17. $7a \times -4b = -28ab$;
$-28ab \times -8c = 224abc$

18. $8ab^2 \times 3ac = 24a^2b^2c$;
$24a^2b^2c \times -4c^2 = -96a^2b^2c^3.$

19.
$$\begin{array}{r} 27ab \\ -39mp \\ \hline 243 \\ 81 \\ \hline -1053abmp \\ 18ap \\ \hline 8424 \\ 1053 \\ \hline -18954a^2bmp^2 \end{array}$$

20.
$$\begin{array}{r} 6ab^2y^3 \\ 2b^3y^3 \\ \hline 12ab^5y^6 \\ -5a^2y \\ \hline -60a^3b^5y^7 \end{array}$$

21.
$$\begin{array}{r} 7m^2x \\ 3mx^2 \\ \hline 21m^3x^3 \\ -2mq \\ \hline -42m^4qx^3 \end{array}$$

22.
$$\begin{array}{r} -3pq^2 \\ 6p^3q \\ \hline -18p^4q^3 \\ 8p^2q^3 \\ \hline -144p^6q^6 \end{array}$$

23. $2a^2m^3x^4 \times 3am^5x^2 = 6a^3m^8x^6$;
$6a^3m^8x^6 \times 4a^3mx^2 = 24a^6m^9x^8.$

24. $6x^2yz^3 \times -9x^2y^2z^2 = -54x^4y^3z^5$;
$-54x^4y^3z^5 \times -3x^4yz = 162x^8y^4z^6.$

25. $3ax \times 2am \times -4mx \times b^2$
$= -24a^2b^2m^2x^2.$

26. $7am^2 \times 3b^2n^2$
$= 21ab^2m^2n^2$;
$21ab^2m^2n^2 \times -4ab$
$= -84a^2b^3m^2n^2$;
$-84a^2b^3m^2n^2 \times a^2bn$
$= -84a^4b^4m^2n^3$;
$-84a^4b^4m^2n^3 \times -2b^2n$
$= 168a^4b^6m^2n^4$;
$168a^4b^6m^2n^4 \times -mn^2$
$= -168a^4b^6m^3n^6.$

Exercise XIII.

5. $(4a^2 - 3b) \times 3ab$
$= 12a^3b - 9ab^2.$

6. $(8a^2 - 9ab) \times 3a^2$
$= 24a^4 - 27a^3b.$

7. $(3x^2 - 4y^2 + 5z^2) \times 2x^2y$
$= 6x^4y - 8x^2y^3 + 10x^2yz^2.$

8. $(a^3x - 5a^2x^2 + ax^3 + 2x^4) \times ax^2y$
$= a^4x^3y - 5a^3x^4y + a^2x^5y + 2ax^6y.$

9. $(-9a^5 + 3a^3b^2 - 4a^2b^3 - b^5) \times -3ab^4$
$= 27a^6b^4 - 9a^4b^6 + 12a^3b^7 + 3ab^9.$

10. $(3x^3 - 2x^2y - 7xy^2 + y^3) \times -5x^2y$
$= -15x^5y + 10x^4y^2 + 35x^3y^3 - 5x^2y^4.$

11. $(-4xy^2 + 5x^2y + 8x^3) \times -3x^2y$
$= 12x^3y^3 - 15x^4y^2 - 24x^5y.$

12. $(-3 + 2ab + a^2b^2) \times -a^4$
$= +3a^4 - 2a^5b - a^6b^2.$

13. $(-z - 2xz^2 + 5x^2yz^2 - 6x^3y^2 + 3x^3y^2z) \times -3x^3yz$
$= 3x^3yz^2 + 6x^4yz^3 - 15x^5y^2z^3 + 18x^6y^3z - 9x^6y^3z^2.$

Exercise XIV.

1.
$$\begin{array}{l} x^2 - 4 \\ \underline{x^2 + 5} \\ x^4 - 4x^2 \\ \underline{\quad + 5x^2 - 20} \\ x^4 + x^2 - 20 \end{array}$$

2.
$$\begin{array}{l} y - 6 \\ \underline{y + 13} \\ y^2 - 6y \\ \underline{\quad + 13y - 78} \\ y^2 + 7y - 78 \end{array}$$

3.
$$\begin{array}{l} a^4 + a^2x^2 + x^4 \\ \underline{a^2 - x^2} \\ a^6 + a^4x^2 + a^2x^4 \\ \underline{\quad - a^4x^2 - a^2x^4 - x^6} \\ a^6 \qquad\qquad\qquad - x^6 \end{array}$$

4.
$$\begin{array}{l} x^2 + xy + y^2 \\ \underline{x - y} \\ x^3 + x^2y + xy^2 \\ \underline{\quad - x^2y - xy^2 - y^3} \\ x^3 \qquad\qquad\quad - y^3 \end{array}$$

5.
$$\begin{array}{l} 2x - y \\ \underline{x + 2y} \\ 2x^2 - xy \\ \underline{\quad + 4xy - 2y^2} \\ 2x^2 + 3xy - 2y^2 \end{array}$$

6.
$$\begin{array}{l} 2x^3 + 4x^2 + 8x + 16 \\ \underline{3x - 6} \\ 6x^4 + 12x^3 + 24x^2 + 48x \\ \underline{\quad - 12x^3 - 24x^2 - 48x - 96} \\ 6x^4 \qquad\qquad\qquad\qquad\qquad - 96 \end{array}$$

7.
$$\begin{array}{l} x^3 + x^2 + x - 1 \\ \underline{x - 1} \\ x^4 + x^3 + x^2 - x \\ \underline{\quad - x^3 - x^2 - x + 1} \\ x^4 \qquad\qquad\quad - 2x + 1 \end{array}$$

8.
$$\begin{array}{l} x^2 - 3ax \\ \underline{x + 3a} \\ x^3 - 3ax^2 \\ \underline{\quad + 3ax^2 - 9a^2x} \\ x^3 \qquad\quad - 9a^2x \end{array}$$

9.
$$\begin{array}{l} 2b^2 + 3ab - a^2 \\ -5b + 7a \\ \hline -10b^3 - 15ab^2 + 5a^2b \\ \qquad\qquad + 14ab^2 + 21a^2b - 7a^3 \\ \hline -10b^3 - ab^2 + 26a^2b - 7a^3 \end{array}$$

10.
$$\begin{array}{l} 2a + b \\ a + 2b \\ \hline 2a^2 + ab \\ \qquad + 4ab + 2b^2 \\ \hline 2a^2 + 5ab + 2b^2 \end{array}$$

11.
$$\begin{array}{l} a^2 + ab + b^2 \\ a - b \\ \hline a^3 + a^2b + ab^2 \\ \qquad - a^2b - ab^2 - b^3 \\ \hline a^3 \qquad\qquad\qquad - b^3 \end{array}$$

12.
$$\begin{array}{l} a^2 - ab + b^2 \\ a + b \\ \hline a^3 - a^2b + ab^2 \\ \qquad + a^2b - ab^2 + b^3 \\ \hline a^3 \qquad\qquad\qquad + b^3 \end{array}$$

13.
$$\begin{array}{l} 2ab - 5b^2 \\ 3a^2 - 4ab \\ \hline 6a^3b - 15a^2b^2 \\ \qquad - 8a^2b^2 + 20ab^3 \\ \hline 6a^3b - 23a^2b^2 + 20ab^3 \end{array}$$

14.
$$\begin{array}{l} -a^3 + 2a^2b - b^3 \\ 4a^2 + 8ab \\ \hline -4a^5 + 8a^4b - 4a^2b^3 \\ \qquad - 8a^4b \qquad\qquad + 16a^3b^2 - 8ab^4 \\ \hline -4a^5 \qquad\qquad - 4a^2b^3 + 16a^3b^2 - 8ab^4 \end{array}$$

15.
$$\begin{array}{l} a^2 + ab + b^2 \\ a^2 - ab + b^2 \\ \hline a^4 + a^3b + a^2b^2 \\ \quad - a^3b - a^2b^2 - ab^3 \\ \qquad\qquad + a^2b^2 + ab^3 + b^4 \\ \hline a^4 \qquad\quad + a^2b^2 \qquad\quad + b^4 \end{array}$$

16.
$$\begin{array}{l} a^3 - 3a^2b + 3ab^2 - b^3 \\ a^2 - 2ab + b^2 \\ \hline a^5 - 3a^4b + 3a^3b^2 - a^2b^3 \\ \quad - 2a^4b + 6a^3b^2 - 6a^2b^3 + 2ab^4 \\ \qquad\qquad + a^3b^2 - 3a^2b^3 + 3ab^4 - b^5 \\ \hline a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 \end{array}$$

17.
$$\begin{array}{l} x + 2y - 3z \\ x - 2y + 3z \\ \hline x^2 + 2xy - 3xz \\ \quad - 2xy \qquad\qquad - 4y^2 + 6yz \\ \qquad\qquad + 3xz \qquad\quad + 6yz - 9z^2 \\ \hline x^2 \qquad\qquad\qquad - 4y^2 + 12yz - 9z^2 \end{array}$$

18.
$$\begin{array}{l}
2x^2+3xy+4y^2\\
\underline{3x^2-4xy+yz}\\
6x^4+9x^3y+12x^2y^2\\
\qquad -8x^3y-12x^2y^2-16xy^3\\
\qquad\qquad\qquad\qquad\qquad +2x^2yz+3xy^2z+4y^3z\\
\hline
6x^4+x^3y\qquad\qquad -16xy^3+2x^2yz+3xy^2z+4y^3z
\end{array}$$

19.
$$\begin{array}{l}
x^2+xy+y^2\\
\underline{x^2+xz+z^2}\\
x^4+x^3y+x^2y^2\\
\qquad\qquad +x^3z+x^2yz+xy^2z\\
\qquad\qquad\qquad\qquad +x^2z^2+xyz^2+y^2z^2\\
\hline
x^4+x^3y+x^2y^2+x^3z+x^2yz+xy^2z+x^2z^2+xyz^2+y^2z^2
\end{array}$$

33.
$$\begin{array}{l}
a^2+ab+b^2\\
\underline{a^2-ab+b^2}\\
a^4+a^3b+a^2b^2\\
\quad -a^3b-a^2b^2-ab^3\\
\qquad\quad +a^2b^2+ab^3+b^4\\
\hline
a^4\qquad +a^2b^2\qquad +b^4\\
\underline{a^4-a^2b^2+\quad b^4}\\
a^8+a^6b^2+a^4b^4\\
\quad -a^6b^2-a^4b^4-a^2b^6\\
\qquad\quad +a^4b^4+a^2b^6+b^8\\
\hline
a^8\qquad +a^4b^4\qquad +b^8
\end{array}$$

34.
$$\begin{array}{l}
4a^3-4a^2b+ab^2\\
\underline{4a^2+3ab\ +b^2}\\
16a^5\ -16a^4b+\ 4a^3b^2\\
\qquad +12a^4b-12a^3b^2+3a^2b^3\\
\qquad\qquad\quad +\ 4a^3b^2-4a^2b^3+ab^4\\
\hline
16a^5\ -\ 4a^4b-\ 4a^3b^2-\ a^2b^3+ab^4\\
\underline{\ 2a^2b+b^3}\\
32a^7b-\ 8a^6b^2-\ 8a^5b^3-2a^4b^4+2a^3b^5\\
\qquad\qquad\quad +16a^5b^3-4a^4b^4-4a^3b^5-a^2b^6+ab^7\\
\hline
32a^7b-\ 8a^6b^2+\ 8a^5b^3-6a^4b^4-2a^3b^5-a^2b^6+ab^7
\end{array}$$

25.
$$\begin{array}{r}
x^5 - 5x^4 + 13x^3 - x^2 - x \qquad\qquad\qquad\qquad\\
x^2 - 2x - 2 \qquad\qquad\qquad\qquad\qquad\qquad\\
\hline
x^7 - 5x^6 + 13x^5 - x^4 - x^3 \qquad\qquad\qquad\qquad\\
- 2x^6 + 10x^5 - 26x^4 + 2x^3 + 2x^2 \qquad\quad\\
- 2x^5 + 10x^4 - 26x^3 + 2x^2 + 2x\\
\hline
x^7 - 7x^6 + 21x^5 - 17x^4 - 25x^3 + 4x^2 + 2x
\end{array}$$

26.
$$\begin{array}{r}
x^3 - 2x^2 + 3x - 4 \qquad\qquad\qquad\qquad\qquad\\
4x^3 + 3x^2 + 2x + 1 \qquad\qquad\qquad\qquad\qquad\\
\hline
4x^6 - 8x^5 + 12x^4 - 16x^3 \qquad\qquad\qquad\qquad\\
+ 3x^5 - 6x^4 + 9x^3 - 12x^2 \qquad\qquad\quad\\
+ 2x^4 - 4x^3 + 6x^2 - 8x \qquad\\
+ x^3 - 2x^2 + 3x - 4\\
\hline
\end{array}$$

13. $2ab - 5b^2$
$$\begin{array}{r}
3a^2 - 4ab \qquad\qquad\qquad\\
\hline
6a^3b - 15a^2b^2 \qquad\qquad\qquad\\
- 8a^2b^2 + 20ab^3\\
\hline
6a^3b - 23a^2b^2 + 20ab^3
\end{array}$$

14.
$$\begin{array}{r}
- a^3 + 2a^2b - b^3 \qquad\qquad\qquad\qquad\\
4a^2 + 8ab \qquad\qquad\qquad\qquad\qquad\\
\hline
- 4a^5 + 8a^4b - 4a^2b^3 \qquad\qquad\qquad\quad\\
- 8a^4b \qquad\qquad + 16a^3b^2 - 8ab^4\\
\hline
- 4a^5 \qquad\qquad - 4a^2b^3 + 16a^3b^2 - 8ab^4
\end{array}$$

15.
$$\begin{array}{r}
a^2 + ab + b^2 \qquad\qquad\qquad\\
a^2 - ab + b^2 \qquad\qquad\qquad\\
\hline
a^4 + a^3b + a^2b^2 \qquad\qquad\qquad\\
- a^3b - a^2b^2 - ab^3 \qquad\quad\\
+ a^2b^2 + ab^3 + b^4\\
\hline
a^4 \qquad\qquad + a^2b^2 \qquad\qquad + b^4
\end{array}$$

16.
$$\begin{array}{r}
a^3 - 3a^2b + 3ab^2 - b^3 \qquad\qquad\qquad\qquad\\
a^2 - 2ab + b^2 \qquad\qquad\qquad\qquad\qquad\\
\hline
a^5 - 3a^4b + 3a^3b^2 - a^2b^3 \qquad\qquad\qquad\\
- 2a^4b + 6a^3b^2 - 6a^2b^3 + 2ab^4 \qquad\quad\\
+ a^3b^2 - 3a^2b^3 + 3ab^4 - b^5\\
\hline
a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5
\end{array}$$

17.
$$\begin{array}{r}
x + 2y - 3z \qquad\qquad\qquad\qquad\qquad\\
x - 2y + 3z \qquad\qquad\qquad\qquad\qquad\\
\hline
x^2 + 2xy - 3xz \qquad\qquad\qquad\qquad\quad\\
- 2xy \qquad\qquad - 4y^2 + 6yz \qquad\quad\\
+ 3xz \qquad\quad + 6yz - 9z^2\\
\hline
x^2 \qquad\qquad\qquad\qquad - 4y^2 + 12yz - 9z^2
\end{array}$$

31.

$$
\begin{array}{rrrrr}
 & & & x & -3 \\
 & & & x & -1 \\
\hline
 & & x^2 & -3x & \\
 & & & -x & +3 \\
\hline
 & & x^2 & -4x & +3 \\
 & & & x & +1 \\
\hline
 & x^3 & -4x^2 & +3x & \\
 & & +x^2 & -4x & +3 \\
\hline
 & x^3 & -3x^2 & -x & +3 \\
 & & & x & +3 \\
\hline
x^4 & -3x^3 & -x^2 & +3x & \\
 & +3x^3 & -9x^2 & -3x & +9 \\
\hline
x^4 & & -10x^2 & & +9
\end{array}
$$

32.

$$
\begin{array}{rrrrrrrrr}
 & & & & & & x^2 & -x & +1 \\
 & & & & & & x^2 & +x & +1 \\
\hline
 & & & & x^4 & -x^3 & +x^2 & & \\
 & & & & & +x^3 & -x^2 & +x & \\
 & & & & & & +x^2 & -x & +1 \\
\hline
 & & & & x^4 & & +x^2 & & +1 \\
 & & & & x^4 & & -x^2 & & +1 \\
\hline
x^8 & & +x^6 & & +x^4 & & & & \\
 & & -x^6 & & -x^4 & & -x^2 & & \\
 & & & & +x^4 & & +x^2 & & +1 \\
\hline
x^8 & & & & +x^4 & & & & +1
\end{array}
$$

33.

$$
\begin{array}{rrrrrrrrr}
 & & & & & & a^2 & +ab & +b^2 \\
 & & & & & & a^2 & -ab & +b^2 \\
\hline
 & & & & a^4 & +a^3b & +a^2b^2 & & \\
 & & & & & -a^3b & -a^2b^2 & -ab^3 & \\
 & & & & & & +a^2b^2 & +ab^3 & +b^4 \\
\hline
 & & & & a^4 & & +a^2b^2 & & +b^4 \\
 & & & & a^4 & & -a^2b^2 & & +b^4 \\
\hline
a^8 & & +a^6b^2 & & +a^4b^4 & & & & \\
 & & -a^6b^2 & & -a^4b^4 & & -a^2b^6 & & \\
 & & & & +a^4b^4 & & +a^2b^6 & & +b^8 \\
\hline
a^8 & & & & +a^4b^4 & & & & +b^8
\end{array}
$$

34.

$$
\begin{array}{rrrrrrr}
 & & & & 4a^3 & -4a^2b & +ab^2 \\
 & & & & 4a^2 & +3ab & +b^2 \\
\hline
 & & 16a^5 & -16a^4b & +4a^3b^2 & & \\
 & & & +12a^4b & -12a^3b^2 & +3a^2b^3 & \\
 & & & & +4a^3b^2 & -4a^2b^3 & +ab^4 \\
\hline
 & & 16a^5 & -4a^4b & -4a^3b^2 & -a^2b^3 & +ab^4 \\
 & & & & & 2a^2b & +b^3 \\
\hline
32a^7b & -8a^6b^2 & -8a^5b^3 & -2a^4b^4 & +2a^3b^5 & & \\
 & & +16a^5b^3 & -4a^4b^4 & -4a^3b^5 & -a^2b^6 & +ab^7 \\
\hline
32a^7b & -8a^6b^2 & +8a^5b^3 & -6a^4b^4 & -2a^3b^5 & -a^2b^6 & +ab^7
\end{array}
$$

35.

$$\begin{array}{l}
x + a \\
x + 2a \\ \hline
x^2 + ax \\
\quad + 2ax + 2a^2 \\ \hline
x^2 + 3ax + 2a^2 \\
x - 3a \\ \hline
x^3 + 3ax^2 + 2a^2x \\
\quad - 3ax^2 - 9a^2x - 6a^3 \\ \hline
x^3 - 7a^2x - 6a^3 \\
x - 4a \\ \hline
x^4 - 7a^2x^2 - 6a^3x \\
\qquad + 28a^3x - 4ax^3 + 24a^4 \\ \hline
x^4 - 7a^2x^2 + 22a^3x - 4ax^3 + 24a^4 \\
x + 5a \\ \hline
x^5 - 7a^2x^3 + 22a^3x^2 - 4ax^4 + 24a^4x \\
\quad - 20a^2x^3 - 35a^3x^2 + 5ax^4 + 110a^4x + 120a^5 \\ \hline
x^5 - 27a^2x^3 - 13a^3x^2 + ax^4 + 134a^4x + 120a^5
\end{array}$$

or $x^5 + ax^4 - 27a^2x^3 - 13a^3x^2 + 134a^4x + 120a^5$

36.

$$\begin{array}{l}
81a^4 - 9a^2b^2 + b^4 \\
9a^2 + b^2 \\ \hline
729a^6 - 81a^4b^2 + 9a^2b^4 \\
\quad + 81a^4b^2 - 9a^2b^4 + b^6 \\ \hline
729a^6 + b^6
\end{array}$$

$$\begin{array}{l}
27a^3 + b^3 \\
27a^3 - b^3 \\ \hline
729a^6 + 27a^3b^3 \\
\quad - 27a^3b^3 - b^6 \\ \hline
729a^6 - b^6 \\
729a^6 + b^6 \\ \hline
531441a^{12} - 729a^6b^6 \\
\quad + 729a^6b^6 - b^{12} \\ \hline
531441a^{12} - b^{12}
\end{array}$$

37.

$$\begin{array}{l}
y^2 - yz - 2z^2 \\
y^2 + yz - 2z^2 \\ \hline
y^4 - y^3z - 2y^2z^2 \\
\quad + y^3z - y^2z^2 - 2yz^3 \\
\qquad - 2y^2z^2 + 2yz^3 + 4z^4 \\ \hline
y^4 - 5y^2z^2 + 4z^4
\end{array}$$

$$\begin{array}{l}
y^2 - 2yz - z^2 \\
y^2 + 2yz - z^2 \\ \hline
y^4 - 2y^3z - y^2z^2 \\
\quad 2y^3z - 4y^2z^2 - 2yz^3 \\
\qquad - y^2z^2 + 2yz^3 + z^4 \\ \hline
y^4 - 6y^2z^2 + z^4 \\
y^4 - 5y^2z^2 + 4z^4 \\ \hline
- y^2z^2 - 3z^4
\end{array}$$

38.

$$\begin{array}{l}
3a^2 - ab - 3b^2 \\
a^2b - 2b^2 \\ \hline
3a^4b - a^3b^2 - 3a^2b^3 \\
\qquad - 6a^2b^2 + 2ab^3 + 6b^4 \\ \hline
3a^4b - a^3b^2 - 3a^2b^3 - 6a^2b^2 + 2ab^3 + 6b^4 \\
\qquad - 2ab^4 - 6b^5 \\ \hline
3a^4b - a^3b^2 - 3a^2b^3 - 6a^2b^2 + 2ab^3 - 2ab^4 + 6b^4 - 6b^5
\end{array}$$

39.

$$
\begin{array}{l}
a+b-c\\
\underline{a-b+c}\\
a^2+ab-ac\\
\quad -ab \quad -b^2+bc\\
\quad\quad +ac \quad +bc-c^2\\
\hline
a^2 \qquad\quad -b^2+2bc-c^2\\
\underline{-a+b+c}\\
-a^3+ab^2-2abc+ac^2\\
\qquad\qquad\qquad +a^2b-b^3 \quad +2b^2c-bc^2\\
\qquad\qquad\qquad\qquad\qquad +a^2c-b^2c+2bc^2-c^3\\
\hline
-a^3+ab^2-2abc+ac^2+a^2b-b^3+a^2c+b^2c+bc^2-c^3\\
a+b+c\\
\hline
-a^4+a^2b^2-2a^2bc+a^2c^2+a^3b-ab^3+a^3c+ab^2c+abc^2-ac^3\\
\quad +a^2b^2+a^2bc \qquad -a^3b+ab^3 \qquad -2ab^2c+abc^2 \qquad -b^4+b^3c+b^2c^2-bc^3\\
\qquad +a^2bc+a^2c^2 \qquad\qquad -a^3c+ab^2c-2abc^2+ac^3 \qquad -b^3c+b^2c^2+bc^3-c^4\\
\hline
-a^4+2a^2b^2 \qquad +2a^2c^2 \qquad\qquad\qquad\qquad\qquad -b^4 \quad +2b^2c^2 \quad -c^4
\end{array}
$$

40.

$$
\begin{array}{l}
a+b\\
\underline{b+c}\\
ab+b^2\\
\underline{\qquad\quad +ac+bc}\\
ab+b^2+ac+bc
\end{array}
\qquad
\begin{array}{l}
c+d\\
\underline{a+d}\\
ac+ad\\
\underline{\qquad\quad +cd+d^2}\\
ac+ad+cd+d^2
\end{array}
\qquad
\begin{array}{l}
a+c\\
\underline{b-d}\\
ab+bc\\
\underline{\qquad\quad -ad-cd}\\
ab+bc-ad-cd
\end{array}
$$

$$
\begin{aligned}
&(ab+b^2+ac+bc)-(ac+ad+cd+d^2)-(ab+bc-ad-cd)\\
&=ab+b^2+ac+bc-ac-ad-cd-d^2-ab-bc+ad+cd\\
&=b^2-d^2.
\end{aligned}
$$

41.

$$
\begin{array}{l}
a+b+c+d\\
\underline{a+b+c+d}\\
a^2+ab+ac+ad\\
\quad +ab \qquad\qquad +b^2+bc+bd\\
\qquad +ac \qquad\qquad +bc \qquad +c^2+cd\\
\qquad\quad +ad \qquad\qquad +bd \qquad +cd+d^2\\
\hline
a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2
\end{array}
$$

$$
\begin{array}{l}
a-b-c+d\\
\underline{a-b-c+d}\\
a^2-ab-ac+ad\\
\quad -ab \qquad\qquad +b^2+bc-bd\\
\qquad -ac \qquad\qquad +bc \qquad +c^2-cd\\
\qquad\quad +ad \qquad\qquad -bd \qquad -cd+d^2\\
\hline
a^2-2ab-2ac+2ad+b^2+2bc-2bd+c^2-2cd+d^2
\end{array}
$$

$$\begin{array}{l}
a - b + c - d \\
\underline{a - b + c - d} \\
a^2 - ab + ac - ad \\
\quad - ab \qquad\qquad + b^2 - bc + bd \\
\qquad + ac \qquad\qquad - bc \qquad + c^2 - cd \\
\qquad\quad - ad \qquad\qquad + bd \qquad - cd + d^2 \\
\hline
a^2 - 2ab + 2ac - 2ad + b^2 - 2bc + 2bd + c^2 - 2cd + d^2
\end{array}$$

$$\begin{array}{l}
a + b - c - d \\
\underline{a + b - c - d} \\
a^2 + ab - ac - ad \\
\quad + ab \qquad\qquad + b^2 - bc - bd \\
\qquad - ac \qquad\qquad - bc \qquad + c^2 + cd \\
\qquad\quad - ad \qquad\qquad - bd \qquad + cd + d^2 \\
\hline
a^2 + 2ab - 2ac - 2ad + b^2 - 2bc - 2bd + c^2 + 2cd + d^2 \\
a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2 \\
a^2 - 2ab - 2ac + 2ad + b^2 + 2bc - 2bd + c^2 - 2cd + d^2 \\
a^2 - 2ab + 2ac - 2ad + b^2 - 2bc + 2bd + c^2 - 2cd + d^2 \\
\hline
4a^2 \qquad\qquad + 4b^2 \qquad\qquad + 4c^2 \qquad\qquad + 4d^2
\end{array}$$

42. $(a + b + c)^2 - a(b + c - a) - b(a + c - b) - c(a + b - c)$
$= a^2+b^2+c^2+2ab+2ac+2bc-ab-ac+a^2-ab-bc+b^2-ac-bc+c^2$
$= 2a^2+2b^2+2c^2.$

43. $(a - b)x - (b - c)a - \{(b - x)(b - a) - (b - c)(b + c)\}$
$= ax - bx - ab + ac - \{(b^2 - bx + ax - ab) - (b^2 - c^2)\}$
$= ax - bx - ab + ac - \{b^2 - bx + ax - ab - b^2 + c^2\}$
$= ax - bx - ab + ac - b^2 + bx - ax + ab + b^2 - c^2$
$= ac - c^2.$

44. $(m + n)m - \{(m - n)^2 - (n - m)n\}$
$= m^2 + mn - \{m^2 - 2mn + n^2 - n^2 + mn\}$
$= m^2 + mn - m^2 + 2mn - n^2 + n^2 - mn$
$= 2mn.$

45. $(a - b + c)^2 - \{a(c - a - b) - [b(a + b + c) - c(a - b - c)]\}$
$= a^2 - 2ab + 2ac - 2bc + b^2 + c^2 - \{(ac - a^2 - ab)$
$\qquad - [(ab + b^2 + bc) - (ac - bc - c^2)]\}$
$= a^2 - 2ab + 2ac - 2bc + b^2 + c^2 - \{(ac - a^2 - ab)$
$\qquad - [ab + b^2 + bc - ac + bc + c^2]\}$
$= a^2 - 2ab + 2ac - 2bc + b^2 + c^2$
$\qquad - \{ac - a^2 - ab - ab - b^2 - 2bc + ac - c^2\}$
$= a^2 - 2ab + 2ac - 2bc + b^2 + c^2 - ac + a^2$
$\qquad + ab + ab + b^2 + 2bc - ac + c^2$
$= 2a^2 + 2b^2 + 2c^2.$

46. $(p^2+q^2)r-(p+q)(p\{r-q\}-q\{r-p\})$
$=p^2r+q^2r-(p+q)(\{pr-pq\}-\{qr-pq\})$
$=p^2r+q^2r-(p+q)(pr-pq-qr+pq)$
$=p^2r+q^2r-(p+q)(pr-qr)$
$=p^2r+q^2r-(p^2r-q^2r)$
$=p^2r+q^2r-p^2r+q^2r$
$=2q^2r.$

47. $(9x^2y^2-4y^4)(x^2-y^2)-\{3xy-2y^2\}\{3x(x^2+y^2)-2y(y^2+3xy-x^2)\}y$
$=9x^4y^2-13x^2y^4+4y^6-\{3xy-2y^2\}\{3x^3+3xy^2-2y^3-6xy^2+2x^2y\}y$
$=9x^4y^2-13x^2y^4+4y^6-\{3xy-2y^2\}\{3x^3y+3xy^3-2y^4-6xy^3+2x^2y^2\}$
$=9x^4y^2-13x^2y^4+4y^6-\{9x^4y^2-13x^2y^4+4y^6\}$
$=9x^4y^2-13x^2y^4+4y^6-9x^4y^2+13x^2y^4-4y^6$
$=0.$

48. $a^2-\{2ab-[-(a+\{b-c\})(a-\{b-c\})+2ab]-4bc\}-(b+c)^2$
$=a^2-\{2ab-[-(a+b-c)(a-b+c)+2ab]-4bc\}-(b+c)^2$
$=a^2-\{2ab-[-a^2+b^2-2bc+c^2+2ab]-4bc\}-(b+c)^2$
$=a^2-\{2ab+a^2-b^2+2bc-c^2-2ab-4bc\}-(b+c)^2$
$=a^2-2ab-a^2+b^2-2bc+c^2+2ab+4bc-b^2-2bc-c^2$
$=0.$

49. $\{ac-(a-b)(b+c)\}-b\{b-(a-c)\}$
$=\{ac-(ab-b^2+ac-bc)\}-b(b-a+c)$
$=ac-ab+b^2-ac+bc-b^2+ab-bc$
$=0.$

50. $5\{(a-b)x-cy\}-2\{a(x-y)-bx\}-\{3ax-(5c-2a)y\}$
$=5\{ax-bx-cy\}-2\{ax-ay-bx\}-\{3ax-5cy+2ay\}$
$=5ax-5bx-5cy-2ax+2ay+2bx-3ax+5cy-2ay$
$=-3bx.$

51. $(x-1)(x-2)-3x(x+3)+2\{(x+2)(x+1)-3\}$
$=x^2-3x+2-3x^2-9x+2\{x^2+3x+2-3\}$
$=x^2-3x+2-3x^2-9x+2x^2+6x+4-6$
$=-6x.$

52. $\{(2a+b)^2+(a-2b)^2\}\{(3a-2b)^2-(2a-3b)^2\}$
$=\{(4a^2+4ab+b^2)+(a^2-4ab+4b^2)\}\{(9a^2-12ab+4b^2)-(4a^2-12ab+9b^2)\}$
$=\{4a^2+4ab+b^2+a^2-4ab+4b^2\}\{9a^2-12ab+4b^2-4a^2+12ab-9b^2\}$
$=\{5a^2+5b^2\}\{5a^2-5b^2\}$
$=25a^4-25b^4.$

53. $4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2)$
$=4(a^2-9b^2)-2(a^2-12ab+36b^2)-2a^2-12b^2$
$=4a^2-36b^2-2a^2+24ab-72b^2-2a^2-12b^2$
$=24ab-120b^2.$

54. $x^2(x^2+y^2)^2-2x^2y^2(x+y)(x-y)-(x^3-y^3)^2$
$=x^2(x^4+2x^2y^2+y^4)-2x^2y^2(x^2-y^2)-(x^3-y^3)^2$
$=x^6+2x^4y^2+x^2y^4-2x^4y^2+2x^2y^4-x^6+2x^3y^3-y^6$
$=3x^2y^4+2x^3y^3-y^6.$

55. $16(a^2+b^2)(a^2-b^2)-(2a-3)(2a+3)(4a^2+9)$
$+(2b-3)(2b+3)(4b^2+9)$
$=16(a^4-b^4)-(4a^2-9)(4a^2+9)+(4b^2-9)(4b^2+9)$
$=16a^4-16b^4-16a^4+81+16b^4-81$
$=0.$

Exercise XV.

7. $(x+y)^2$
$=x^2+2xy+y^2.$

8. $(y-z)^2$
$=y^2-2yz+z^2.$

9. $(2x+1)^2$
$=4x^2+4x+1.$

10. $(2a+5b)^2$
$=4a^2+20ab+25b^2.$

11. $(1-x^2)^2$
$=1-2x^2+x^4.$

12. $(3ax-4x^2)^2$
$=9a^2x^2-24ax^3+16x^4.$

13. $(1-7a)^2$
$=1-14a+49a^2.$

14. $(5xy+2)^2$
$=25x^2y^2+20xy+4.$

15. $(ab+cd)^2$
$=a^2b^2+2abcd+c^2d^2.$

16. $(3mn-4)^2$
$=9m^2n^2-24mn+16.$

17. $(12+5x)^2$
$=144+120x+25x^2.$

18. $(4xy^2-yz^2)^2$
$=16x^2y^4-8xy^3z^2+y^2z^4.$

19. $(3abc-bcd)^2$
$=9a^2b^2c^2-6ab^2c^2d+b^2c^2d^2.$

20. $(4x^3-xy^2)^2$
$=16x^6-8x^4y^2+x^2y^4.$

21. $(x+y)(x-y)$
$=x^2-y^2.$

22. $(2a+b)(2a-b)$
$=4a^2-b^2.$

23. $(3-x)(3+x)$
$=9-x^2.$

24. $(3ab+2b^2)(3ab-2b^2)$
$=9a^2b^2-4b^4.$

25. $(4x^2-3y^2)(4x^2+3y^2)$
$=16x^4-9y^4.$

26. $(a^3x^2-by^4)(a^3x^2+by^4)$
$=a^6x^4-b^2y^8.$

27. $(6xy-5y^2)(6xy+5y^2)$
$=36x^2y^2-25y^4.$

28. $(4x^5-1)(4x^5+1)$
$=16x^{10}-1.$

29. $(1+3ab^3)(1-3ab^3)$
$=1-9a^2b^6.$

30. $(ax+by)(ax-by)(a^2x^2+b^2y^2)$
$=a^4x^4-b^4y^4.$

Exercise XVI.

1. $(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz.$

2. $(x-y+z)^2=x^2+y^2+z^2-2xy+2xz-2yz.$

3. $(m+n-p-q)^2 = m^2+n^2+p^2+q^2+2mn-2mp-2mq-2np-2nq+2pq.$

4. $(x^2+2x-3)^2 = x^4+4x^3-2x^2-12x+9.$

5. $(x^2-6x+7)^2 = x^4-12x^3+50x^2-84x+49.$

6. $(2x^2-7x+9)^2 = 4x^4-28x^3+85x^2-126x+81.$

7. $(x^2+y^2-z^2)^2 = x^4+y^4+z^4+2x^2y^2-2x^2z^2-2y^2z^2.$

8. $(x^4-4x^2y^2+y^4)^2 = x^8+18x^4y^4+y^8-8x^6y^2-8x^2y^6.$

9. $(a^3+b^3+c^3)^2 = a^6+b^6+c^6+2a^3b^3+2a^3c^3+2b^3c^3.$

10. $(x^3-y^3-z^3)^2 = x^6+y^6+z^6-2x^3y^3-2x^3z^3+2y^3z^3.$

11. $(x+2y-3z)^2 = x^2+4y^2+9z^2+4xy-6xz-12yz.$

12. $(x^2-2y^2+5z^2)^2 = x^4+4y^4+25z^4-4x^2y^2+10x^2z^2-20y^2z^2.$

13. $(x^2+2x-2)^2 = x^4+4x^3-8x+4.$

14. $(x^2-5x+7)^2 = x^4+39x^2-10x^3-70x+49.$

15. $(2x^2-3x-4)^2 = 4x^4-12x^3-7x^2+24x+16.$

16. $(x+2y+3z)^2 = x^2+4y^2+9z^2+4xy+6xz+12yz.$

Exercise XVII.

1. $(x+2)(x+3) = x^2+5x+6.$
2. $(x+1)(x+5) = x^2+6x+5.$
3. $(x-3)(x-6) = x^2-9x+18.$
4. $(x-8)(x-1) = x^2-9x+8.$
5. $(x-8)(x+1) = x^2-7x-8.$
6. $(x-2)(x+5) = x^2+3x-10.$
7. $(x-3)(x+7) = x^2+4x-21.$
8. $(x-2)(x-4) = x^2-6x+8.$
9. $(x+1)(x+11) = x^2+12x+11.$
10. $(x-2a)(x+3a) = x^2+ax-6a^2.$
11. $(x-c)(x-d)$
$= x^2-(c+d)x+cd.$
12. $(x-4y)(x+y) = x^2-3xy-4y^2.$
13. $(a-2b)(a-5b) = a^2-7ab+10b^2.$
14. $(x^2+2y^2)(x^2+y^2)$
$= x^4+3x^2y^2+2y^4.$
15. $(x^2-3xy)(x^2+xy)$
$= x^4-2x^3y-3x^2y^2.$
16. $(ax-9)(ax+6)$
$= a^2x^2-3ax-54.$
17. $(x+a)(x-b)$
$= x^2+(a-b)x-ab.$
18. $(x-11)(x+4) = x^2-7x-44.$
19. $(x+12)(x-11) = x^2+x-132.$
20. $(x-10)(x-5) = x^2-15x+50$

Exercise XVIII.

1. $(x+a)^3 = x^3+3x^2a+3xa^2+a^3.$
2. $(x-a)^3 = x^3-3x^2a+3xa^2-a^3.$
3. $(x+1)^3 = x^3+3x^2+3x+1.$
4. $(x-1)^3 = x^3-3x^2+3x-1.$
5. $(x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4.$
6. $(x-a)^4 = x^4-4x^3a+6x^2a^2-4xa^3+a^4.$

7. $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1.$

8. $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1.$

9. $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$

10. $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.$

11. $(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$

12. $(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1.$

Exercise XIX.

1. $\dfrac{+264}{+4} = 66.$

2. $\dfrac{-4648}{-8} = 581.$

3. $\dfrac{+3840}{-30} = -128.$

4. $\dfrac{-2568}{+12} = -214.$

5.
$$\begin{array}{r} -21.7 \\ -49.\overline{)1063.3} \\ 98 \\ \hline 83 \\ 49 \\ \hline 343 \\ 343 \\ \hline \end{array}$$

6.
$$\begin{array}{r} -1.23 \\ 345\overline{)-424.35} \\ 345 \\ \hline 793 \\ 690 \\ \hline 1035 \\ 1035 \\ \hline \end{array}$$

7.
$$\begin{array}{r} -11 \\ +24\overline{)-264} \\ 24 \\ \hline 24 \\ 24 \\ \hline \end{array}$$

8.
$$\begin{array}{r} +43\tfrac{3}{17} \\ -85\overline{)-3670} \\ 340 \\ \hline 270 \\ 255 \\ \hline \tfrac{15}{85} = \tfrac{3}{17} \end{array}$$

9.
$$\begin{array}{r} -0.1123 \\ -61\overline{)6.8503} \\ 61 \\ \hline 75 \\ 61 \\ \hline 140 \\ 122 \\ \hline 183 \\ 183 \\ \hline \end{array}$$

10.
$$\begin{array}{r} -0.022\tfrac{7}{81} \\ +324\overline{)-7.1560} \\ 648 \\ \hline 676 \\ 648 \\ \hline \tfrac{28}{324} = \tfrac{7}{81} \end{array}$$

11.
$$\begin{array}{r} 0.31831+ \\ -314159.\overline{)-100000.0} \\ 942477 \\ \hline 575230 \\ 314159 \\ \hline 2610710 \\ 2513272 \\ \hline 974380 \\ 942477 \\ \hline 319030 \\ 314159 \\ \hline 4871 \end{array}$$

12.
$$\begin{array}{r} 0.0101321+ \\ -314159\overline{)-3183.10} \\ 314159 \\ \hline 415100 \\ 314159 \\ \hline 1009410 \\ 942477 \\ \hline 669330 \\ 628318 \\ \hline 410120 \\ 314159 \\ \hline 95961 \end{array}$$

Exercise XX.

5. $\dfrac{6mx}{2x} = 3m.$

6. $\dfrac{12a^4}{-3a} = -4a^3.$

7. $\dfrac{10ab}{2bc} = \dfrac{5a}{c}.$

8. $\dfrac{x^3}{-x^5} = -\dfrac{1}{x^2}.$

9. $\dfrac{-12am}{-2m} = 6a.$

10. $\dfrac{35abcd}{5bd} = 7ac.$

11. $\dfrac{abx}{5aby} = \dfrac{x}{5y}.$

12. $\dfrac{27a^7}{-3a^3} = -9a^4.$

13. $\dfrac{-3bmx}{4ax^2} = -\dfrac{3bm}{4ax}.$

14. $\dfrac{ab^2c^3}{abc} = bc^2.$

15. $\dfrac{m^5p^2x^4}{mp^2x^2} = m^4x^2.$

16. $\dfrac{-51abdy^2}{3bdy} = -17ay.$

17. $\dfrac{225m^2y}{25my^2} = \dfrac{9m}{y}.$

18. $\dfrac{30x^2y^3}{-5x^3y} = -\dfrac{6y^2}{x}.$

19. $\dfrac{4a^2m^4x^5}{5a^5m^3x} = \dfrac{4mx^4}{5a^3}.$

20. $\dfrac{42x^3y^2z^4}{7xy^2z^3} = 6x^2z.$

21. $\dfrac{-3a^2b^3c^4d^5}{-a^4b^2cd^3} = \dfrac{3bc^3d^2}{a^2}.$

22. $\dfrac{12am^5n^4p^3q^2}{4m^2n^3p^4q^5} = \dfrac{3am^3n}{pq^3}.$

23. $(4a^2bz^3 \times 10a^2b^3z) \div 5a^3b^2z^2$
$= 40a^4b^4z^4 \div 5a^3b^2z^2$
$= 8ab^2z^2.$

24. $(21x^3y^4z^6 \div 3xy^2z)(-2x^3y^2z)$
$= (7xy^2z^5)(-2x^3y^2z)$
$= -14x^4y^4z^6.$

25. $104ab^3x^9 \div (91a^5b^6x^7 \div 7a^4b^4x)$
$= 104ab^3x^9 \div 13ab^2x^6.$
$= 8bx^3.$

26. $(24a^5b^3x \div 3a^2b^2)$
$+ (35a^6b^2x^2 \div -5a^3bx)$
$= (8a^3bx) + (-7a^3bx)$
$= a^3bx.$

27. $85a^{4m+1} \div 5a^{4m-2}$
$= 17a^{4m+1-(4m-2)}$
$= 17a^3.$

28. $\dfrac{84a^{n-4}}{12a^2}$
$= 7a^{n-4-2}$
$= 7a^{n-6}.$

Exercise XXI.

3. $(18amy - 27bny + 36cpy) \div -9y = -2am + 3bn - 4cp.$

4. $(21ax - 18bx + 15cx) \div -3x = -7a + 6b - 5c.$

5. $(12x^5 - 8x^3 + 4x) \div 4x = 3x^4 - 2x^2 + 1.$

6. $(3x^3 - 6x^5 + 9x^7 - 12x^9) \div 3x^2 = x - 2x^3 + 3x^5 - 4x^7.$

7. $(35m^3y + 28m^2y^2 - 14my^3) \div -7my = -5m^2 - 4my + 2y^2.$

8. $(4a^4b - 6a^3b^2 + 12a^2b^3) \div 2a^2b = 2a^2 - 3ab + 6b^2.$

9. $(12x^3y^3 - 15x^4y^2 - 24x^5y) \div -3x^2y = -4xy^2 + 5x^2y + 8x^3.$

10. $(12x^5y^4 - 24x^4y^2 + 36x^3y^3 - 12x^2y^2) \div 12x^2y^2 = x^3y^2 - 2x^2 + 3xy - 1.$

11. $(3a^4 - 2a^5b - a^6b^2) \div a^4 = 3 - 2ab - a^2b^2.$

12. $(3x^3yz^2 + 6x^4yz^3 - 15x^5y^2z^3 + 18x^6y^3z) \div -3x^3yz$
$= -z - 2xz^2 + 5x^2yz^2 - 6x^3y^2.$

13. $(-16a^3b^2c^5 + 8a^4b^2c^4 - 12a^5b^3c^3) \div -4a^2b^2c^3$
$= 4ac^2 - 2a^2c + 3a^3bc.$

Exercise XXII.

6.
$$\begin{array}{l|l} x^2 - 7x + 12 & x - 3 \\ \underline{x^2 - 3x} & \overline{x - 4} \\ \quad -4x + 12 & \\ \quad \underline{-4x + 12} & \end{array}$$

7.
$$\begin{array}{l|l} x^2 + x - 72 & x + 9 \\ \underline{x^2 + 9x} & \overline{x - 8} \\ \quad -8x - 72 & \\ \quad \underline{-8x - 72} & \end{array}$$

8.
$$\begin{array}{l|l} 2x^3 - x^2 + 3x - 9 & 2x - 3 \\ \underline{2x^3 - 3x^2} & \overline{x^2 + x + 3} \\ \quad 2x^2 + 3x - 9 & \\ \quad \underline{2x^2 - 3x} & \\ \qquad 6x - 9 & \\ \qquad \underline{6x - 9} & \end{array}$$

9.
$$\begin{array}{l|l} 6x^3 + 14x^2 - 4x + 24 & 2x + 6 \\ \underline{6x^3 + 18x^2} & \overline{3x^2 - 2x + 4} \\ \quad -4x^2 - 4x + 24 & \\ \quad \underline{-4x^2 - 12x} & \\ \qquad 8x + 24 & \\ \qquad \underline{8x + 24} & \end{array}$$

10.
$$\begin{array}{l|l} 9x^3 + 3x^2 + x - 1 & 3x - 1 \\ \underline{9x^3 - 3x^2} & \overline{3x^2 + 2x + 1} \\ \quad 6x^2 + x - 1 & \\ \quad \underline{6x^2 - 2x} & \\ \qquad 3x - 1 & \\ \qquad \underline{3x - 1} & \end{array}$$

11.
$$\begin{array}{l|l} 7x^3 - 24x^2 + 58x - 21 & 7x - 3 \\ \underline{7x^3 - 3x^2} & \overline{x^2 - 3x + 7} \\ \quad -21x^2 + 58x - 21 & \\ \quad \underline{-21x^2 + 9x} & \\ \qquad 49x - 21 & \\ \qquad \underline{49x - 21} & \end{array}$$

12.
$$\begin{array}{l|l} x^6 - 1 & x - 1 \\ \underline{x^6 - x^5} & \overline{x^5 + x^4 + x^3 + x^2 + x + 1} \\ \quad x^5 - 1 & \\ \quad \underline{x^5 - x^4} & \\ \qquad x^4 - 1 & \\ \qquad \underline{x^4 - x^3} & \\ \qquad\quad x^3 - 1 & \\ \qquad\quad \underline{x^3 - x^2} & \\ \qquad\qquad x^2 - 1 & \\ \qquad\qquad \underline{x^2 - x} & \\ \qquad\qquad\quad x - 1 & \\ \qquad\qquad\quad \underline{x - 1} & \end{array}$$

13.
$$\begin{array}{l|l} a^3 - 2ab^2 + b^3 & a - b \\ \underline{a^3 - a^2b} & \overline{a^2 + ab - b^2} \\ \quad a^2b - 2ab^2 + b^3 & \\ \quad \underline{a^2b - ab^2} & \\ \qquad -ab^2 + b^3 & \\ \qquad \underline{-ab^2 + b^3} & \end{array}$$

14.
$$\begin{array}{l|l}
x^4-81y^4 & x-3y \\ \hline
\underline{x^4-3x^3y} & x^3+3x^2y+9xy^2+27y^3 \\
\quad 3x^3y-81y^4 & \\
\quad \underline{3x^3y-9x^2y^2} & \\
\qquad 9x^2y^2-81y^4 & \\
\qquad \underline{9x^2y^2-27xy^3} & \\
\qquad\quad 27xy^3-81y^4 & \\
\qquad\quad \underline{27xy^3-81y^4} &
\end{array}$$

15.
$$\begin{array}{l|l}
x^5-y^5 & x-y \\ \hline
\underline{x^5-x^4y} & x^4+x^3y+x^2y^2+xy^3+y^4 \\
\quad x^4y-y^5 & \\
\quad \underline{x^4y-x^3y^2} & \\
\qquad x^3y^2-y^5 & \\
\qquad \underline{x^3y^2-x^2y^3} & \\
\qquad\quad x^2y^3-y^5 & \\
\qquad\quad \underline{x^2y^3-xy^4} & \\
\qquad\qquad xy^4-y^5 & \\
\qquad\qquad xy^4-y^5 &
\end{array}$$

16.
$$\begin{array}{l|l}
a^5+32b^5 & a+2b \\ \hline
\underline{a^5+2a^4b} & a^4-2a^3b+4a^2b^2-8ab^3+16b^4 \\
\quad -2a^4b+32b^5 & \\
\quad \underline{-2a^4b-4a^3b^2} & \\
\qquad 4a^3b^2+32b^5 & \\
\qquad \underline{4a^3b^2+8a^2b^3} & \\
\qquad\quad -8a^2b^3+32b^5 & \\
\qquad\quad \underline{-8a^2b^3-16ab^4} & \\
\qquad\qquad 16ab^4+32b^5 & \\
\qquad\qquad \underline{16ab^4+32b^5} &
\end{array}$$

17.
$$\begin{array}{l|l}
2a^4+27ab^3-81b^4 & a+3b \\ \hline
\underline{2a^4+6a^3b} & 2a^3-6a^2b+18ab^2-27b^3 \\
\quad -6a^3b+27ab^3-81b^4 & \\
\quad \underline{-6a^3b-18a^2b^2} & \\
\qquad 18a^2b^2+27ab^3-81b^4 & \\
\qquad \underline{18a^2b^2+54ab^3} & \\
\qquad\quad -27ab^3-81b^4 & \\
\qquad\quad \underline{-27ab^3-81b^4} &
\end{array}$$

18.
$$\begin{array}{l|l}
x^4-5x^3+11x^2-12x+6 & x^2-3x+3 \\
\underline{x^4-3x^3+3x^2} & x^2-2x+2 \\
\quad -2x^3+8x^2-12x & \\
\quad \underline{-2x^3+6x^2-6x} & \\
\qquad 2x^2-6x+6 & \\
\qquad \underline{2x^2-6x+6} &
\end{array}$$

19.
$$\begin{array}{l|l}
x^4+x^3-9x^2-16x-4 & x^2+4x+4 \\
\underline{x^4+4x^3+4x^2} & x^2-3x-1 \\
\quad -3x^3-13x^2-16x-4 & \\
\quad \underline{-3x^3-12x^2-12x} & \\
\qquad -x^2-4x-4 & \\
\qquad \underline{-x^2-4x-4} &
\end{array}$$

20.
$$\begin{array}{l|l}
x^4-13x^2+36 & x^2+5x+6 \\
\underline{x^4+5x^3+6x^2} & x^2-5x+6 \\
\quad -5x^3-19x^2+36 & \\
\quad \underline{-5x^3-25x^2-30x} & \\
\qquad\qquad 6x^2+30x+36 & \\
\qquad\qquad \underline{6x^2+30x+36} &
\end{array}$$

21.
$$\begin{array}{l|l}
x^4+64 & x^2+4x+8 \\
\underline{x^4+4x^3+\ \ 8x^2} & x^2-4x+8 \\
\quad -4x^3-\ \ 8x^2+64 & \\
\quad \underline{-4x^3-16x^2-32x} & \\
\qquad\qquad 8x^2+32x+64 & \\
\qquad\qquad \underline{8x^2+32x+64} &
\end{array}$$

22.
$$\begin{array}{l|l}
x^4+\ \ x^3-24x^2-35x+57 & x^2+2x-3 \\
\underline{x^4+2x^3-\ \ 3x^2} & x^2-x-19 \\
\quad -\ \ x^3-21x^2-35x & \\
\quad \underline{-\ \ x^3-\ \ 2x^2+\ \ 3x} & \\
\qquad\qquad -19x^2-38x+57 & \\
\qquad\qquad \underline{-19x^2-38x+57} &
\end{array}$$

23.
$$\begin{array}{l|l}
1-\ \ x-3x^2-x^5 & 1+2x+x^2 \\
\underline{1+2x+\ \ x^2} & 1-3x+2x^2-x^3 \\
\quad -3x-4x^2-x^5 & \\
\quad \underline{-3x-6x^2-3x^3} & \\
\qquad\qquad 2x^2+3x^3-x^5 & \\
\qquad\qquad \underline{2x^2+4x^3+2x^4} & \\
\qquad\qquad\qquad -\ \ x^3-2x^4-x^5 & \\
\qquad\qquad\qquad \underline{-\ \ x^3-2x^4-x^5} &
\end{array}$$

24.
$$\begin{array}{l|l}
x^6-2x^3+1 & x^2-2x+1 \\
\underline{x^6-2x^5+x^4} & x^4+2x^3+3x^2+2x+1 \\
\quad 2x^5-x^4\ \ -2x^3+1 & \\
\quad \underline{2x^5-4x^4+2x^3} & \\
\qquad 3x^4-4x^3+1 & \\
\qquad \underline{3x^4-6x^3+3x^2} & \\
\qquad\qquad 2x^3-3x^2+1 & \\
\qquad\qquad \underline{2x^3-4x^2+2x} & \\
\qquad\qquad\qquad x^2-2x+1 & \\
\qquad\qquad\qquad \underline{x^2-2x+1} &
\end{array}$$

25.
$$\begin{array}{l|l}
a^4+2a^2b^2+9b^4 & a^2-2ab+3b^2 \\
\underline{a^4-2a^3b+3a^2b^2} & a^2+2ab+3b^2 \\
\quad 2a^3b-\ \ a^2b^2+9b^4 & \\
\quad \underline{2a^3b-4a^2b^2+6ab^3} & \\
\qquad\qquad 3a^2b^2-6ab^3+9b^4 & \\
\qquad\qquad \underline{3a^2b^2-6ab^3+9b^4} &
\end{array}$$

26.
$$\begin{array}{l|l}
4x^5-\ \ x^3+4x & 2x^2+3x+2 \\
\underline{4x^5+6x^4+4x^3} & 2x^3-3x^2+2x \\
\quad -6x^4-5x^3+4x & \\
\quad \underline{-6x^4-9x^3-6x^2} & \\
\qquad\qquad 4x^3+6x^2+4x & \\
\qquad\qquad \underline{4x^3+6x^2+4x} &
\end{array}$$

27.
$$\begin{array}{l|l}
a^5-243 & a-3 \\
\underline{a^5-3a^4} & a^4+3a^3+9a^2+27a+81 \\
\quad 3a^4-243 & \\
\quad \underline{3a^4-9a^3} & \\
\qquad 9a^3-243 & \\
\qquad \underline{9a^3-27a^2} & \\
\qquad\qquad 27a^2-243 & \\
\qquad\qquad \underline{27a^2-81a} & \\
\qquad\qquad\qquad 81a-243 & \\
\qquad\qquad\qquad \underline{81a-243} &
\end{array}$$

28.
$$\begin{array}{l|l}
18x^4 - 45x^3 + 82x^2 - 67x + 40 & 3x^2 - 4x + 5 \\ \cline{2-2}
\underline{18x^4 - 24x^3 + 30x^2} & 6x^2 - 7x + 8 \\
\quad -21x^3 + 52x^2 - 67x & \\
\quad \underline{-21x^3 + 28x^2 - 35x} & \\
\qquad 24x^2 - 32x + 40 & \\
\qquad \underline{24x^2 - 32x + 40} &
\end{array}$$

29.
$$\begin{array}{l|l}
x^4 - 9x^2 - 6xy - y^2 & x^2 + 3x + y \\ \cline{2-2}
\underline{x^4 + 3x^3 + x^2y} & x^2 - 3x - y \\
\quad -3x^3 - 9x^2 - x^2y - 6xy - y^2 & \\
\quad \underline{-3x^3 - 9x^2 \qquad - 3xy} & \\
\qquad -x^2y - 3xy - y^2 & \\
\qquad \underline{-x^2y - 3xy - y^2} &
\end{array}$$

30.
$$\begin{array}{l|l}
x^4 - 6x^3y + 9x^2y^2 - 4y^4 & x^2 - 3xy + 2y^2 \\ \cline{2-2}
\underline{x^4 - 3x^3y + 2x^2y^2} & x^2 - 3xy - 2y^2 \\
\quad -3x^3y + 7x^2y^2 - 4y^4 & \\
\quad \underline{-3x^3y + 9x^2y^2 - 6xy^3} & \\
\qquad -2x^2y^2 + 6xy^3 - 4y^4 & \\
\qquad \underline{-2x^2y^2 + 6xy^3 - 4y^4} &
\end{array}$$

31.
$$\begin{array}{l|l}
x^4 + x^2y^2 + y^4 & x^2 - xy + y^2 \\ \cline{2-2}
\underline{x^4 - x^3y + x^2y^2} & x^2 + xy + y^2 \\
\quad x^3y + y^4 & \\
\quad \underline{x^3y - x^2y^2 + xy^3} & \\
\qquad x^2y^2 - xy^3 + y^4 & \\
\qquad \underline{x^2y^2 - xy^3 + y^4} &
\end{array}$$

32.
$$\begin{array}{l|l}
x^5 + x^3 + x^4y - x^3y^2 - 2xy^2 + y^3 & x^3 + x - y \\ \cline{2-2}
\underline{x^5 + x^3 - x^2y} & x^2 + xy - y^2 \\
\quad x^4y + x^2y - x^3y^2 - 2xy^2 + y^3 & \\
\quad \underline{x^4y + x^2y \qquad - xy^2} & \\
\qquad -x^3y^2 - xy^2 + y^3 & \\
\qquad \underline{-x^3y^2 - xy^2 + y^3} &
\end{array}$$

33.
$$\begin{array}{l|l}
2x^2 + xy - xz - 3y^2 - 4yz - z^2 & 2x + 3y + z \\ \cline{2-2}
\underline{2x^2 + 3xy + xz} & x - y - z \\
\quad -2xy - 2xz - 3y^2 - 4yz - z^2 & \\
\quad \underline{-2xy \qquad - 3y^2 - yz} & \\
\qquad -2xz \qquad - 3yz - z^2 & \\
\qquad \underline{-2xz \qquad - 3yz - z^2} &
\end{array}$$

34.
$$\begin{array}{l|l}
12-38x+82x^2-112x^3+106x^4-70x^5 & 3-5x+7x^2 \\ \cline{2-2}
\underline{12-20x+28x^2} & 4-6x+8x^2-10x^3 \\
\quad -18x+54x^2-112x^3 & \\
\quad \underline{-18x+30x^2-\ 42x^3} & \\
\qquad\qquad 24x^2-\ 70x^3+106x^4 & \\
\qquad\qquad \underline{24x^2-\ 40x^3+\ 56x^4} & \\
\qquad\qquad\qquad -\ 30x^3+\ 50x^4-70x^5 & \\
\qquad\qquad\qquad \underline{-\ 30x^3+\ 50x^4-70x^5} &
\end{array}$$

35.
$$\begin{array}{l|l}
x^5 \qquad\qquad\qquad\qquad\qquad + y^5 & x^4-x^3y+x^2y^2-xy^3+y^4 \\ \cline{2-2}
\underline{x^5-x^4y+x^3y^2-x^2y^3+xy^4} & x+y \\
\qquad x^4y-x^3y^2+x^2y^3-xy^4+y^5 & \\
\qquad \underline{x^4y-x^3y^2+x^2y^3-xy^4+y^5} &
\end{array}$$

36.
$$\begin{array}{l|l}
2x^4-7x^3y+2x^2y^2-2xy^3-y^4 & 2x^2-xy+y^2 \\ \cline{2-2}
\underline{2x^4-\ x^3y+\ x^2y^2} & x^2-3xy-y^2 \\
\quad -6x^3y+\ x^2y^2-2xy^3 & \\
\quad \underline{-6x^3y+3x^2y^2-3xy^3} & \\
\qquad\qquad -2x^2y^2+\ xy^3-y^4 & \\
\qquad\qquad \underline{-2x^2y^2+\ xy^3-y^4} &
\end{array}$$

37.
$$\begin{array}{l|l}
16x^4+4x^2y^2+y^4 & 4x^2-2xy+y^2 \\ \cline{2-2}
\underline{16x^4-8x^3y\ +4x^2y^2} & 4x^2+2xy+y^2 \\
\qquad 8x^3y+y^4 & \\
\qquad \underline{8x^3y-4x^2y^2+2xy^3} & \\
\qquad\qquad 4x^2y^2-2xy^3+y^4 & \\
\qquad\qquad \underline{4x^2y^2-2xy^3+y^4} &
\end{array}$$

38.
$$\begin{array}{l|l}
32a^5b-56a^4b^2+\ 8a^3b^3-4a^2b^4-ab^5 & -4a^2b+6ab^2+b^3 \\ \cline{2-2}
\underline{32a^5b-48a^4b^2-\ 8a^3b^3} & -8a^3+2a^2b-ab^2 \\
\qquad -\ 8a^4b^2+16a^3b^3-4a^2b^4 & \\
\qquad \underline{-\ 8a^4b^2+12a^3b^3+2a^2b^4} & \\
\qquad\qquad\qquad 4a^3b^3-6a^2b^4-ab^5 & \\
\qquad\qquad\qquad \underline{4a^3b^3-6a^2b^4-ab^5} &
\end{array}$$

39.
$$\begin{array}{l|l}
1+5x^3-6x^4 & 1-x+3x^2 \\ \cline{2-2}
\underline{1-x+3x^2} & 1+x-2x^2 \\
\quad x-3x^2+5x^3-6x^4 & \\
\quad \underline{x-\ x^2+3x^3} & \\
\qquad -2x^2+2x^3-6x^4 & \\
\qquad \underline{-2x^2+2x^3-6x^4} &
\end{array}$$

40.
$$\begin{array}{l|l}
1 - 51a^3b^3 - 52a^4b^4 & -1 + 3ab + 4a^2b^2 \\
\underline{1 - 3ab - 4a^2b^2} & -1 - 3ab - 13a^2b^2 \\
\quad 3ab + 4a^2b^2 - 51a^3b^3 - 52a^4b^4 & \\
\quad \underline{3ab - 9a^2b^2 - 12a^3b^3} & \\
\qquad 13a^2b^2 - 39a^3b^3 - 52a^4b^4 & \\
\qquad \underline{13a^2b^2 - 39a^3b^3 - 52a^4b^4} &
\end{array}$$

41.
$$\begin{array}{l|l}
x^7y - xy^7 & x^3y - 2x^2y^2 + 2xy^3 - y^4 \\
\underline{x^7y - 2x^6y^2 + 2x^5y^3 - x^4y^4} & x^4 + 2x^3y + 2x^2y^2 + xy^3 \\
\quad 2x^6y^2 - 2x^5y^3 + x^4y^4 - xy^7 & \\
\quad \underline{2x^6y^2 - 4x^5y^3 + 4x^4y^4 - 2x^3y^5} & \\
\qquad 2x^5y^3 - 3x^4y^4 + 2x^3y^5 - xy^7 & \\
\qquad \underline{2x^5y^3 - 4x^4y^4 + 4x^3y^5 - 2x^2y^6} & \\
\qquad\quad x^4y^4 - 2x^3y^5 + 2x^2y^6 - xy^7 & \\
\qquad\quad \underline{x^4y^4 - 2x^3y^5 + 2x^2y^6 - xy^7} &
\end{array}$$

42.
$$\begin{array}{l|l}
x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6 & x^3 - 3x^2y + 3xy^2 - y^3 \\
\underline{x^6 - 3x^5y + 3x^4y^2 - x^3y^3} & x^3 - 3x^2y + 3xy^2 - y^3 \\
\quad -3x^5y + 12x^4y^2 - 19x^3y^3 + 15x^2y^4 & \\
\quad \underline{-3x^5y + 9x^4y^2 - 9x^3y^3 + 3x^2y^4} & \\
\qquad 3x^4y^2 - 10x^3y^3 + 12x^2y^4 - 6xy^5 & \\
\qquad \underline{3x^4y^2 - 9x^3y^3 + 9x^2y^4 - 3xy^5} & \\
\qquad\quad -x^3y^3 + 3x^2y^4 - 3xy^5 + y^6 & \\
\qquad\quad \underline{-x^3y^3 + 3x^2y^4 - 3xy^5 + y^6} &
\end{array}$$

43.
$$\begin{array}{l|l}
a^7 - 2a^6b - 2a^4b^3 + 2a^3b^4 - 6a^2b^5 - 3ab^6 & a^3 - 2a^2b - ab^2 \\
\underline{a^7 - 2a^6b - a^5b^2} & a^4 + a^2b^2 + 3b^4 \\
\quad a^5b^2 - 2a^4b^3 + 2a^3b^4 - 6a^2b^5 - 3ab^6 & \\
\quad \underline{a^5b^2 - 2a^4b^3 - a^3b^4} & \\
\qquad 3a^3b^4 - 6a^2b^5 - 3ab^6 & \\
\qquad \underline{3a^3b^4 - 6a^2b^5 - 3ab^6} &
\end{array}$$

44.
$$\begin{array}{l|l}
81x^6y - 54x^5y^2 - 18x^3y^4 + 18x^2y^5 - 18xy^6 - 9y^7 & 3x^4 + x^2y^2 + y^4 \\
\underline{81x^6y + 27x^4y^3 + 27x^2y^5} & 27x^2y - 18xy^2 - 9y^3 \\
\quad -54x^5y^2 - 27x^4y^3 - 18x^3y^4 - 9x^2y^5 - 18xy^6 - 9y^7 & \\
\quad \underline{-54x^5y^2 - 18x^3y^4 - 18xy^6} & \\
\qquad -27x^4y^3 - 9x^2y^5 - 9y^7 & \\
\qquad \underline{-27x^4y^3 - 9x^2y^5 - 9y^7} &
\end{array}$$

45.
$$\begin{array}{l|l}
a^4 + 2a^3b + 8a^2b^2 + 8ab^3 + 16b^4 & a^2 + 4b^2 \\
\underline{a^4 \qquad\qquad + 4a^2b^2} & \overline{a^2 + 2ab + 4b^2} \\
\quad 2a^3b + 4a^2b^2 + 8ab^3 & \\
\quad \underline{2a^3b \qquad\qquad + 8ab^3} & \\
\qquad 4a^2b^2 \qquad\qquad + 16b^4 & \\
\qquad \underline{4a^2b^2 \qquad\qquad + 16b^4} &
\end{array}$$

46.
$$\begin{array}{l|l}
-x^6 + 21x^3y^3 - 24xy^5 + 8y^6 & -x^2 + 3xy - y^2 \\
\underline{-x^6 + 3x^5y - x^4y^2} & \overline{x^4 + 3x^3y + 8x^2y^2 - 8y^4} \\
\quad -3x^5y + x^4y^2 + 21x^3y^3 - 24xy^5 + 8y^6 & \\
\quad \underline{-3x^5y + 9x^4y^2 - 3x^3y^3} & \\
\qquad -8x^4y^2 + 24x^3y^3 - 24xy^5 + 8y^6 & \\
\qquad \underline{-8x^4y^2 + 24x^3y^3 - 8x^2y^4} & \\
\qquad\qquad 8x^2y^4 - 24xy^5 + 8y^6 & \\
\qquad\qquad \underline{8x^2y^4 - 24xy^5 + 8y^6} &
\end{array}$$

47.
$$\begin{array}{l|l}
16a^4 \qquad\qquad + 8a^2b^2 + 9b^4 & 4a^2 - 4ab + 3b^2 \\
\underline{16a^4 - 16a^3b + 12a^2b^2} & \overline{4a^2 + 4ab + 3b^2} \\
\quad 16a^3b - 4a^2b^2 + 9b^4 & \\
\quad \underline{16a^3b - 16a^2b^2 + 12ab^3} & \\
\qquad 12a^2b^2 - 12ab^3 + 9b^4 & \\
\qquad \underline{12a^2b^2 - 12ab^3 + 9b^4} &
\end{array}$$

48.
$$\begin{array}{l|l}
a^3 \qquad\qquad - 3abc + b^3 + c^3 & a + b + c \\
\underline{a^3 + a^2b + a^2c} & \overline{a^2 - ab - ac + b^2 - bc + c^2} \\
\quad -a^2b - a^2c - 3abc + b^3 + c^3 & \\
\quad \underline{-a^2b - ab^2 - abc} & \\
\qquad -a^2c + ab^2 - 2abc + b^3 + c^3 & \\
\qquad \underline{-a^2c \qquad - abc - ac^2} & \\
\qquad\quad +ab^2 - abc + ac^2 + b^3 + c^3 & \\
\qquad\quad \underline{+ab^2 \qquad\qquad + b^3 + b^2c} & \\
\qquad\qquad -abc + ac^2 - b^2c + c^3 & \\
\qquad\qquad \underline{-abc \qquad - b^2c - bc^2} & \\
\qquad\qquad\quad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\quad \underline{ac^2 + bc^2 + c^3} &
\end{array}$$

49.
$$\begin{array}{l|l}
a^3 \qquad\qquad -6abc+8b^3+c^3 & a^2-2ab-ac+4b^2-2bc+c^2 \\
\underline{a^3-2a^2b-a^2c+4ab^2-2abc+ac^2} & \overline{a+2b+c} \\
\quad +2a^2b+a^2c-4ab^2-4abc-ac^2+8b^3+c^3 & \\
\quad \underline{+2a^2b \qquad -4ab^2-2abc \qquad +8b^3-4b^2c+2bc^2} & \\
\qquad +a^2c \qquad -2abc-ac^2 \qquad +4b^2c-2bc^2+c^3 & \\
\qquad \underline{+a^2c \qquad -2abc-ac^2 \qquad +4b^2c-2bc^2+c^3} &
\end{array}$$

50.
$$\begin{array}{l|l}
a^3 + 3a^2b + 3ab^2 + b^3 + c^3 & a + b + c \\ \cline{2-2}
\underline{a^3 + a^2b + a^2c} & a^2 + 2ab + b^2 - ac - bc + c^2 \\
\quad 2a^2b + 3ab^2 - a^2c + b^3 + c^3 & \\
\quad \underline{2a^2b + 2ab^2 \qquad + 2abc} & \\
\qquad ab^2 - a^2c - 2abc + b^3 + c^3 & \\
\qquad \underline{ab^2 \qquad\qquad\quad + b^3 + b^2c} & \\
\qquad\quad - a^2c - 2abc - b^2c + c^3 & \\
\qquad\quad \underline{- a^2c - abc \qquad - ac^2} & \\
\qquad\qquad - abc - b^2c + ac^2 + c^3 & \\
\qquad\qquad \underline{- abc - b^2c \qquad - bc^2} & \\
\qquad\qquad\qquad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\qquad \underline{ac^2 + bc^2 + c^3} &
\end{array}$$

Exercise XXIII.

1.
$$\begin{array}{l|l}
a^2(b + c) + b^2(a - c) + c^2(a - b) + abc & a + b + c \\ \cline{2-2}
\underline{a^2(b + c) + b^2(a \quad\;) + c^2(a \quad\;) + 2abc} & a(b + c) - bc \\
- abc \qquad + b^2(\;\; - c) + c^2(\;\; - b) & \\
\underline{- abc \qquad + b^2(\;\; - c) + c^2(\;\; - b)} &
\end{array}$$

2.
$$\begin{array}{l|l}
x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc & x^2 - (a + b)x + ab \\ \cline{2-2}
\underline{x^3 - (a + b \quad\;)x^2 + (ab \qquad\qquad)x} & x - c \\
\qquad - cx^2 \qquad + (ac + bc)x - abc & \\
\qquad \underline{- cx^2 \qquad + (ac + bc)x - abc} &
\end{array}$$

3.
$$\begin{array}{l|l}
x^3 - 2ax^2 + (a^2 + ab - b^2)x - a^2b + ab^2 & x - a + b \\ \cline{2-2}
\underline{x^3 - ax^2 + bx^2 \qquad\qquad\qquad\qquad\quad} & x^2 - (a + b)x + ab \\
\quad - (a + b)x^2 + (a^2 + ab - b^2)x - a^2b + ab^2 & \\
\quad \underline{- (a + b)x^2 + (a^2 \qquad - b^2)x} & \\
\qquad\qquad\qquad + abx - a^2b + ab^2 & \\
\qquad\qquad\qquad \underline{+ abx - a^2b + ab^2} &
\end{array}$$

4.
$$\begin{array}{l|l}
x^4 - (a^2 - b - c)x^2 - (b - c)ax + bc & x^2 - ax + c \\ \cline{2-2}
\underline{x^4 + (\qquad + c)x^2 - ax^3} & x^2 + ax + b \\
ax^3 - (a^2 - b \quad\;)x^2 - (b - c)ax + bc & \\
\underline{ax^3 - (a^2 \qquad\;)x^2 \qquad + cax} & \\
\qquad + bx^2 \qquad - bax + bc & \\
\qquad \underline{+ bx^2 \qquad - bax + bc} &
\end{array}$$

5.
$$\begin{array}{l|l}
y^3-(m+n+p)y^2+(mn+mp+np)y-mnp & y-p \\
\underline{y^3-(\qquad +p)y^2} & y^2-(m+n)y+mn \\
\quad -(m+n\qquad)y^2+(mn+mp+np)y & \\
\quad \underline{-(m+n\qquad)y^2+(\quad +mp+np)y} & \\
\qquad\qquad\qquad mny-mnp & \\
\qquad\qquad\qquad \underline{mny-mnp} &
\end{array}$$

6.
$$\begin{array}{l|l}
x^4+(5+a)x^3-(4-5a+b)x^2-(4a+5b)x+4b & x^2+5x-4 \\
\underline{x^4+(5\quad)x^3-(4\qquad\quad)x^2} & x^2+ax-b \\
\qquad ax^3-(\quad -5a+b)x^2-(4a+5b)x & \\
\qquad \underline{ax^3+(\quad +5a\quad)x^2-(4a\quad)x} & \\
\qquad\qquad -bx^2\qquad -5bx+4b & \\
\qquad\qquad \underline{-bx^2\qquad -5bx+4b} &
\end{array}$$

7.
$$\begin{array}{l|l}
 & x^2-(a+c)x+ac \\
 & x^2-(b+d)x+bd
\end{array}$$
$$\begin{array}{l}
x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x+abcd \\
\underline{x^4-(a\quad +c\quad)x^3+(\quad +ac\qquad)x^2} \\
\quad -(\ +b\quad +d)x^3+(ab\quad +ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x \\
\quad \underline{-(\ +b\quad +d)x^3+(ab\quad +ad+bc\quad +cd)x^2-(abc\quad +acd\quad)x} \\
\qquad\qquad bdx^2-(\quad +abd\quad +bcd)x+abcd \\
\qquad\qquad \underline{bdx^2-(\quad +abd\quad +bcd)x+abcd}
\end{array}$$

8.
$$\begin{array}{l|l}
x^5-(m-c)x^4+(n-cm+d)x^3+(r+cn-dm)x^2+(cr+dn)x+dr & x^3+mx^2+nx+r \\
\underline{x^5-(m\quad)x^4+(n\qquad)x^3+(r\qquad)x^2} & x^2+cx+d \\
\quad cx^4+(\ -cm+d)x^3+(\ +cn-dm)x^2+(cr+dn)x & \\
\quad \underline{cx^4+(\ -cm\quad)x^3+(\ +cn\quad)x^2+(cr\quad)x} & \\
\qquad\qquad dx^3\qquad -dmx^2\qquad +dnx+dr & \\
\qquad\qquad \underline{dx^3\qquad -dmx^2\qquad +dnx+dr} &
\end{array}$$

9.
$$\begin{array}{l|l}
x^5-mx^4+nx^3-nx^2+mx-1 & x-1 \\
\underline{x^5-x^4} & x^4+(1-m)x^3+(1-m+n)x^2+(1-m)x+1 \\
(1-m)x^4+nx^3-nx^2+mx-1 & \\
\underline{(1-m)x^4-(1-m)x^3} & \\
\qquad (1-m+n)x^3-nx^2+mx-1 & \\
\qquad \underline{(1-m+n)x^3-(1-m+n)x^2} & \\
\qquad\qquad (1-m)x^2+mx-1 & \\
\qquad\qquad \underline{(1-m)x^2-(1-m)x} & \\
\qquad\qquad\qquad x-1 & \\
\qquad\qquad\qquad \underline{x-1} &
\end{array}$$

10.
$$\begin{array}{l|l}
(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3 & (x+y)^2+2(x+y)z+z^2 \\
\underline{(x+y)^3+2(x+y)^2z+\quad(x+y)z^2} & x+y+z \\
\qquad (x+y)^2z+2(x+y)z^2+z^3 & \\
\qquad \underline{(x+y)^2z+2(x+y)z^2+z^3} &
\end{array}$$

EXERCISE XXIV.

1. $(y^3-1)\div(y-1)$
$=y^2+y+1.$

2. $(b^3-125)\div(b-5)$
$=b^2+5b+25.$

3. $(a^3-216)\div(a-6)$
$=a^2+6a+36.$

4. $(x^3-343)\div(x-7)$
$=x^2+7x+49.$

5. $(x^5-y^5)\div(x-y)$
$=x^4+x^3y+x^2y^2+xy^3+y^4.$

6. $(a^5-1)\div(a-1)$
$=a^4+a^3+a^2+a+1.$

7. $(1-8x^3)\div(1-2x)$
$=1+2x+4x^2.$

8. $(x^5-32b^5)\div(x-2b)$
$=x^4+2x^3b+4x^2b^2+8xb^3+16b^4.$

9. $(8a^3x^3-1)\div(2ax-1)$
$=4a^2x^2+2ax+1.$

10. $(1-27x^3y^3)\div(1-3xy)$
$=1+3xy+9x^2y^2.$

11. $(64a^3b^3-27x^3)\div(4ab-3x)$
$=16a^2b^2+12abx+9x^2.$

12. $(243a^5-1)\div(3a-1)$
$=81a^4+27a^3+9a^2+3a+1.$

13. $(32a^5-243b^5)\div(2a-3b)$
$=16a^4+24a^3b+36a^2b^2+54ab^3+81b^4.$

EXERCISE XXV.

1. $(x^3+y^3)\div(x+y)$
$=x^2-xy+y^2.$

2. $(x^5+y^5)\div(x+y)$
$=x^4-x^3y+x^2y^2-xy^3+y^4.$

3. $(1+8a^3)\div(1+2a)$
$=1-2a+4a^2.$

4. $(27a^3+b^3)\div(3a+b)$
$=9a^2-3ab+b^2.$

5. $(8a^3x^3+1)\div(2ax+1)$
$=4a^2x^2-2ax+1.$

6. $(x^3+27y^3)\div(x+3y)$
$=x^2-3xy+9y^2.$

7. $(a^5+32b^5)\div(a+2b)$
$=a^4-2a^3b+4a^2b^2-8ab^3+16b^4.$

8. $(512x^3y^3+z^3)\div(8xy+z)$
$=64x^2y^2-8xyz+z^2.$

9. $(729a^3+216b^3)\div(9a+6b)$
$=81a^2-54ab+36b^2.$

10. $(64a^3+1000b^3)\div(4a+10b)$
$=16a^2-40ab+100b^2.$

11. $(64a^3b^3+27x^3)\div(4ab+3x)$
$=16a^2b^2-12abx+9x^2.$

12. $(x^3+343)\div(x+7)$
$=x^2-7x+49.$

13. $(27x^3y^3+8z^3)\div(3xy+2z)$
$=9x^2y^2-6xyz+4z^2.$

14. $(1024a^5+243b^5)\div(4a+3b)$
$=256a^4-192a^3b+144a^2b^2-108ab^3+81b^4.$

Exercise XXVI.

1. $(x^4 - y^4) \div (x - y)$
$= x^3 + x^2y + xy^2 + y^3.$

2. $(x^4 - y^4) \div (x + y)$
$= x^3 - x^2y + xy^2 - y^3.$

3. $(a^6 - x^6) \div (a - x)$
$= a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5.$

4. $(a^6 - x^6) \div (a + x)$
$= a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5.$

5. $(x^4 - 81\,y^4) \div (x - 3\,y)$
$= x^3 + 3\,x^2y + 9\,xy^2 + 27y^3.$

6. $(x^4 - 81\,y^4) \div (x + 3\,y)$
$= x^3 - 3\,x^2y + 9\,xy^2 - 27y^3.$

7. $(16\,x^4 - 1) \div (2\,x - 1)$
$= 8\,x^3 + 4\,x^2 + 2\,x + 1.$

8. $(16\,x^4 - 1) \div (2\,x + 1)$
$= 8\,x^3 - 4\,x^2 + 2\,x - 1.$

9. $(81\,a^4x^4 - 1) \div (3\,ax - 1)$
$= 27a^3x^3 + 9\,a^2x^2 + 3\,ax + 1.$

10. $(81\,a^4x^4 - 1) \div (3\,ax + 1)$
$= 27a^3x^3 - 9\,a^2x^2 + 3\,ax - 1.$

11. $(64\,a^6 - b^6) \div (2\,a - b)$
$= 32\,a^5 + 16\,a^4b + 8\,a^3b^2 + 4\,a^2b^3 + 2\,ab^4 + b^5.$

12. $(64\,a^6 - b^6) \div (2\,a + b)$
$= 32\,a^5 - 16\,a^4b + 8\,a^3b^2 - 4\,a^2b^3 + 2\,ab^4 - b^5.$

13. $(x^6 - 729\,y^6) \div (x - 3\,y)$
$= x^5 + 3\,x^4y + 9\,x^3y^2 + 27x^2y^3 + 81\,xy^4 + 243\,y^5.$

14. $(x^6 - 729\,y^6) \div (x + 3\,y)$
$= x^5 - 3\,x^4y + 9\,x^3y^2 - 27\,x^2y^3 + 81\,xy^4 - 243\,y^5.$

15. $(81\,a^4 - 16\,c^4) \div (3\,a - 2\,c)$
$= 27a^3 + 18a^2c + 12\,ac^2 + 8\,c^3.$

16. $(81\,a^4 - 16\,c^4) \div (3\,a + 2\,c)$
$= 27a^3 - 18\,a^2c + 12\,ac^2 - 8\,c^3.$

17. $(256\,a^4 - 10{,}000) \div (4\,a - 10)$
$= 64\,a^3 + 160\,a^2 + 400\,a + 1000.$

18. $(256\,a^4 - 10{,}000) \div (4\,a + 10)$
$= 64\,a^3 - 160\,a^2 + 400\,a - 1000$

19. $(625\,x^4 - 1) \div (5\,x - 1)$
$= 125\,x^3 + 25\,x^2 + 5\,x + 1.$

Exercise XXVII.

1. $(x^6 + y^6) \div (x^2 + y^2)$
$= x^4 - x^2y^2 + y^4.$

2. $(a^6 + 1) \div (a^2 + 1)$
$= a^4 - a^2 + 1.$

3. $(a^{10} + y^{10}) \div (a^2 + y^2)$
$= a^8 - a^6y^2 + a^4y^4 - a^2y^6 + y^8.$

4. $(b^{10} + 1) \div (b^2 + 1)$
$= b^8 - b^6 + b^4 - b^2 + 1.$

5. $(a^{12} + b^{12}) \div (a^4 + b^4)$
$= a^8 - a^4b^4 + b^8.$

6. $(x^{12} + 1) \div (x^4 + 1)$
$= x^8 - x^4 + 1.$

7. $(64\,x^6 + y^6) \div (4\,x^2 + y^2)$
$= 16\,x^4 - 4\,x^2y^2 + y^4.$

8. $(64 + a^6) \div (4 + a^2)$
$= 16 - 4\,a^2 + a^4.$

9. $(729\,a^6 + b^6) \div (9\,a^2 + b^2)$
$= 81\,a^4 - 9\,a^2b^2 + b^4.$

10. $(729\,c^6 + 1) \div (9\,c^2 + 1)$
$= 81\,c^4 - 9\,c^2 + 1.$

Exercise XXVIII.

1.
$$\begin{aligned} 5x - 1 &= 19, \\ 5x &= 19 + 1, \\ 5x &= 20, \\ x &= 4. \end{aligned}$$

2.
$$\begin{aligned} 3x + 6 &= 12, \\ 3x &= 12 - 6, \\ 3x &= 6, \\ x &= 2. \end{aligned}$$

3.
$$\begin{aligned} 24x &= 7x + 34, \\ 24x - 7x &= 34, \\ 17x &= 34, \\ x &= 2. \end{aligned}$$

4.
$$\begin{aligned} 8x - 29 &= 26 - 3x, \\ 8x + 3x &= 26 + 29, \\ 11x &= 55, \\ x &= 5. \end{aligned}$$

5.
$$\begin{aligned} 12 - 5x &= 19 - 12x, \\ -5x + 12x &= 19 - 12, \\ 7x &= 7, \\ x &= 1. \end{aligned}$$

6.
$$\begin{aligned} 3x + 6 - 2x &= 7x, \\ 3x - 2x - 7x &= -6, \\ -6x &= -6, \\ x &= 1. \end{aligned}$$

7.
$$\begin{aligned} 5x + 50 &= 4x + 56, \\ 5x - 4x &= 56 - 50, \\ x &= 6. \end{aligned}$$

8.
$$\begin{aligned} 16x - 11 &= 7x + 70, \\ 16x - 7x &= 70 + 11, \\ 9x &= 81, \\ x &= 9. \end{aligned}$$

9.
$$\begin{aligned} 24x - 49 &= 19x - 14, \\ 24x - 19x &= 49 - 14, \\ 5x &= 35, \\ x &= 7. \end{aligned}$$

10.
$$\begin{aligned} 3x + 23 &= 78 - 2x, \\ 3x + 2x &= 78 - 23, \\ 5x &= 55, \\ x &= 11. \end{aligned}$$

11.
$$\begin{aligned} 26 - 8x &= 80 - 14x, \\ 14x - 8x &= 80 - 26, \\ 6x &= 54, \\ x &= 9. \end{aligned}$$

12.
$$\begin{aligned} 13 - 3x &= 5x - 3, \\ -5x - 3x &= -3 - 13, \\ -8x &= -16, \\ x &= 2. \end{aligned}$$

13.
$$\begin{aligned} 3x - 22 &= 7x + 6, \\ 3x - 7x &= 6 + 22, \\ -4x &= 28, \\ x &= -7. \end{aligned}$$

14.
$$\begin{aligned} 8 + 4x &= 12x - 16, \\ 4x - 12x &= -16 - 8, \\ -8x &= -24, \\ x &= 3. \end{aligned}$$

15.
$$\begin{aligned} 5x - (3x - 7) &= 4x - (6x - 35), \\ 5x - 3x + 7 &= 4x - 6x + 35, \\ -4x + 5x - 3x + 6x &= 35 - 7, \\ 4x &= 28, \\ x &= 7. \end{aligned}$$

16.
$$\begin{aligned} 6x - 2(9 - 4x) + 3(5x - 7) &= 10x - (4 + 16x + 35), \\ 6x - 18 + 8x + 15x - 21 &= 10x - 4 - 16x - 35, \\ 6x + 8x + 15x - 10x + 16x &= 18 + 21 - 4 - 35, \\ 35x &= 0, \\ x &= 0. \end{aligned}$$

17.
$$\begin{aligned} 9x - 3(5x-6) + 30 &= 0,\\ 9x - 15x + 18 + 30 &= 0,\\ 9x - 15x &= -18 - 30,\\ -6x &= -48,\\ x &= 8.\end{aligned}$$

18.
$$\begin{aligned} x - 7(4x-11) &= 14(x-5) - 19(8-x) - 61,\\ x - 28x + 77 &= 14x - 70 - 152 + 19x - 61,\\ x - 28x - 14x - 19x &= -70 - 152 - 61 - 77,\\ -60x &= -360,\\ x &= 6.\end{aligned}$$

19.
$$\begin{aligned} (x+7)(x-3) &= (x-5)(x-15),\\ x^2 + 4x - 21 &= x^2 - 20x + 75,\\ 4x + 20x &= 75 + 21,\\ 24x &= 96,\\ x &= 4.\end{aligned}$$

20.
$$\begin{aligned} (x-8)(x+12) &= (x+1)(x-6),\\ x^2 + 4x - 96 &= x^2 - 5x - 6,\\ 4x + 5x &= -6 + 96,\\ 9x &= 90,\\ x &= 10.\end{aligned}$$

21.
$$\begin{aligned} (x-2)(7-x) + (x-5)(x+3) - 2(x-1) + 12 &= 0,\\ 9x - 14 - x^2 + x^2 - 2x - 15 - 2x + 2 + 12 &= 0,\\ 9x - 2x - 2x &= 14 + 15 - 2 - 12,\\ 5x &= 15,\\ x &= 3.\end{aligned}$$

22.
$$\begin{aligned} (2x-7)(x+5) &= (9-2x)(4-x) + 229,\\ 2x^2 + 3x - 35 &= 36 - 17x + 2x^2 + 229,\\ 20x &= 300,\\ x &= 15.\end{aligned}$$

23.
$$\begin{aligned} 14 - x - 5(x-3)(x+2) + (5-x)(4-5x) &= 45x - 76,\\ 14 - x - 5x^2 + 5x + 30 + 5x^2 - 29x + 20 &= 45x - 76,\\ 5x - 29x - 45x - x &= -76 - 20 - 30 - 14,\\ -70x &= -140,\\ x &= 2.\end{aligned}$$

24.
$$\begin{aligned} (x+5)^2 - (4-x)^2 &= 21x,\\ (x^2 + 10x + 25) - (16 - 8x + x^2) &= 21x,\\ x^2 + 10x + 25 - 16 + 8x - x^2 &= 21x,\\ 10x + 8x - 21x &= -25 + 16,\\ -3x &= -9,\\ x &= 3.\end{aligned}$$

25.
$$\begin{aligned} 5(x-2)^2 + 7(x-3)^2 &= (3x-7)(4x-19) + 42,\\ 5(x^2 - 4x + 4) + 7(x^2 - 6x + 9) &= 12x^2 - 85x + 133 + 42,\\ 5x^2 - 20x + 20 + 7x^2 - 42x + 63 &= 12x - 85x + 133 + 42,\\ 23x &= 92,\\ x &= 4.\end{aligned}$$

EXERCISE XXIX.

6. To the double of a certain number I add 14, and obtain as a result 154. What is the number?

Let $x =$ the number.
Then $2x =$ its double,
and $2x + 14 =$ its double increased by 14.
But $154 =$ its double increased by 14.
Therefore, $2x + 14 = 154$, $2x = 140$, $x = 70$.

7. To four times a certain number I add 16, and obtain as a result 188. What is the number?

Let $x =$ the number.
Then $4x = 4$ times the number,
and $4x + 16 = 4$ times the number increased by 16.
But $188 = 4$ times the number increased by 16.
Therefore, $4x + 16 = 188$, $4x = 172$, $x = 43$.

8. By adding 46 to a certain number, I obtain as a result a number three times as large as the original number. Find the original number.

Let $x =$ the original number.
Then $3x = 3$ times the original number.
But $x + 46 = 3$ times the original number.
Therefore, $3x = x + 46$, $2x = 46$, $x = 23$.

9. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

Let $x =$ the smaller number.
Then $3x =$ the larger number,
and $16 - x = 16$ diminished by the smaller number;
also, $30 - 3x = 30$ diminished by the larger number.
Therefore, $16 - x = 30 - 3x$, $2x = 14$, $x = 7$, $3x = 21$.

10. Divide the number 92 into four parts, such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

Let $x =$ the first part.
Then $x - 10 =$ the second part,
$x - 18 =$ the third part,
$x - 24 =$ the fourth part.
and $4x - 52 =$ the whole number.
But $92 =$ the whole number.
$\therefore 4x - 52 = 92$, $4x = 144$, $x = 36$, $x - 10 = 26$, $x - 18 = 18$, $x - 24 = 12$.

11. The sum of two numbers is 20; and, if three times the smaller number is added to five times the greater, the sum is 84. What are the numbers?

Let $x =$ the greater number.
Then $20 - x =$ the smaller number,
$5x = 5$ times the greater number,
$3(20 - x) = 3$ times the smaller number,
$5x + 3(20 - x) = 5$ times the greater $+ 3$ times the smaller.
But $84 = 5$ times the greater $+ 3$ times the smaller.
$\therefore 5x + 3(20 - x) = 84,\ 5x + 60 - 3x = 84,\ 2x = 24,\ x = 12,\ 20 - x = 8.$

12. The joint ages of a father and son are 80 years. If the age of the son were doubled, he would be 10 years older than his father. What is the age of each?

Let $x =$ number of years of father's age.
Then $80 - x =$ number of years of son's age,
$2(80 - x) =$ number of years of father's age $+ 10$,
$x + 10 =$ number of years of father's age $+ 10$.
$\therefore 2(80 - x) = x + 10,\ 160 - 2x = x + 10,\ -3x = -150,\ x = 50,\ 80 - x = 30.$

13. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

Let $x =$ number of years of age of youngest.
Then $x + 4 =$ number of years of age of second,
$x + 8 =$ number of years of age of third,
$x + 12 =$ number of years of age of fourth,
$x + 16 =$ number of years of age of fifth,
$x + 20 =$ number of years of age of sixth.
$3x = 3$ times age of youngest.
$\therefore 3x = x + 20,\ 2x = 20,\ x = 10,\ x + 4 = 14,\ x + 8 = 18,$
$x + 12 = 22,\ x + 16 = 26,\ x + 20 = 30.$

14. Add \$24 to a certain sum and the amount will be as much above \$80 as the sum is below \$80. What is the sum?

Let $x =$ number of dollars in sum.
Then $x + 24 - 80 =$ number of dollars above 80,
and $80 - x =$ number of dollars below 80.
$\therefore x + 24 - 80 = 80 - x,\ 2x = 136,\ x = 68.$

15. Thirty yards of cloth and 40 yards of silk together cost \$330; and the silk twice as much a yard as the cloth. How much did each cost a yard?

Let $x =$ number of dollars one yard of cloth cost.
Then $2x =$ number of dollars one yard of silk cost.
$30x + 80x =$ number of dollars all cost.
But $330 =$ number of dollars all cost.
$\therefore 30x + 80x = 330,\ 110x = 330,\ x = 3,\ 2x = 6.$

16. **Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.**

Let $x =$ the number.
Then $2x + 24 =$ its double increased by 24,
$2x + 24 - 80 =$ excess over 80,
$100 - x =$ difference between the number and 100.

$$\therefore 2x + 24 - 80 = 100 - x, \quad 3x = 156, \quad x = 52.$$

17. **The sum of \$500 is divided among A, B, C, and D. A and B have together \$280, A and C \$260, and A and D \$220. How much does each receive?**

Let $x =$ number of dollars A has.
Then $280 - x =$ number of dollars B has,
$260 - x =$ number of dollars C has,
$220 - x =$ number of dollars D has,
$760 - 2x =$ number of dollars all have.
But $500 =$ number of dollars all have.

$$\therefore 760 - 2x = 500, \quad -2x = -260, \quad x = 130,$$
$$280 - x = 150, \quad 260 - x = 130, \quad 220 - x = 90.$$

18. **In a company of 266 persons composed of men, women, and children, there are twice as many men as women, and twice as many women as children. How many are there of each?**

Let $x =$ number of children.
Then $2x =$ number of women,
and $4x =$ number of men,
$7x =$ whole number.
But $266 =$ whole number.

$$\therefore 7x = 266, \quad x = 38, \quad 2x = 76, \quad 4x = 152.$$

19. **Find two numbers differing by 8, such that four times the less may exceed twice the greater by 10.**

Let $x =$ greater number.
Then $x - 8 =$ smaller number.
$4(x - 8) - 2x = 10.$

$$\therefore 4x - 32 - 2x = 10, \quad 2x = 42, \quad x = 21, \quad x - 8 = 13.$$

20. **A is 58 years older than B, and A's age is as much above 60 as B's age is below 50. Find the age of each.**

Let $x =$ number of years of B's age.
Then $x + 58 =$ number of years of A's age,
$(x + 58) - 60 =$ number of years of A's age above 60,
$50 - x =$ number of years of B's age below 50.

$$\therefore (x + 58) - 60 = 50 - x, \quad 2x = 52, \quad x = 26, \quad x + 58 = 84.$$

21. A man leaves his property, amounting to $7500, to be divided among his wife, his two sons, and three daughters, as follows: a son is to have twice as much as a daughter, and the wife $500 more than all the children together. How much was the share of each?

Let $x =$ number of dollars in a daughter's share.
Then $2x =$ number of dollars in a son's share,
and $3x =$ number of dollars given to all the daughters.
also, $4x =$ number of dollars given to all the sons.
$7x =$ number of dollars given to all sons and daughters,
$7x + 500 =$ number of dollars given to wife,
$7x + 7x + 500 =$ number of dollars in whole estate.
But $7500 =$ number of dollars in whole estate.

$\therefore 7x + 7x + 500 = 7500, \quad 14x = 7000, \quad x = 500,$
$2x = 1000, \quad 7x + 500 = 4000.$

22. A vessel containing some water was filled by pouring in 42 gallons, and there was then in the vessel seven times as much as at first. How much did the vessel hold?

Let $x =$ number of gallons the vessel holds.
Then $x - 42 =$ number of gallons there were in the vessel,
$7(x-42) =$ 7 times number of gallons there were at first.

$\therefore x = 7(x-42), \quad x = 7x - 294, \quad -6x = -294, \quad x = 49.$

23. A has $72 and B has $52. B gives A a certain sum; then A has three times as much as B. How much did A receive from B?

Let $x =$ number of dollars A receives from B.
Then $52 - x =$ number of dollars B has left,
and $72 + x =$ number of dollars A has.

$\therefore 72 + x = 3(52 - x), \quad 72 + x = 156 - 3x, \quad 4x = 84, \quad x = 21.$

24. Divide 90 into two such parts that four times one part may be equal to five times the other.

Let $x =$ larger number.
Then $90 - x =$ smaller number.

$\therefore 4x = 5(90-x), \quad 4x = 450 - 5x, \quad 9x = 450, \quad x = 50, \quad 90 - x = 40.$

25. Divide 60 into two such parts that one part exceeds the other by 24.

Let $x =$ lesser part.
Then $x + 24 =$ greater part.

$\therefore x + x + 24 = 60, \quad 2x = 36, \quad x = 18, \quad x + 24 = 42.$

26. Divide 84 into two such parts that one part may be less than the other by 36.

Let $x =$ lesser part.
Then $x + 36 =$ greater part.
$\therefore x + x + 36 = 84,\ \ 2x = 48,\ \ x = 24,\ \ 84 - x = 60.$

27. A is twice as old as B, and 22 years ago he was three times as old as B. What is A's age?

Let $x =$ number of years of B's age.
Then $2x =$ number of years of A's age;
also, $x - 22 =$ number of years of B's age 22 years ago,
and $2x - 22 =$ number of years of A's age 22 years ago.
$\therefore 3(x - 22) = 2x - 22,\ \ 3x - 66 = 2x - 22,\ \ x = 44,\ \ 2x = 88.$

28. A father is 30 and his son 6 years old. In how many years will the father be just twice as old as the son?

Let $x =$ number of years.
Then $x + 30 =$ number of years of father's age x years hence,
and $x + 6 =$ number of years of son's age x years hence.
$\therefore 30 + x = 2(x + 6),\ \ 30 + x = 2x + 12,\ \ x = 18.$

29. A is twice as old as B, and 20 years since he was three times as old. What is B's age?

Let $x =$ B's age.
Then $2x =$ A's age;
also, $x - 20 =$ B's age 20 years since,
and $2x - 20 =$ A's age 20 years since.
$\therefore 2x - 20 = 3(x - 20),\ \ 2x - 20 = 3x - 60,\ \ x = 40.$

30. A is three times as old as B, and 19 years hence he will be only twice as old as B. What is the age of each?

Let $x =$ number of years of B's age.
Then $3x =$ number of years of A's age;
also, $x + 19 =$ number of years of B's age 19 years hence,
and $3x + 19 =$ number of years of A's age 19 years hence.
$\therefore 3x + 19 = 2(x + 19),\ \ 3x + 19 = 2x + 38,\ \ x = 19,\ \ 3x = 57.$

31. A man has three nephews; his age is 50, and the joint ages of the nephews is 42. How long will it be before the joint ages of the nephews will be equal to that of the uncle?

Let $x =$ the number of years.
Then $50 + x =$ number of years of uncle's age x years hence.
$3x + 42 =$ number of years of nephews' age x years hence
$\therefore 3x + 42 = 50 + x,\ \ 2x = 8,\ \ x = 4.$

32. A sum of money consists of dollars and twenty-five-cent pieces, and amounts to $20. The number of coins is 50. How many are there of each sort?

Let $x =$ number of dollars.
Then $50 - x =$ number of quarters,
and $x + \dfrac{50-x}{4} =$ sum in dollars.
But $20 =$ sum in dollars.

$$\therefore x + \frac{50-x}{4} = 20,\ 4x + 50 - x = 80,\ 3x = 30,\ x = 10,\ 50 - x = 40.$$

33. A person bought 30 pounds of sugar of two different kinds, and paid for the whole $2.94. The better kind cost 10 cents a pound, and the poorer kind 7 cents a pound. How many pounds were there of each kind?

Let $x =$ number of pounds of the better kind.
Then $30 - x =$ number of pounds of the poorer kind,
and $10x + 7(30 - x) =$ number of cents he paid for all.
But $294 =$ number of cents he paid for all.

$$\therefore 10x + 7(30 - x) = 294,\ \ 10x + 210 - 7x = 294,$$
$$3x = 84,\ \ x = 28,\ \ 30 - x = 2.$$

34. A workman was hired for 40 days, at $1 for every day he worked, but with the condition that for every day he did not work he was to pay 45 cents for his board. At the end of the time he received $22.60. How many days did he work?

Let $x =$ number of days he was idle.
Then $40 - x =$ number of days he worked.
and $45x =$ number of cents he paid for board;
also, $4000 - 100x =$ number of cents he received for work,
$(4000 - 100x) - 45x =$ number of cents cleared.
But $2260 =$ number of cents cleared.

$$\therefore 4000 - 100x - 45x = 2260,\ \ -145x = -1740,\ \ x = 12,\ \ 40 - x = 28.$$

35. A wine merchant has two kinds of wine; one worth 50 cents a quart, and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons, worth $2.40 a gallon. How many gallons must he take of each kind?

Let $x =$ number of gallons at $2.
Then $100 - x =$ number of gallons at $3,
and $2x =$ number of dollars one part cost;
also, $3(100 - x) =$ number of dollars the other part cost,
and $2x + 3(100 - x) =$ number of dollars all cost.
But $240 =$ number of dollars all cost.

$$\therefore 2x + 3(100 - x) = 240,\ \ 2x + 300 - 3x = 240,\ \ x = 60,\ \ 100 - x = 40.$$

36. A gentleman gave some children 10 cents each, and had a dollar left. He found that he would have required one dollar more to enable him to give them 15 cents each. How many children were there?

$$\begin{array}{lrl} \text{Let} & x &= \text{number of children.} \\ \text{Then} & 10x &= \text{number of cents given,} \\ \text{and} & 10x + 200 &= \text{number of cents required to give each 15 cts.} \\ \text{But} & 15x &= \text{number of cents required to give each 15 cts.} \end{array}$$

$$\therefore 10x + 200 = 15x, \quad -5x = -200, \quad x = 40.$$

37. Two casks contain equal quantities of vinegar: from the first cask 34 quarts are drawn; from the second, 20 gallons; the quantity remaining in one vessel is now twice that in the other. How much did each cask contain at first?

$$\begin{array}{lrl} \text{Let} & x &= \text{number of quarts each contained at first.} \\ \text{Then} & x - 34 &= \text{number of quarts first now contains,} \\ \text{and} & x - 80 &= \text{number of quarts second now contains.} \\ & 2(x - 80) &= \text{twice the No. quarts second now contains.} \end{array}$$

$$\therefore 2(x - 80) = x - 34, \quad 2x - 160 = x - 34, \quad x = 126.$$

38. A gentleman hired a man for 12 months, at the wages of $90 and a suit of clothes. At the end of 7 months the man quits his service, and receives $33.75 and the suit of clothes. What was the price of the suit of clothes?

$$\begin{array}{lrl} \text{Let} & x &= \text{number of dollars the suit cost.} \\ \text{Then} & x + 90 &= \text{number of dollars he receives by the year.} \\ \text{and} & \dfrac{x + 90}{12} &= \text{number of dollars he receives by the month.} \\ \text{and} & \dfrac{7(x + 90)}{12} &= \text{number of dollars he receives for 7 months.} \\ \text{But} & x + 33.75 &= \text{number of dollars he receives for 7 months.} \end{array}$$

$$\therefore \frac{7(x + 90)}{12} = x + 33.75, \quad 7x + 630 = 405 + 12x, \quad 5x = 225, \quad x = 45.$$

39. A man has three times as many quarters as half-dollars, four times as many dimes as quarters, and twice as many half-dimes as dimes. The whole sum is $7.30. How many coins has he in all?

$$\begin{array}{lrl} \text{Let} & x &= \text{number of half-dollar pieces.} \\ \text{Then} & 3x &= \text{number of quarter-dollar pieces,} \\ & 12x &= \text{number of dimes,} \\ & 24x &= \text{number of half-dimes.} \end{array}$$

$$\frac{x}{2} + \frac{3x}{4} + \frac{12x}{10} + \frac{24x}{20} = \text{the whole sum in dollars.}$$

$$\text{But} \quad 7.30 = \text{the whole sum in dollars.}$$

$$\therefore \frac{x}{2} + \frac{3x}{4} + \frac{12x}{10} + \frac{24x}{20} = 7.30, \quad 10x + 15x + 24x + 24x = 146,$$

$$73x = 146, \quad x = 2, \quad 3x = 6, \quad 12x = 24, \quad 24x = 48$$

40. A person paid a bill of \$15.25 with quarters and half-dollars, and gave 51 pieces of money all together. How many of each kind were there?

$$
\begin{aligned}
\text{Let}\qquad x &= \text{number of half-dollars.}\\
\text{Then}\qquad 51-x &= \text{number of quarter-dollars.}\\
50x &= \text{number of cents in half-dollars,}\\
25(51-x) &= \text{number of cents in quarter-dollars,}\\
50x+25(51-x) &= \text{number of cents in all.}\\
\text{But}\qquad 1525 &= \text{number of cents in all.}\\
\therefore 50x+25(51-x) &= 1525,\quad 50x+1275-25x=1525,\\
25x &= 250,\quad x=10,\quad 51-x=41.
\end{aligned}
$$

41. A bill of £100 was paid with guineas (21 shillings) and half-crowns ($2\frac{1}{2}$ shillings), and 48 more half-crowns than guineas were used. How many of each were paid?

$$
\begin{aligned}
\text{Let}\qquad x &= \text{number of guineas.}\\
\text{Then}\qquad x+48 &= \text{number of half-crowns,}\\
21x+2\tfrac{1}{2}(x+48) &= \text{number of shillings in the lot.}\\
\text{But}\qquad 2000 &= \text{number of shillings in the lot.}\\
\therefore 21x+2\tfrac{1}{2}(x+48) &= 2000,\quad 21x+\frac{5x+240}{2}=2000,\\
42x+5x+240 &= 4000,\quad 47x=3760,\quad x=80,\quad x+48=128.
\end{aligned}
$$

Exercise XXX.

1. $5a^2-15a$
$=5a(a-3).$

2. $6a^3+18a^2-12a$
$=6a(a^2+3a-2).$

3. $49x^2-21x+14$
$=7(7x^2-3x+2).$

4. $4x^3y-12x^2y^2+8xy^3$
$=4xy(x^2-3xy+2y^2).$

5. $y^4-ay^3+by^2+cy$
$=y(y^3-ay^2+by+c).$

6. $6a^5b^3-21a^4b^2+27a^3b^4$
$=3a^3b^2(2a^2b-7a+9b^2).$

7. $54x^2y^6+108x^4y^8-243x^6y^9$
$=27x^2y^6(2+4x^2y^2-9x^4y^3).$

8. $45x^7y^{10}-90x^5y^7-360x^4y^8$
$=45x^4y^7(x^3y^3-2x-8y).$

9. $70a^3y^4-140a^2y^5+210ay^6$
$=70ay^4(a^2-2ay+3y^2).$

10. $32a^3b^6+96a^6b^8-128a^8b^9$
$=32a^3b^6(1+3a^3b^2-4a^5b^3)$

Exercise XXXI.

1. $x^2-ax-bx+ab$
$=(x-a)(x-b).$

2. $ab+ay-by-y^2$
$=(a-y)(b+y).$

3. $bc+bx-cx-x^2$
$=(b-x)(c+x).$

4. $mx+mn+ax+an$
$=(m+a)(x+n).$

5. $cdx^2 - cxy + dxy - y^2$
$= (cx + y)(dx - y)$.

6. $abx - aby + pqx - pqy$
$= (ab + pq)(x - y)$.

7. $cdx^2 + adxy - bcxy - aby^2$
$= (cx + ay)(dx - by)$.

8. $abcy - b^2dy - acdx + bd^2x$
$= (ac - bd)(by - dx)$.

9. $ax - ay - bx + by$
$= (a - b)(x - y)$.

10. $cdz^2 - cyz + dyz - y^2$
$= (cz + y)(dz - y)$.

Exercise XXXII.

1. $x^2 + 11x + 24$
$= (x + 8)(x + 3)$.

2. $x^2 + 11x + 30$
$= (x + 6)(x + 5)$.

3. $y^2 + 17y + 60$
$= (y + 12)(y + 5)$.

4. $z^2 + 13z + 12$
$= (z + 12)(z + 1)$.

5. $x^2 + 21x + 110$
$= (x + 11)(x + 10)$.

6. $y^2 + 35y + 300$
$= (y + 20)(y + 15)$.

7. $b^2 + 23b + 102$
$= (b + 17)(b + 6)$.

8. $x^2 + 3x + 2$
$= (x + 2)(x + 1)$.

9. $x^2 + 7x + 6$
$= (x + 6)(x + 1)$.

10. $a^2 + 9ab + 8b^2$
$= (a + 8b)(a + b)$.

11. $x^2 + 13ax + 36a^2$
$= (x + 9a)(x + 4a)$.

12. $y^2 + 19py + 48p^2$
$= (y + 16p)(y + 3p)$.

13. $z^2 + 29qz + 100q^2$
$= (z + 25q)(z + 4q)$.

14. $a^4 + 5a^2 + 6$
$= (a^2 + 3)(a^2 + 2)$.

15. $z^6 + 4z^3 + 3$
$= (z^3 + 3)(z^3 + 1)$.

16. $a^2b^2 + 18ab + 32$
$= (ab + 16)(ab + 2)$.

17. $x^8y^4 + 7x^4y^2 + 12$
$= (x^4y^2 + 4)(x^4y^2 + 3)$.

18. $z^{10} + 10z^5 + 16$
$= (z^5 + 8)(z^5 + 2)$.

19. $a^2 + 9ab + 20b^2$
$= (a + 5b)(a + 4b)$.

20. $x^6 + 9x^3 + 20$
$= (x^3 + 5)(x^3 + 4)$.

21. $a^2x^2 + 14abx + 33b^2$
$= (ax + 11b)(ax + 3b)$.

22. $a^2c^2 + 7acx + 10x^2$
$= (ac + 5x)(ac + 2x)$.

23. $x^2y^2z^2 + 19xyz + 48$
$= (xyz + 16)(xyz + 3)$.

24. $b^2c^2 + 18abc + 65a^2$
$= (bc + 13a)(bc + 5a)$.

25. $r^2s^2 + 23rsz + 90z^2$
$= (rs + 18z)(rs + 5z)$.

26. $m^4n^4 + 20m^2n^2pq + 51p^2q^2$
$= (m^2n^2 + 17pq)(m^2n^2 + 3pq)$.

Exercise XXXIII.

1. $x^2 - 7x + 10$
$= (x - 5)(x - 2)$.

2. $x^2 - 29x + 190$
$= (x - 19)(x - 10)$.

3. $a^2 - 23a + 132$
$= (a - 12)(a - 11)$.

4. $b^2 - 30b + 200$
$= (b - 20)(b - 10)$.

5. $z^2 - 43z + 460$
$= (z - 23)(z - 20)$.

6. $x^2 - 7x + 6$
$= (x - 6)(x - 1)$.

7. $x^4 - 4a^2x^2 + 3a^4$
$= (x^2 - 3a^2)(x^2 - a^2)$.

8. $x^2 - 8x + 12$
$= (x - 6)(x - 2)$.

9. $z^2 - 57z + 56$
$= (z - 56)(z - 1)$.

10. $y^6 - 7y^3 + 12$
$= (y^3 - 4)(y^3 - 3)$.

11. $x^2y^2 - 27xy + 26$
$= (xy - 26)(xy - 1)$.

12. $a^4b^6 - 11a^2b^3 + 30$
$= (a^2b^3 - 6)(a^2b^3 - 5)$.

13. $a^2b^2c^2 - 13abc + 22$
$= (abc - 11)(abc - 2)$.

14. $x^2 - 15x + 50$
$= (x - 10)(x - 5)$.

15. $x^2 - 20x + 100$
$= (x - 10)(x - 10)$.

16. $a^2x^2 - 21ax + 54$
$= (ax - 18)(ax - 3)$.

17. $a^2x^2 - 16abx + 39b^2$
$= (ax - 13b)(ax - 3b)$.

18. $a^2c^2 - 24acz + 143z^2$
$= (ac - 13z)(ac - 11z)$.

19. $x^2 - 20x + 91$
$= (x - 13)(x - 7)$.

20. $x^2 - 23x + 120$
$= (x - 15)(x - 8)$.

21. $z^2 - 53z + 360$
$= (z - 45)(z - 8)$.

22. $x^2 - (a + c)x + ac$
$= (x - a)(x - c)$.

23. $y^2z^2 - 28abyz + 187a^2b^2$
$= (yz - 17ab)(yz - 11ab)$.

24. $c^2d^2 - 30abcd + 221a^2b^2$
$= (cd - 17ab)(cd - 13ab)$

Exercise XXXIV.

1. $x^2 + 6x - 7$
$= (x + 7)(x - 1)$.

2. $x^2 + 5x - 84$
$= (x + 12)(x - 7)$.

3. $y^2 + 7y - 60$
$= (y + 12)(y - 5)$.

4. $y^2 + 12y - 45$
$= (y + 15)(y - 3)$.

5. $z^2 + 11z - 12$
$= (z + 12)(z - 1)$.

6. $z^2 + 13z - 140$
$= (z + 20)(z - 7)$.

7. $a^2 + 13a - 300$
$= (a + 25)(a - 12)$.

8. $a^2 + 25a - 150$
$= (a + 30)(a - 5)$.

9. $b^8 + 3b^4 - 4$
$= (b^4 + 4)(b^4 - 1)$.

10. $b^2c^2 + 3bc - 154$
$= (bc + 14)(bc - 11)$.

11. $c^{10} + 15c^5 - 100$
$= (c^5 + 20)(c^5 - 5)$.

12. $c^2 + 17c - 390$
$= (c + 30)(c - 13)$.

13. $a^2 + a - 132$
$= (a + 12)(a - 11)$.

14. $x^2y^2z^2 + 9xyz - 22$
$= (xyz + 11)(xyz - 2)$.

Exercise XXXV.

1. $x^2 - 3x - 28$
$= (x - 7)(x + 4)$.

2. $y^2 - 7y - 18$
$= (y - 9)(y + 2)$.

3. $x^2 - 9x - 36$
$= (x - 12)(x + 3)$.

4. $z^2 - 11z - 60$
$= (z - 15)(z + 4)$.

5. $z^2 - 13z - 14$
$= (z - 14)(z + 1)$.

6. $a^2 - 15a - 100$
$= (a - 20)(a + 5)$.

7. $c^{10} - 9c^5 - 10$
$= (c^5 - 10)(c^5 + 1)$.

8. $x^2 - 8x - 20$
$= (x - 10)(x + 2)$.

9. $y^2 - 5ay - 50a^2$
$= (y - 10a)(y + 5a)$.

10. $a^2b^2 - 3ab - 4$
$= (ab - 4)(ab + 1)$.

11. $a^2x^2 - 3ax - 54$
$= (ax - 9)(ax + 6)$.

12. $c^2d^2 - 24cd - 180$
$= (cd - 30)(cd + 6)$.

13. $a^6c^2 - a^3c - 2$
$= (a^3c - 2)(a^3c + 1)$.

14. $y^8z^4 - 5y^4z^2 - 84$
$= (y^4z^2 - 12)(y^4z^2 + 7)$.

15. $a^2b^2 - 16ab - 36$
$= (ab - 18)(ab + 2)$.

16. $x^2 - (a - b)x - ab$
$= (x - a)(x + b)$.

Exercise XXXVI.

1. $x^2 + 12x + 36$
$= (x + 6)^2$.

2. $x^2 + 28x + 196$
$= (x + 14)^2$.

3. $x^2 + 34x + 289$
$= (x + 17)^2$.

4. $z^2 + 2z + 1$
$= (z + 1)^2$.

5. $y^2 + 200y + 10{,}000$
$= (y + 100)^2$.

6. $z^4 + 14z^2 + 49$
$= (z^2 + 7)^2$.

7. $x^2 + 36xy + 324y^2$
$= (x + 18y)^2$.

8. $y^4 + 16y^2z^2 + 64z^4$
$= (y^2 + 8z^2)^2$.

9. $y^6 + 24y^3 + 144$
$= (y^3 + 12)^2$.

10. $x^2z^2 + 162xz + 6561$
$= (xz + 81)^2$.

11. $4a^2 + 12ab^2 + 9b^4$
$= (2a + 3b^2)^2.$

12. $9x^2y^4 + 30xy^2z + 25z^2$
$= (3xy^2 + 5z)^2.$

13. $9x^2 + 12xy + 4y^2$
$= (3x + 2y)^2.$

14. $4a^4x^2 + 20a^2x^3y + 25x^4y^2$
$= (2a^2x + 5x^2y)^2.$

Exercise XXXVII.

1. $a^2 - 8a + 16$
$= (a - 4)^2.$

2. $a^2 - 30a + 225$
$= (a - 15)^2.$

3. $x^2 - 38x + 361$
$= (x - 19)^2.$

4. $x^2 - 40x + 400$
$= (x - 20)^2.$

5. $y^2 - 100y + 2500$
$= (y - 50)^2.$

6. $y^4 - 20y^2 + 100$
$= (y^2 - 10)^2.$

7. $y^2 - 50yz + 625z^2$
$= (y - 25z)^2.$

8. $x^4 - 32x^2y^2 + 256y^4$
$= (x^2 - 16y^2)^2.$

9. $z^6 - 34z^3 + 289$
$= (z^3 - 17)^2.$

10. $4x^4y^2 - 20x^2y^3z + 25y^4z^2$
$= (2x^2y - 5y^2z)^2.$

11. $16x^2y^4 - 8xy^3z^2 + y^2z^4$
$= (4xy^2 - yz^2)^2.$

12. $9a^2b^2c^2 - 6ab^2c^2d + b^2c^2d^2$
$= (3abc - bcd)^2.$

13. $16x^6 - 8x^4y^2 + x^2y^4$
$= (4x^3 - xy^2)^2.$

14. $a^6x^4 - 2a^3bx^2y^4 + b^2y^8$
$= (a^3x^2 - by^4)^2.$

15. $36x^2y^2 - 60xy^3 + 25y^4$
$= (6xy - 5y^2)^2.$

16. $1 - 6ab^3 + 9a^2b^6$
$= (1 - 3ab^3)^2.$

17. $9m^2n^2 - 24mn + 16$
$= (3mn - 4)^2.$

18. $4b^2x^2 - 12bx^2y + 9x^2y^2$
$= (2bx - 3xy)^2.$

19. $49a^2 - 112ab + 64b^2$
$= (7a - 8b)^2.$

20. $64x^4y^6 - 160x^4y^3z + 100x^4z^2$
$= (8x^2y^3 - 10x^2z)^2.$

21. $49a^2b^2c^2 - 28abcx + 4x^2$
$= (7abc - 2x)^2.$

22. $121x^4 - 286x^2y + 169y^2$
$= (11x^2 - 13y)^2.$

23. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2$
$= (17xyz - 3yzd)^2.$

24. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2$
$= (19xyz - 2abc)^2.$

Exercise XXXVIII.

1. $a^2 - b^2$
$= (a + b)(a - b).$

2. $a^2 - 16$
$= (a + 4)(a - 4).$

3. $4a^2 - 25$
$= (2a + 5)(2a - 5).$

4. $a^4 - b^4$
$= (a^2 + b^2)(a + b)(a - b).$

5. $a^4 - 1$
$= (a^2 + 1)(a + 1)(a - 1).$

6. $a^8 - b^8$
$= (a^4+b^4)(a^2+b^2)(a+b)(a-b).$

7. $a^8 - 1$
$= (a^4+1)(a^2+1)(a+1)(a-1).$

8. $36x^2 - 49y^2$
$= (6x + 7y)(6x - 7y).$

9. $100x^2y^2 - 121a^2b^2$
$= (10xy+11ab)(10xy-11ab).$

10. $1 - 49x^2$
$= (1 + 7x)(1 - 7x).$

11. $a^4 - 25b^2$
$= (a^2 + 5b)(a^2 - 5b).$

12. $(a - b)^2 - c^2$
$= \{(a - b) - c\}\{(a - b) + c\}$
$= (a - b - c)(a - b + c).$

13. $x^2 - (a - b)^2$
$= \{x - (a - b)\}\{x + (a - b)\}$
$= (x - a + b)(x + a - b).$

14. $(a + b)^2 - (c + d)^2$
$= [(a + b) + (c + d)][(a + b) - (c + d)]$
$= (a + b + c + d)(a + b - c - d).$

15. $(x + y)^2 - (x - y)^2$
$= \{(x + y) + (x - y)\}\{(x + y) - (x - y)\}$
$= (x + y + x - y)(x + y - x + y)$
$= 4xy.$

16. $2ab - a^2 - b^2 + 1$
$= 1 - (a^2 - 2ab + b^2)$
$= 1 - (a - b)^2$
$= \{1 + (a - b)\}\{1 - (a - b)\}$
$= (1 + a - b)(1 - a + b).$

17. $x^2 - 2yz - y^2 - z^2$
$= x^2 - (y^2 + 2yz + z^2)$
$= x^2 - (y + z)^2$
$= \{x + (y + z)\}\{x - (y + z)\}$
$= (x + y + z)(x - y - z).$

18. $x^2 - 2xy + y^2 - z^2$
$= (x^2 - 2xy + y^2) - z^2$
$= (x - y)^2 - z^2$
$= (x - y + z)(x - y - z).$

19. $a^2 + 12bc - 4b^2 - 9c^2$
$= a^2 - (4b^2 - 12bc + 9c^2)$
$= a^2 - (2b - 3c)^2$
$= \{a + (2b - 3c)\}\{a - (2b - 3c)\}$
$= (a + 2b - 3c)(a - 2b + 3c).$

20. $a^2 - 2ay + y^2 - x^2 - 2xz - z^2$
$= (a^2 - 2ay + y^2) - (x^2 + 2xz + z^2)$
$= (a - y)^2 - (x + z)^2$
$= \{(a - y) + (x + z)\}\{(a - y) - (x + z)\}$
$= (a - y + x + z)(a - y - x - z).$

21. $2xy - x^2 - y^2 + z^2$
$= z^2 - (x^2 - 2xy + y^2)$
$= z^2 - (x - y)^2$
$= \{z + (x - y)\}\{z - (x - y)\}$
$= (z + x - y)(z - x + y).$

22. $x^2 + y^2 - z^2 - d^2 - 2xy - 2dz$
$= (x^2 - 2xy + y^2) - (d^2 + 2dz + z^2)$
$= (x - y)^2 - (d + z)^2$
$= \{(x - y) - (d + z)\}\{(x - y) + (d + z)\}$
$= (x - y - d - z)(x - y + d + z).$

23. $x^2 - y^2 + z^2 - a^2 - 2xz + 2ay$
$= (x^2 - 2xz + z^2) - (a^2 - 2ay + y^2)$
$= (x - z)^2 - (a - y)^2$
$= \{(x - z) - (a - y)\}\{(x - z) + (a - y)\}$
$= (x - z - a + y)(x - z + a - y).$

24. $2ab + a^2 + b^2 - c^2$
$= (a^2 + 2ab + b^2) - c^2$
$= (a + b)^2 - c^2$
$= (a + b + c)(a + b - c).$

25. $2xy - x^2 - y^2 + a^2 + b^2 - 2ab$
$= (a^2 - 2ab + b^2) - (x^2 - 2xy + y^2)$
$= (a - b)^2 - (x - y)^2$
$= \{(a - b) + (x - y)\}\{(a - b) - (x - y)\}$
$= (a - b + x - y)(a - b - x + y).$

26. $(ax + by)^2 - 1$
$= (ax + by + 1)(ax + by - 1).$

27. $1 - x^2 - y^2 + 2xy$
$= 1 - (x^2 - 2xy + y^2)$
$= 1 - (x - y)^2$
$= (1 + x - y)(1 - x + y).$

28. $(5a - 2)^2 - (a - 4)^2$
$= \{(5a - 2) + (a - 4)\}\{(5a - 2) - (a - 4)\}$
$= (5a - 2 + a - 4)(5a - 2 - a + 4)$
$= (6a - 6)(4a + 2) = 12(a - 1)(2a + 1).$

29. $a^2 - 2ab + b^2 - x^2$
$= (a - b)^2 - x^2$
$= (a - b + x)(a - b - x).$

30. $(x + 1)^2 - (y + 1)^2$
$= (x + 1 + y + 1)(x + 1 - y - 1)$
$= (x + y + 2)(x - y).$

31. $(x + 1)^2 - (y - 1)^2$
$= (x + 1 + y - 1)(x + 1 - y + 1)$
$= (x + y)(x - y + 2).$

32. $d^2 - x^2 + 4xy - 4y^2$
$= d^2 - (x^2 - 4xy + 4y^2)$
$= d^2 - (x - 2y)^2$
$= (d + x - 2y)(d - x + 2y).$

33. $a^2 - b^2 - 2bc - c^2$
$= a^2 - (b^2 + 2bc + c^2)$
$= a^2 - (b + c)^2$
$= (a + b + c)(a - b - c).$

34. $4x^4 - 9x^2 + 6x - 1$
$= 4x^4 - (9x^2 - 6x + 1)$
$= 4x^4 - (3x - 1)^2$
$= (2x^2 + 3x - 1)(2x^2 - 3x + 1).$

Exercise XXXIX.

1. $a^3 - b^3$
$= (a - b)(a^2 + ab + b^2).$

2. $x^3 - 8$
$= (x - 2)(x^2 + 2x + 4).$

3. $x^3 - 343$
$= (x - 7)(x^2 + 7x + 49).$

4. $y^3 - 125$
$= (y - 5)(y^2 + 5y + 25).$

5. $y^3 - 216$
$= (y - 6)(y^2 + 6y + 36).$

6. $8x^3 - 27y^3$
$= (2x - 3y)(4x^2 + 6xy + 9y^2).$

7. $64y^3 - 1000z^3 = (4y - 10z)(16y^2 + 40yz + 100z^2).$

8. $729x^3 - 512y^3 = (9x - 8y)(81x^2 + 72xy + 64y^2).$

9. $27a^3 - 1728 = (3a - 12)(9a^2 + 36a + 144).$

10. $1000a^3 - 1331b^3 = (10a - 11b)(100a^2 + 110ab + 121b^2).$

Exercise XL.

1. $x^3 + y^3$
$= (x + y)(x^2 - xy + y^2).$

2. $x^3 + 8$
$= (x + 2)(x^2 - 2x + 4).$

3. $x^3 + 216$
$= (x + 6)(x^2 - 6x + 36).$

4. $y^3 + 64z^3$
$= (y + 4z)(y^2 - 4yz + 16z^2).$

5. $64b^3 + 125c^3$
$= (4b+5c)(16b^2-20bc+25c^2).$

6. $216a^3 + 512c^3$
$= (6a+8c)(36a^2-48ac+64c^2).$

7. $729x^3 + 1728y^3 = (9x + 12y)(81x^2 - 108xy + 144y^2).$

8. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$

9. $x^7 + y^7 = (x + y)(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6).$

10. $32b^5 + 243c^5 = (2b + 3c)(16b^4 - 24b^3c + 36b^2c^2 - 54bc^3 + 81c^4).$

Exercise XLI.

1. $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$

2. $a^{10} + b^{10} = (a^2 + b^2)(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8).$

3. $x^{12} + y^{12}$
$= (x^4 + y^4)(x^8 - x^4y^4 + y^8).$

4. $b^6 + 64c^6$
$= (b^2 + 4c^2)(b^4 - 4b^2c^2 + 16c^4).$

5. $x^6 + 1$
$= (x^2 + 1)(x^4 - x^2 + 1).$

6. $a^{12} + 1$
$= (a^4 + 1)(a^8 - a^4 + 1).$

7. $64a^6 + x^6$
$= (4a^2+x^2)(16a^4-4a^2x^2+x^4).$

8. $729 + c^6$
$= (9 + c^2)(81 - 9c^2 + c^4).$

Exercise XLII.

1. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$

2. $9x^4 + 3x^2y^2 + 4y^4 = (3x^2 + 3xy + 2y^2)(3x^2 - 3xy + 2y^2).$

3. $16x^4 - 17x^2y^2 + y^4 = (4x^2 + 3xy - y^2)(4x^2 - 3xy - y^2).$

4. $81a^4 + 23a^2b^2 + 16b^4 = (9a^2 + 7ab + 4b^2)(9a^2 - 7ab + 4b^2).$

5. $81a^4 - 28a^2b^2 + 16b^4 = (9a^2 + 10ab + 4b^2)(9a^2 - 10ab + 4b^2)$

6. $9x^4 + 38x^2y^2 + 49y^4 = (3x^2 + 2xy + 7y^2)(3x^2 - 2xy + 7y^2).$

7. $25a^4 - 9a^2b^2 + 16b^4 = (5a^2 + 7ab + 4b^2)(5a^2 - 7ab + 4b^2).$

8. $49m^4 + 110m^2n^2 + 81n^4 = (7m^2 + 4mn + 9n^2)(7m^2 - 4mn + 9n^2)$.

9. $9a^4 + 21a^2c^2 + 25c^4 = (3a^2 + 3ac + 5c^2)(3a^2 - 3ac + 5c^2)$.

10. $49a^4 - 15a^2b^2 + 121b^4 = (7a^2 + 13ab + 11b^2)(7a^2 - 13ab + 11b^2)$.

11. $64x^4 + 128x^2y^2 + 81y^4 = (8x^2 + 4xy + 9y^2)(8x^2 - 4xy + 9y^2)$.

12. $4x^4 - 37x^2y^2 + 9y^4 = (2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2)$.

13. $25x^4 - 41x^2y^2 + 16y^4 = (5x^2 + xy - 4y^2)(5x^2 - xy - 4y^2)$.

14. $81x^4 - 34x^2y^2 + y^4 = (9x^2 + 4xy - y^2)(9x^2 - 4xy - y^2)$.

Exercise XLIII.

1. $12x^2 - 5x - 2$
$= (4x + 1)(3x - 2)$.

2. $12x^2 - 7x + 1$
$= (3x - 1)(4x - 1)$.

3. $12x^2 - x - 1$
$= (4x + 1)(3x - 1)$.

4. $3x^2 - 2x - 5$
$= (x + 1)(3x - 5)$.

5. $3x^2 + 4x - 4$
$= (x + 2)(3x - 2)$.

6. $6x^2 + 5x - 4$
$= (3x + 4)(2x - 1)$.

7. $4x^2 + 13x + 3$
$= (4x + 1)(x + 3)$.

8. $4x^2 + 11x - 3$
$= (x + 3)(4x - 1)$.

9. $4x^2 - 4x - 3$
$= (2x + 1)(2x - 3)$.

10. $x^2 - 3ax + 2a^2$
$= (x - a)(x - 2a)$.

11. $12a^4 + a^2x^2 - x^4$
$= (3a^2 + x^2)(4a^2 - x^2)$

12. $2x^2 + 5xy + 2y^2$
$= (2x + y)(x + 2y)$.

13. $6a^2x^2 + ax - 1$
$= (2ax + 1)(3ax - 1)$.

14. $6b^2 - 7bx - 3x^2$
$= (3b + x)(2b - 3x)$.

15. $4x^2 + 8x + 3$
$= (2x + 1)(2x + 3)$.

16. $a^2 - ax - 6x^2$
$= (a + 2x)(a - 3x)$.

17. $8a^2 + 14ab - 15b^2$
$= (2a + 5b)(4a - 3b)$.

18. $6a^2 - 19ac + 10c^2$
$= (3a - 2c)(2a - 5c)$.

19. $8x^2 + 34xy + 21y^2$
$= (4x + 3y)(2x + 7y)$.

20. $8x^2 - 22xy - 21y^2$
$= (4x + 3y)(2x - 7y)$.

21. $6x^2 + 19xy - 7y^2$
$= (2x + 7y)(3x - y)$.

22. $11a^2 - 23ab + 2b^2$
$= (11a - b)(a - 2b)$.

23. $2c^2 - 13cd + 6d^2$
$= (2c - d)(c - 6d)$.

24. $6y^2 + 7yz - 3z^2$
$= (2y + 3z)(3y - z)$.

Exercise XLIV.

1. $a^3 + 3a^2b + 3ab^2 + b^3$
$= (a + b)^3.$

2. $a^3 + 3a^2 + 3a + 1$
$= (a + 1)^3.$

3. $a^3 - 3a^2 + 3a - 1$
$= (a - 1)^3.$

4. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$= (x + y)^4.$

5. $x^4 - 4x^3 + 6x^2 - 4x + 1$
$= (x - 1)^4.$

6. $a^4 - 4a^3c + 6a^2c^2 - 4ac^3 + c^4$
$= (a - c)^4.$

7. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$
$= (x + y + z)^2.$

8. $x^2 - 2xy + y^2 - 2xz + 2yz + z^2$
$= (x - y - z)^2.$

9. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$
$= (a + b - c)^2.$

Exercise XLV.

1. $2x^2 - 5xy + 2y^2 - 17x + 13y + 21.$

$$2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y),$$
$$2x^2 - 17x + 21 = (x - 7)(2x - 3),$$
$$2y^2 + 13y + 21 = (-y - 3)(-2y - 7).$$
$$x - 2y,\quad x - 7,\quad -2y - 7;$$
$$2x - y,\quad 2x - 3,\quad -y - 3.$$
$$(x - 2y - 7)(2x - y - 3).$$

2. $6x^2 - 37xy + 6y^2 - 5x - 5y - 1.$

$$6x^2 - 5x - 1 = (6x + 1)(x - 1),$$
$$6y^2 - 5y - 1 = (6y + 1)(y - 1),$$
$$6x^2 - 37xy + 6y^2 = (6x - y)(x - 6y).$$
$$6x - y,\quad 6x + 1,\quad 1 - y;$$
$$x - 6y,\quad -6y - 1,\quad x - 1.$$
$$(6x - y + 1)(x - 6y - 1).$$

3. $6x^2 - 5xy - 6y^2 - x - 5y - 1.$

$$6x^2 - 5xy - 6y^2 = (2x - 3y)(3x + 2y),$$
$$6x^2 - x - 1 = (3x + 1)(2x - 1),$$
$$-6y^2 - 5y - 1 = (-3y - 1)(2y + 1).$$
$$2x - 3y,\quad 2x - 1,\quad -3y - 1;$$
$$3x + 2y,\quad 3x + 1,\quad 2y + 1.$$
$$(2x - 3y - 1)(3x + 2y + 1).$$

4. $5x^2 - 8xy + 3y^2 + 7x - 5y + 2.$

$$5x^2 - 8xy + 3y^2 = (5x - 3y)(x - y),$$
$$5x^2 + 7x + 2 = (5x + 2)(x + 1),$$
$$3y^2 - 5y + 2 = (3y - 2)(y - 1).$$
$$5x - 3y, \quad -3y + 2, \quad 5x + 2;$$
$$x - y, \quad -y + 1, \quad x + 1.$$
$$(5x - 3y + 2)(x - y + 1).$$

5. $2x^2 - xy - 3y^2 - 8x + 7y + 6.$

$$2x^2 - 8x - 3y^2 = (2x - 3y)(x + y),$$
$$2x^2 - 8x + 6 = (2x - 2)(x - 3),$$
$$-3y^2 + 7y + 6 = (-3y - 2)(y - 3),$$
$$2x - 3y, \quad 2x - 2, \quad -3y - 2;$$
$$x + y, \quad x - 3, \quad y - 3.$$
$$(2x - 3y - 2)(x + y - 3).$$

6. $x^2 - 25y^2 - 10x - 20y + 21.$

$$x^2 - 10x + 21 = (x - 7)(x - 3),$$
$$-25y^2 - 20y + 21 = (5y - 3)(-5y - 7),$$
$$x^2 - 25y^2 = (x + 5y)(x - 5y).$$
$$x - 7, \quad x - 5y, \quad -5y - 7;$$
$$x - 3, \quad x + 5y, \quad 5y - 3.$$
$$(x - 5y - 7)(x + 5y - 3).$$

7. $2x^2 - 5xy + 2y^2 - xz - yz - z^2.$

$$2x^2 - 5xy + 2y^2 = (2x - y)(x - 2y),$$
$$2x^2 - xz - z^2 = (2x + z)(x - z),$$
$$2y^2 - yz - z^2 = (2y + z)(y - z).$$
$$2x - y, \quad 2x + z, \quad -y + z;$$
$$x - 2y, \quad x - z, \quad -2y - z.$$
$$(2x - y + z)(x - 2y - z).$$

8. $6x^2 + xy - y^2 - 3xz + 6yz - 9z^2.$

$$6x^2 + xy - y^2 = (3x - y)(2x + y),$$
$$6x^2 - 3xz - 9z^2 = (3x + 3z)(2x - 3z),$$
$$-y^2 - 6yz + 9z^2 = (-y + 3z)(y - 3z).$$
$$3x - y, \quad 3x + 3z, \quad -y + 3z;$$
$$2x + y, \quad 2x - 3z, \quad y - y - 3z.$$
$$(3x + 3z - y)(2x - 3z + y).$$

9. $6x^2 - 7xy + y^2 + 35xz - 5yz - 6z^2.$

$$6x^2 - 7xy + y^2 = (6x - y)(x - y),$$
$$6x^2 + 35xz - 6z^2 = (6x - z)(x + 6z),$$
$$y^2 - 5yz - 6z^2 = (y + z)(y - 6z).$$
$$6x - y, \quad 6x - z, \quad -y - z;$$
$$x - y, \quad x + 6z, \quad -y + 6z.$$
$$(6x - y - z)(x - y + 6z).$$

10. $5x^2 - 8xy + 3y^2 - 3xz + yz - 2z^2.$

$$\begin{aligned} 5x^2 - 8xy + 3y^2 &= (5x - 3y)(x - y), \\ 5x^2 - 3xz - 2z^2 &= (5x + 2z)(x - z), \\ 3y^2 + yz - 2z^2 &= (3y - 2z)(y + z). \end{aligned}$$

$$\begin{array}{lll} 5x - 3y, & 5x + 2z, & -3y + 2z; \\ x - y, & x - z, & -y - z. \end{array}$$

$$(5x - 3y + 2z)(x - y - z).$$

11. $2x^2 - xy - 3y^2 - 5yz - 2z^2.$

$$\begin{aligned} 2x^2 - 2xy - 3y^2 &= (2x - 3y)(x + y), \\ -3y - 5yz - 2z^2 &= (-3y - 2z)(y + z), \\ 2x^2 - 2z^2 &= (2x - 2z)(x + z). \end{aligned}$$

$$\begin{array}{lll} 2x - 3y, & -3y - 2z, & 2x - 2z; \\ x + y, & y + z, & x + z. \end{array}$$

$$(2x - 3y - 2z)(x + y + z).$$

12. $6x^2 - 13xy + 6y^2 + 12xz - 13yz + 6z^2.$

$$\begin{aligned} 6x^2 - 13xy + 6y^2 &= (3x - 2y)(2y - 3y), \\ 6x^2 + 12xz + 6z^2 &= (3x + 3z)(2x + 2z), \\ 6y^2 - 13yz + 6z^2 &= (3y - 2z)(2y - 3z). \end{aligned}$$

$$\begin{array}{lll} 3x - 2y, & 3x + 3z, & -2y + 3z; \\ 2x - 3y, & 2x + 2z, & -3y + 2z. \end{array}$$

$$(3x - 2y + 3z)(2x - 3y + 2z).$$

13. $x^2 - 2xy + y^2 + 5x - 5y$

$$\begin{aligned} &= (x^2 - 2xy + y^2) + (5x - 5y) \\ &= (x - y)^2 + 5(x - y) \\ &= (x - y)(x - y + 5). \end{aligned}$$

14. $2x^2 + 5xy - 3y^2 - 4xz + 2yz$

$$\begin{aligned} &= (2x^2 + 5xy - 3y^2) - (4xz - 2yz) \\ &= (2x - y)(x + 3y) - 2z(2x - y) \\ &= (x + 3y - 2z)(2x - y). \end{aligned}$$

EXERCISE XLVI.

1. $5x^2 - 15x - 20$

$$\begin{aligned} &= 5(x^2 - 3x - 4) \\ &= 5(x + 1)(x - 4). \end{aligned}$$

2. $2x^5 - 16x^4 + 24x^3$

$$\begin{aligned} &= 2x^3(x^2 - 8x + 12) \\ &= 2x^3(x - 2)(x - 6). \end{aligned}$$

3. $3a^2b^2 - 9ab - 12$

$$\begin{aligned} &= 3(a^2b^2 - 3ab - 4) \\ &= 3(ab - 4)(ab + 1). \end{aligned}$$

4. $a^2 + 2ax + x^2 + 4a + 4x$

$$\begin{aligned} &= (a^2 + 2ax + x^2) + (4a + 4x) \\ &= (a + x)^2 + 4(a + x) \\ &= (a + x)(a + x + 4). \end{aligned}$$

5. $a^2 - 2ab + b^2 - c^2$

$$\begin{aligned} &= (a^2 - 2ab + b^2) - c^2 \\ &= (a - b)^2 - c^2 \\ &= (a - b + c)(a - b - c). \end{aligned}$$

6. $x^2 - 2xy + y^2 - c^2 + 2cd - d^2$

$$\begin{aligned} &= (x^2 - 2xy + y^2) - (c^2 - 2cd + d^2) \\ &= (x - y)^2 - (c - d)^2 \\ &= \{(x - y) + (c - d)\}\{(x - y) - (c - d)\} \\ &= (x - y + c - d)(x - y - c + d). \end{aligned}$$

7. $4 - x^2 - 2x^3 - x^4$
$= 4 - (x^2 + 2x^3 + x^4)$
$= 4 - (x + x^2)^2$
$= (2 + x + x^2)(2 - x - x^2).$

8. $a^2 - b^2 - a - b$
$= (a^2 - b^2) - (a + b)$
$= (a + b)(a - b) - (a + b)$
$= (a + b)(a - b - 1).$

9. $a^4 + a^2 + 1$
$= (a^4 + 2a^2 + 1) - a^2$
$= (a^2 + 1)^2 - a^2$
$= (a^2 + a + 1)(a^2 - a + 1).$

10. $x^2 - y^2 - xz + yz$
$= (x^2 - y^2) - (xz - yz)$
$= (x + y)(x - y) - z(x - y)$
$= (x - y)(x + y - z).$

11. $ab - ac - b^2 + bc$
$= (ab - ac) - (b^2 - bc)$
$= a(b - c) - b(b - c)$
$= (a - b)(b - c).$

12. $3x^2 - 3xz - xy + yz$
$= (3x^2 - 3xz) - (xy - yz)$
$= 3x(x - z) - y(x - z)$
$= (3x - y)(x - z).$

13. $a^2 - x^2 - ab - bx$
$= (a^2 - x^2) - (ab + bx)$
$= (a + x)(a - x) - b(a + x)$
$= (a + x)(a - x - b).$

14. $a^2 - 2ax + x^2 + a - x$
$= (a^2 - 2ax + x^2) + (a - x)$
$= (a - x)(a - x) + 1(a - x)$
$= (a - x)(a - x + 1).$

15. $3x^2 - 3y^2 - 2x + 2y$
$= (3x - 3y^2) - (2x - 2y)$
$= 3(x^2 - y^2) - 2(x - y)$
$= 3(x - y)(x + y) - 2(x - y)$
$= (x - y)(3x + 3y - 2).$

16. $x^4 + x^3 + x^2 + x$
$= x^3(x + 1) + x(x + 1)$
$= (x^3 + x)(x + 1)$
$= x(x^2 + 1)(x + 1).$

17. $a^4x^4 - a^3x^3 - a^2x^2 + 1$
$= a^3x^3(ax-1) - (ax+1)(ax-1)$
$= (ax - 1)(a^3x^3 - ax - 1).$

18. $3x^3 - 2x^2y - 27xy^2 + 18y^3$
$= x^2(3x - 2y) - 9y^2(3x - 2y)$
$= (x^2 - 9y^2)(3x - 2y)$
$= (x - 3y)(x + 3y)(3x - 2y).$

19. $4x^4 - x^2 + 2x - 1$
$= 4x^4 - (x^2 - 2x + 1)$
$= 4x^4 - (x - 1)^2$
$= (2x^2 + x - 1)(2x^2 - x + 1).$
$= (2x - 1)(x + 1)(2x^2 - x + 1).$

20. $x^6 - y^6$
$= (x^3 + y^3)(x^3 - y^3)$
$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$

21. $x^6 + y^6$
$= (x^2 + y^2)(x^4 - x^2y^2 + y^4).$

22. $729 - x^6$
$= (27 + x^3)(27 - x^3)$
$= (3 + x)(9 - 3x + x^2)(3 - x)(9 + 3x + x^2).$

23. $x^{12}y + y^{13}$
$= y(x^{12} + y^{12})$
$= y(x^4 + y^4)(x^8 - x^4y^4 + y^8).$

24. $c(a^4 - c^4)$
$= c(a^2 + c^2)(a^2 - c^2)$
$= c(a^2 + c^2)(a + c)(a - c).$

25. $x^2 + 4x - 21$
$= (x + 7)(x - 3).$

26. $3a^2 - 21ab + 30b^2$
$= 3(a^2 - 7ab + 10b^2)$
$= 3(a - 2b)(a - 5b).$

27. $2x^4 - 4x^3y - 6x^2y^2$
$= 2x^2(2x^2 - 2xy - 3y^2)$
$= 2x^2(x - 3y)(x + y).$

28. $4a^2 - 4ab + b^2$
$= (2a - b)^2.$

29. $16x^2 - 80xy + 100y^2$
$= 4(4x^2 - 20xy + 25y^2$
$= 4(2x - 5y)^2.$

30. $36a^2x^3y^2 - 25b^2x^5y^2$
$= x^3y^2(36a^2 - 25b^2x^2)$
$= x^3y^2(6a + 5bx)(6a - 5bx).$

31. $9x^2y^4 - 30xy^2z + 25z^2$
$= (3xy^2 - 5z)^2.$

32. $16x^5 - x$
$= x(16x^4 - 1)$
$= x(4x^2 + 1)(4x^2 - 1)$
$= x(4x^2 + 1)(2x + 1)(2x - 1).$

33. $x^2 - 2xy - 2xz + y^2 + 2yz + z^2.$
$x^2 - 2xy + y^2 = (x - y)(x - y),$
$x^2 - 2xz + z^2 = (x - z)(x - z),$
$y^2 + 2yz + z^2 = (y + z)(y + z).$
$x - y,\ x - z,\ -y - z;$
$x - y,\ x - z,\ -y - z.$
$(x - y - z)(x - y - z).$

34. $a^2 - ab - 6b^2 - 4a + 12b$
$= (a - ab - 6b^2) - 4(a - 3b)$
$= (a - 3b)(a + 2b) - 4(a - 3b)$
$= (a - 3b)(a + 2b - 4).$

35. $x^2 + 2xy + y^2 - x - y - 6.$
$x^2 + 2x + y^2 = (x + y)^2,$
$x^2 - x - 6 = (x + 2)(x - 3),$
$y^2 - y - 6 = (y + 2)(y - 3).$
$x + y,\ x + 2,\ y + 2;$
$x + y,\ x - 3,\ y - 3.$
$(x + y + 2)(x + y - 3).$

36. $(a + b)^4 - c^4$
$= \{(a + b)^2 + c^2\}\{(a + b)^2 - c^2\}$
$= \{(a + b)^2 + c^2\}(a + b + c)(a + b - c)$
$= (a^2 + 2ab + b^2 + c^2)(a + b + c)(a + b - c).$

37. $x^2 - xy - 6y^2 - 4x + 12y$
$= (x^2 - xy - 6y^2) - 4(x - 3y)$
$= (x + 2y)(x - 3y) - 4(x - 3y)$
$= (x + 2y - 4)(x - 3y).$

38. $1 - x + x^2 - x^3$
$= (1 - x) + x^2(1 - x)$
$= (1 - x)(1 + x^2).$

39. $3x^2 - 11xy + 6y^2$
$= (3x - 2y)(x - 3y).$

40. $x^2 + 20x + 91$
$= (x + 7)(x + 13).$

41. $(x - y)(x^2 - z^2) - (x - z)(x^2 - y^2)$
$= (x - y)(x + z)(x - z) - (x - z)(x - y)(x + y)$
$= (x - y)(x - z)(z - y).$

42. $x^2 - 5x - 24$
$= (x - 8)(x + 3).$

43. $(x^2 - y^2 - z^2)^2 - 4y^2z^2$
$= \{(x^2 - y^2 - z^2) + 2yz\}\{(x^2 - y^2 - z^2) - 2yz\}$
$= (x^2 - y^2 - z^2 + 2yz)(x^2 - y^2 - z^2 - 2yz)$
$= \{x^2 - (y^2 - 2yz + z^2)\}\{x^2 - (y^2 + 2yz + z^2)\}$
$= \{x^2 - (y - z)^2\}\{x^2 - (y + z)^2\}$
$= (x + y - z)(x - y + z)(x + y + z)(x - y - z).$

44. $5x^3y^2 + 5x^2yz - 60xz^2$
$= 5x(x^2y^2 + xyz - 12z^2)$
$= 5x(xy + 4z)(xy - 3z)$.

45. $3x^3 - x^2 + 3x - 1$
$= x^2(3x - 1) + (3x - 1)$
$= (x^2 + 1)(3x - 1)$.

46. $x^2 - 2mx + m^2 - n^2$
$= (x^2 - 2mx + m^2) - n^2$
$(x - m)^2 - n^2$
$= (x - m + n)(x - m - n)$.

47. $4a^2b^2 - (a^2 + b^2 - c^2)^2$
$= \{2ab + (a^2 + b^2 - c^2)\}\{2ab - (a^2 + b^2 - c^2)\}$
$= \{2ab + a^2 + b^2 - c^2\}\{2ab - a^2 - b^2 + c^2\}$
$= \{(a^2 + 2ab + b^2) - c^2\}\{c^2 - (a^2 - 2ab + b^2)\}$
$= \{(a + b)^2 - c^2\}\{c^2 - (a - b)^2\}$
$= (a + b + c)(a + b - c)(c + a - b)(c - a + b)$.

48. $a^7 + a^5$
$= a^5(a^2 + 1)$.

49. $1 - 14a^3x + 49a^6x^2$
$= (1 - 7a^3x)^2$.

50. $y^2 - 4y - 117$
$= (y - 13)(y + 9)$.

51. $x^2 + 6x - 135$
$= (x + 15)(x - 9)$.

52. $4a^2 - 12ab + 9b^2 - 4c^2$
$= (4a^2 - 12ab + 9b^2) - 4c^2$
$= (2a - 3b)^2 - 4c^2$
$= (2a - 3b + 2c)(2a - 3b - 2c)$.

53. $(a + 3b)^2 - 9(b - c)^2$
$= \{(a + 3b) + 3(b - c)\}\{(a + 3b) - 3(b - c)\}$
$= (a + 3b + 3b - 3c)(a + 3b - 3b + 3c)$
$= (a + 6b - 3c)(a + 3c)$.

54. $9x^2 - 4y^2 + 4yz - z^2$
$= 9x^2 - (4y^2 - 4yz + z^2)$
$= 9x^2 - (2y - z)^2$
$= (3x + 2y - z)(3x - 2y + z)$.

55. $6b^2x^2 - 7bx^3 - 3x^4$
$= x^2(6b^2 - 7bx - 3x^2)$
$= x^2(3b + x)(2b - 3x)$.

56. $a^3 - b^3 - 3ab(a - b)$
$= a^3 - 3a^2b + 3ab^2 - b^3$
$= (a - b)^3$.

57. $x^3 + y^3 + 3xy(x + y)$
$= x^3 + 3x^2y + 3xy^2 + y^2$
$= (x + y)^3$.

58. $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$
$= a^3 - b^3 - a^3 + ab^2 + a^2b - 2ab^2 + b^3$
$= a^2b - ab^2$
$= ab(a - b)$.

59. $9x^2y^2 - 3xy^3 - 6y^4$
$= 3y^2(3x^2 - xy - 2y^2)$
$= 3y^2(x - y)(3x + 2y)$.

60. $6x^2 + 13xy + 6y^2$
$= (3x + 2y)(2x + 3y)$.

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61. $$\begin{aligned}&6a^2b^2 - ab^3 - 12b^4\\ &= b^2(6a^2 - ab - 12b^2)\\ &= b^2(3a + 4b)(2a - 3b).\end{aligned}$$

62. $$\begin{aligned}&a^2 + 2ad + d^2 - 4b^2 + 12bc - 9c^2\\ &= (a^2 + 2ad + d^2) - (4b^2 - 12bc - 9c^2)\\ &= a + d)^2 - (2b - 3c)^2\\ &= \{(a + d) + (2b - 3c)\}\{(a + d) - (2b - 3c)\}\\ &= (a + d + 2b - 3c)(a + d - 2b + 3c)\\ &= (a + 2b - 3c + d)(a - 2b + 3c + d).\end{aligned}$$

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63. $$\begin{aligned}&x^3 - 2x^2y + 4xy^2 - 8y^3\\ &= x^2(x - 2y) + 4y^2(x - 2y)\\ &= (x^2 + 4y^2)(x - 2y).\end{aligned}$$

64. $$\begin{aligned}&4a^2x^2 - 8abx + 3b^2\\ &= (2ax - b)(2ax - 3b).\end{aligned}$$

65. $$\begin{aligned}&18x^2 - 24xy + 8y^2 + 9x - 6y\\ &= (18x^2 - 24xy + 8y^2) + (9x - 6y)\\ &= 2(9x^2 - 12xy + 4y^2) + 3(3x - 2y)\\ &= 2(3x - 2y)^2 + 3(3x - 2y)\\ &= (6x - 4y + 3)(3x - 2y).\end{aligned}$$

66. $$\begin{aligned}&2x^2 + 2xy - 12y^2 + 6xz + 18yz\\ &= 2(x^2 + xy - 6y^2 + 3xz + 9yz)\\ &= 2(x^2 + xy - 6y^2) + 3z(x + 3y)\\ &= 2(x + 3y)(x - 2y) + 3z(x + 3y)\\ &= 2(x + 3y)(x - 2y + 3z).\end{aligned}$$

67. $$\begin{aligned}&(x + y)^2 - 1 - xy(x + y + 1)\\ &= (x + y + 1)(x + y - 1) - xy(x + y + 1)\\ &= (x + y + 1)(x + y - xy - 1).\\ &= (x + y + 1)(1 - x)(y - 1).\end{aligned}$$

68. $$\begin{aligned}&x^2 - y^2 - z^2 + 2yz + x + y - z\\ &= x^2 - (y^2 - 2yz + z^2) + x + y - z\\ &= x^2 - (y - z)^2 + (x + y - z)\\ &= (x + y - z)(x - y + z) + (x + y - z)\\ &= (x + y - z)(x - y + z + 1).\end{aligned}$$

69. $$\begin{aligned}&2x^2 + 4xy + 2y^2 + 2ax + 2ay\\ &= 2(x^2 + 2xy + y^2) + 2a(x+y)\\ &= 2(x + y)^2 + 2a(x + y)\\ &= 2(x + y + a)(x + y).\end{aligned}$$

70. $$\begin{aligned}&16a^2b + 32abc + 12bc^2\\ &= 4b(4a^2 + 8ac + 3c^2)\\ &= 4b(2a + 3c)(2a + c).\end{aligned}$$

71. $$\begin{aligned}&m^2p - m^2q - n^2p + n^2q\\ &= (m^2p - m^2q) - (n^2p - n^2q)\\ &= m^2(p - q) - n^2(p - q)\\ &= (m^2 - n^2)(p - q)\\ &= (m + n)(m - n)(p - q).\end{aligned}$$

72. $$\begin{aligned}&12ax^2 - 14axy - 6ay^2\\ &= 2a(6x^2 - 7xy - 3y^2)\\ &= 2a(3x + y)(2x - 3y).\end{aligned}$$

73. $$\begin{aligned}&2x^3 + 4x^2 - 70x\\ &= 2x(x^2 + 2x - 35)\\ &= 2x(x + 7)(x - 5).\end{aligned}$$

74. $$\begin{aligned}&16a^3x - 2x^4\\ &= 2x(8a^3 - x^3)\\ &= 2x(2a - x)(4a^2 + 2ax + x^2)\end{aligned}$$

75. $$\begin{aligned}&32bx^3 - 4by^3\\ &= 4b(8x^3 - y^3)\\ &= 4b(2x - y)(4x^2 + 2xy + y^2).\end{aligned}$$

76. $x - 27x^4$
$= x(1 - 27x^3)$
$= x(1 - 3x)(1 + 3x + 9x^2).$

77. $x^{12} - y^{12}$
$= (x^6 + y^6)(x^6 - y^6)$
$= (x^6 + y^6)(x^3 + y^3)(x^3 - y^3)$
$= (x^2+y^2)(x^4-x^2y^2+y^4)(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2).$

78. $49m^2 - 121n^2$
$= (7m + 11n)(7m - 11n).$

79. $16 - 81y^4$
$= (4 + 9y^2)(4 - 9y^2)$
$= (4 + 9y^2)(2 + 3y)(2 - 3y).$

80. $12z^4 - z^2 - 6$
$= (3z^2 + 2)(4z^2 - 3).$

81. $x^3 - x^2 + x - 1$
$= (x^3 - x^2) + (x - 1)$
$= x^2(x - 1) + (x - 1)$
$= (x^2 + 1)(x - 1).$

82. $x^2 + 2x + 1 - y^2$
$= (x + 2x + 1) - y^2$
$= (x + 1)^2 - y^2$
$= (x + 1 + y)(x + 1 - y).$

83. $49(a - b)^2 - 64(m - n)^2$
$= \{7(a - b) + 8(m - n)\}\{7(a - b) - 8(m - n)\}$
$= (7a - 7b + 8m - 8n)(7a - 7b - 8m + 8n).$

84. $1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2$
$$= \left(1 + \frac{a^2 + b^2 - c^2}{2ab}\right)\left(1 - \frac{a^2 + b^2 - c^2}{2ab}\right)$$
$$= \left(\frac{2ab + a^2 + b^2 - c^2}{2ab}\right)\left(\frac{2ab - a^2 - b^2 + c^2}{2ab}\right)$$
$$= \left(\frac{(a^2 + 2ab + b^2) - c^2}{2ab}\right)\left(\frac{c^2 - (a^2 - 2ab + b^2)}{2ab}\right)$$
$$= \left(\frac{(a + b)^2 - c^2}{2ab}\right)\left(\frac{c^2 - (a - b)^2}{2ab}\right)$$
$$= \left(\frac{(a + b + c)(a + b - c)}{2ab}\right)\left(\frac{(c + a - b)(c - a + b)}{2ab}\right).$$

85. $x^2 - 53x + 360$
$= (x - 8)(x - 45).$

86. $x^3 - 2x^2y + x^2 - 4x + 8y - 4$
$= (x^3 - 2x^2y + x^2) - (4x - 8y + 4)$
$= x^2(x - 2y + 1) - 4(x - 2y + 1)$
$= (x^2 - 4)(x - 2y + 1)$
$= (x + 2)(x - 2)(x - 2y + 1).$

87. $2ab - 2bc - ae + ce + 2b^2 - be$
$= (2ab - 2bc + 2b^2) - (ae - ce + be)$
$= 2b(a - c + b) - e(a - c + b)$
$= (2b - e)(a + b - c).$

88. $125x^5 + 350x^3y^2 + 245xy^4$
$= 5x(25x^4 + 70x^2y^2 + 49y^4)$
$= 5x(5x^2 + 7y^2)^2.$

89. $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5$
$= a(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$
$= a\{a^3(a^2 + ab + b^2) + b^3(a^2 + ab + b^2)\}$
$= a(a^3 + b^3)(a^2 + ab + b^2)$
$= a(a + b)(a^2 - ab + b^2)(a^2 + ab + b^2).$

90. $2a^4x - 2a^3cx + 2ac^3x - 2c^4x$
$= 2a^3x(a - c) + 2c^3x(a - c)$
$= (2a^3x + 2c^3x)(a - c)$
$= 2x(a^3 + c^3)(a - c)$
$= 2x(a + c)(a^2 - ac + c^2)(a - c).$

91. $6x^2 - 5xy - 6y^2 + 3xz + 15yz - 9z^2.$
$6x^2 - 5xy - 6y^2 = (3x + 2y)(2x - 3y),$
$6x^2 + 3xz - 9z^2 = (3x - 3z)(2x + 3z),$
$-6y^2 + 15yz - 9z^2 = (2y - 3z)(-3y + 3z).$
$3x + 2y,\ \ 3x - 3z,\ \ 2y - 3z;$
$2x - 3y,\ \ 2x + 3z,\ \ -3y + 3z.$
$(3x + 2y - 3z)(2x - 3y + 3z).$

92. $4x^2 - 9xy + 2y^2 - 3xz - yz - z^2.$
$4x^2 - 9xy + 2y^2 = (4x - y)(x - 2y),$
$4x^2 - 3xz - z^2 = (4x + z)(x - z),$
$2y^2 - yz - z^2 = (-2y - z)(-y + z).$
$4x - y,\ \ 4x + z,\ \ -y + z;$
$x - 2y,\ \ x - z,\ \ -2y - z.$
$(4x - y + z)(x - 2y - z).$

93. $3a^2 - 7ab + 2b^2 + 5ac - 5bc + 2c^2.$
$3a^2 - 7ab + 2b^2 = (3a - b)(a - 2b),$
$3a^2 + 5ac + 2c^2 = (3a + 2c)(a + c),$
$2b^2 - 5bc + 2c^2 = (-2b + c)(-b + 2c).$
$3a - b,\ \ 3a + 2c,\ \ -b + 2c;$
$a - 2b,\ \ a + c,\ \ -2b + c.$
$(3a - b + 2c)(a - 2b + c).$

94. $x^4 - 2x^3 + x^2 - 8x + 8$
$= x^4 - 2x^3 + x^2 - (8x - 8)$
$= x^2(x^2 - 2x + 1) - 8(x - 1)$
$= x^2(x - 1)^2 - 8(x - 1)$
$= (x^3 - x^2 - 8)(x - 1).$

95. $5x^2 - 8xy + 3y^2 - 5x + 3y$
$= (5x^2 - 8xy + 3y^2) - (5x - 3y)$
$= (5x - 3y)(x - y) - (5x - 3y)$
$= (5x - 3y)(x - y - 1).$

96. $a^2 - 2ad + d^2 - 4b^2 + 12bc - 9c^2$
$= (a^2 - 2ad + d^2) - (4b^2 - 12bc + 9c^2)$
$= (a - d)^2 - (2b - 3c)^2$
$= \{(a - d) + (2b - 3c)\}\{(a - d) - (2b - 3c)\}$
$= (a - d + 2b - 3c)(a - d - 2b + 3c)$
$= (a + 2b - 3c - d)(a - 2b + 3c - d).$

97. $(x^2 - x - 6)(x^2 - x - 20)$
$= (x - 3)(x + 2)(x - 5)(x + 4).$

EXERCISE XLVII.

1. $18ab^2c^2d = 3^2 \times 2ab^2c^2d,$
$36a^2bcd^2 = 3^2 \times 2^2a^2bcd^2.$
$\therefore$ H.C.F. $= 18abcd.$

2. $17pq^2 = 17pq^2,$
$34p^2q = 17 \times 2p^2q,$
$51p^3q^3 = 17 \times 3p^3q^3.$
$\therefore$ H.C.F. $= 17pq.$

3. $8x^2y^3z^4 = 2^3 \times x^2y^3z^4,$
$12x^3y^2z^3 = 2^2 \times 3x^3y^2z^3,$
$20x^4y^3z^2 = 2^2 \times 5x^4y^3z^2.$
$\therefore$ H.C.F. $= 4x^2y^2z^2.$

4. $30x^4y^5 = 2 \times 3 \times 5x^4y^5,$
$90x^2y^3 = 2 \times 3^2 \times 5x^2y^3,$
$120x^3y^4 = 2^3 \times 3 \times 5x^3y^4.$
$\therefore$ H.C.F. $= 30x^2y^3.$

5. $a^2 - b^2 = (a+b)(a-b),$
$a^3 - b^3 = (a-b)(a^2+ab+b^2).$
$\therefore$ H.C.F. $= a - b.$

6. $a^2 - x^2 = (a+x)(a-x),$
$(a-x)^2 = (a-x)(a-x).$
$\therefore$ H.C.F. $= a - x.$

7. $a^3 + x^3 = (a+x)(a^2-ax+x^2)$
$(a+x)^3 = (a+x)^3.$
$\therefore$ H.C.F. $= a + x.$

8. $9x^2 - 1 = (3x+1)(3x-1)$
$(3x+1)^2 = (3x+1)^2.$
$\therefore$ H.C.F. $= 3x + 1.$

9. $7x^2 - 4x = x(7x-4),$
$7a^2x - 4a^2 = a^2(7x-4).$
$\therefore$ H.C.F. $= 7x - 4.$

10. $12a^3x^2y - 4a^3xy^2 = 4a^3xy(3x-y),$
$30a^2x^3y^2 - 10a^2x^2y^3 = 10a^2x^2y^2(3x-y).$
$\therefore$ H.C.F. $= 2a^2xy(3x-y).$

11. $8a^3b^2c - 12a^2bc^3 = 4a^2bc(2ab-3c^2),$
$6ab^4c + 4ab^3c^2 = 2ab^3c(3b+2c).$
$\therefore$ H.C.F. $= 2abc.$

12. $x^2 - 2x - 3 = (x-3)(x+1),$
$x^2 + x - 12 = (x-3)(x+4).$
$\therefore$ H.C.F. $= x - 3.$

13. $2a^3 - 2ab^2 = 2a(a+b)(a-b),$
$4b(a+b)^2 = 4b(a+b)(a+b).$
$\therefore$ H.C.F. $= 2(a+b).$

14. $12x^3y(x-y)(x-3y) = 2^2 \times 3x^3y(x-y)(x-3y),$
$18x^2(x-y)(3x-y) = 2 \times 3^2x^2(x-y)(3x-y).$
$\therefore$ H.C.F. $= 6x^2(x-y).$

15. $3x^3 + 6x^2 - 24x = 3x(x^2+2x-8)$
$= 3x(x+4)(x-2),$
$6x^3 - 96x = 6x(x^2-16)$
$= 6x(x-4)(x+4).$
$\therefore$ H.C.F. $= 3x(x+4).$

16. $ac(a-b)(a-c) = ac(a-b)(a-c),$
$bc(b-a)(b-c) = bc(a-b)(c-b).$
$\therefore$ H.C.F. $= c(a-b).$

17. $10x^3y - 60x^2y^2 + 5xy^3 = 5xy(2x^2 - 12xy + y^2),$
$5x^2y^2 - 5xy^3 - 100y^4 = 5y^2(x^2 - xy - 20y^2)$
$= 5y^2(x-5)(x+4).$
$\therefore$ H.C.F. $= 5y.$

18. $x(x+1)^2 = x(x+1)^2,$
$x^2(x^2-1) = x^2(x+1)(x-1),$
$2x(x^2-x-2) = 2x(x-2)(x+1).$
$\therefore$ H.C.F. $= x(x+1).$

19. $3x^2-6x+3 = 3(x-1)^2,$
$6x^2+6x-12 = 6(x+2)(x-1),$
$12x^2-12 = 12(x-1).$
$\therefore$ H.C.F. $= 3(x-1).$

20. $6(a-b)^4 = 6(a-b)^4,$
$8(a^2-b^2)^2 = 8(a+b)^2(a-b)^2,$
$10(a^4-b^4) = 10(a^2+b^2)(a+b)(a-b).$
$\therefore$ H.C.F. $= 2(a-b).$

21. $x^2-y^2 = (x+y)(x-y).$
$(x+y)^2 = (x+y)^2,$
$x^2+3xy+2y^2 = (x+y)(x+2y).$
$\therefore$ H.C.F. $= x+y.$

22. $x^2-y^2 = (x+y)(x-y),$
$x^3-y^3 = (x-y)(x^2+xy+y^2),$
$x^2-7xy+6y^2 = (x-y)(x-6y).$
$\therefore$ H.C.F. $= x-y.$

23. $x^2-1 = (x-1)(x+1),$
$x^3-1 = (x-1)(x^2+x+1),$
$x^2+x-2 = (x-1)(x+2).$
$\therefore$ H.C.F. $= x-1.$

EXERCISE XLVIII.

1.

$$\begin{array}{r|r|l}
5x^2+4x-1 & 20x^2+21x-5 & 4 \\
\underline{5x^2-x} & \underline{20x^2+16x-4} & \\
5x-1 & 5x-1 & x+1 \\
\underline{5x-1} & & 1
\end{array}$$

$\therefore$ H.C.F. $= 5x-1.$

2.

$$\begin{array}{r|r|l}
2x^3-4x^2-13x-7 & 6x^3-11x^2-37x-20 & 3 \\
\underline{2x^3+4x^2+2x} & \underline{6x^3-12x^2-39x-21} & \\
-8x^2-15x-7 & x^2+2x+1 & 2x-8 \\
\underline{-8x^2-16x-8} & \underline{x^2+x} & x+1 \\
x+1 & x+1 & \\
 & \underline{x+1} &
\end{array}$$

$\therefore$ H.C.F. $= x+1.$

3.

$$\begin{array}{r|r|l}
a\underline{)6a^4+25a^3-21a^2+4a} & 2a\underline{)24a^4+112a^3-94a^2+18a} & \text{Reserve } a. \\
6a^3+25a^2-21a+4 & 12a^3+56a^2-47a+9 & 2 \\
\underline{6a^3-5a^2+a} & \underline{12a^3+50a^2-42a+8} & \\
30a^2-22a+4 & 6a^2-5a+1 & a+5 \\
\underline{30a^2-25a+5} & \underline{6a^2-2a} & 2a-1 \\
3a-1 & -3a+1 & \\
 & \underline{-3a+1} &
\end{array}$$

$\therefore$ H.C.F. $= a(3a-1).$

4.

$$\begin{array}{r|r|l}
9x^3+9x^2-4x-4 & 45x^3+54x^2-20x-24 & 5 \\
\underline{9x^3-4x} & \underline{45x^3+45x^2-20x-20} & \\
9x^2-4 & 9x^2-4 & x+1 \\
\underline{9x^2-4} & & 1
\end{array}$$

$\therefore$ H.C.F. $= 9x^2-4.$

5.

$$\begin{array}{r}
3x^2\underline{)\,27x^6-3x^4+6x^3-3x^2} \\
9x^4-\ \ x^2+2x\ -1 \\
\underline{9x^4+6x^3+3x\qquad\quad} \\
-6x^3-\ \ x^2-x-1 \\
\underline{-6x^3-4x^2\qquad -2} \\
3x^2-x+1
\end{array}
\qquad
\begin{array}{r|l}
6x\underline{)\,162x^6+48x^3-18x^2+6x} & \text{Reserve } 3x. \\
27x^5+8x^2-3x\ +1 & 3x \\
\underline{27x^5-3x^3+6x^2-3x} & \\
3x^3+2x^2\qquad +1 & 3x-2 \\
\underline{3x^3-\ \ x^2+x\quad} & x+1 \\
3x^2-x+1 & \\
\underline{3x^2-x+1} &
\end{array}$$

$\therefore$ H.C.F. $=3x(3x^2-x+1)$.

6.

$$\begin{array}{r}
10\underline{)\,20x^3-60x^2+50x-20} \\
2x^3-\ \ 6x^2+\ \ 5x-\ \ 2 \\
\underline{2x^3-\ \ 6x^2+\ \ 4x\qquad} \\
x-\ \ 2 \\
\underline{x-\ \ 2}
\end{array}
\qquad
\begin{array}{r|l}
4x\underline{)\,32x^4-92x^3+68x^2-24x} & \text{Reserve } 2. \\
8x^3-23x^2+17x-\ 6 & 4 \\
\underline{8x^3-24x^2+20x-\ 8} & \\
x^2-\ 3x+\ 2 & 2x \\
\underline{x^2-\ 2x\qquad\ } & x-1 \\
-\quad x+\ 2 &
\end{array}$$

$\therefore$ H.C.F. $=2(x-2)$.

7.

$$\begin{array}{r}
4x^2-\ \ 8x-5 \\
\underline{4x^2-10x\quad\ } \\
2x-5 \\
\underline{2x-5}
\end{array}
\qquad
\begin{array}{r|l}
12x^2-\ \ 4x-65 & 3 \\
\underline{12x^2-24x-15} & \\
10\underline{)\,20x-50} & \\
2x-\ \ 5 & 2x+1
\end{array}$$

$\therefore$ H.C.F. $=2x-5$.

8.

$$\begin{array}{r}
a\underline{)\,3a^3-5a^2x-2ax^2} \\
3a^2-5ax\ -\ \ 2x^2 \\
\underline{3a^2-6ax\qquad\quad\ } \\
ax\ -\ \ 2x^2 \\
\underline{ax\ -\ \ 2x^2}
\end{array}
\qquad
\begin{array}{r|l}
a\underline{)\,9a^3-8a^2x-20ax^2} & \text{Reserve } a. \\
9a^2-\ \ 8ax-20x^2 & 3 \\
\underline{9a^2-15ax-\ \ 6x^2} & \\
7x\underline{)\,7ax-14x^2} & 3a+x \\
a\ -\ \ 2x &
\end{array}$$

$\therefore$ H.C.F. $=a(a-2x)$.

9.

$$\begin{array}{r}
10x^3+\ \ x^2-9x+24 \\
\underline{10x^3-5x^2\qquad\ +15} \\
3\underline{)\,6x^2-9x+\ \ 9} \\
2x^2-3x+\ \ 3
\end{array}
\qquad
\begin{array}{r|l}
20x^4-17x^2+48x-3\qquad & 2x \\
\underline{20x^4+2x^3-18x^2+48x} & \\
-2x^3+\ \ x^2\qquad -3 & -5 \\
\underline{-2x^3+\ 3x^2-3x\quad} & -x-1 \\
-2x^2+3x-3 & \\
\underline{-2x^2+3x-3} &
\end{array}$$

$\therefore$ H.C.F. $=2x^2-3x+3$.

10.

$$
\begin{array}{r|r|l}
2\underline{)8x^3-4x^2-32x-182} & 3\underline{)36x^3-84x^2-111x-126} & \\
4x^3-2x^2-16x-\ \ 91 & 12x^3-28x^2-\ 37x-\ \ 42 & 3 \\
\underline{4x^3-2x^2-42x\qquad\ } & \underline{12x^3-\ \ 6x^2-\ 48x-273} & \\
13\underline{)26x-\ \ 91} & -11\underline{)-22x^2+\ 11x+231} & \\
2x-\ \ \ 7 & 2x^2-\qquad x-\ \ 21 & 2x \\
 & \underline{2x^2-\ \ \ 7x\qquad\quad} & x+3 \\
 & 6x-\ \ 21 & \\
 & \underline{6x-\ \ 21} & \\
\end{array}
$$

$\therefore$ H.C.F. $= 2x-7$.

11.

$$
\begin{array}{r|r|l}
5x^2(12x^3+4x^2+17x-3) & 10x\,(24x^3-52x^2+14x-1) & \text{Reserve } 5x. \\
12x^3+4x^2+17x-3 & 24x^3-52x^2+14x-1 & 2 \\
\underline{12x^3+4x^2-\quad x\quad\ } & \underline{24x^3+\ \ 8x^2+34x-6} & \\
3\underline{)18x-3} & -5\underline{)-60x^2-20x+5} & \\
6x-1 & 12x^2+\ \ 4x-1 & x \\
 & \underline{12x^2-\ \ 2x\qquad} & 2x+1 \\
 & 6x-1 & \\
 & \underline{6x-1} & \\
\end{array}
$$

$\therefore$ H.C.F. $= 5x(6x-1)$.

12.

$$
\begin{array}{r|r|l}
2y\underline{)18x^3y-18x^2y^2-2xy^3-8y^4} & xy\underline{)9x^4y-x^2y^3-20xy^4} & \text{Reserve } y. \\
9x^3-9x^2y-xy^2-\ 4y^3 & 9x^3-\qquad xy^2-20y^3 & 1 \\
\underline{9x^3\qquad\quad -xy^2-20y^3} & \underline{9x^3-16xy^2\qquad\quad} & -x \\
y\underline{)-9x^2y\qquad\quad +16y^3} & 5y^2\underline{)15xy^2-20y^3} & \\
-9x^2\qquad\quad +16y^2 & 3x\quad-\quad 4y & -3x-4y \\
\underline{-9x^2+12xy\qquad\quad} & & \\
-12xy+16y^2 & & \\
\underline{-12xy+16y^2} & & \\
\end{array}
$$

$\therefore$ H.C.F. $= y(3x-4y)$

13.

$$
\begin{array}{r|r|l}
6x^2-\quad x-15 & 9x^2-3x-20 & \\
\underline{6x^2-10x\qquad} & \underline{2\qquad\qquad\qquad} & \\
9x-15 & 18x^2-6x-40 & 3 \\
\underline{9x-15} & \underline{18x^2-3x-45} & \\
 & -3x+\ \ 5 & -2x-3 \\
\end{array}
$$

$\therefore$ H.C.F $= 3x-5$.

14.

$$\begin{array}{r|r|l}
12x^3 - 9x^2 + 5x + 2 & 24x^2 + 10x + 1 & x + 7 \\
2 & 24x^2 + 6x & 6x + 1 \\
\hline
24x^3 - 18x^2 + 10x + 4 & 4x + 1 & \\
24x^3 + 10x^2 + x & 4x + 1 & \\
\hline
-28x^2 + 9x + 4 & & \\
-6 & & \\
\hline
168x^2 - 54x - 24 & & \\
168x^2 + 70x + 7 & & \\
\hline
-31)\underline{-124x - 31} & & \\
4x + 1 & &
\end{array}$$

$\therefore$ H.C.F. $= 4x + 1$

15.

$$\begin{array}{r|r|l}
3)\underline{6x^3 + 15x^2 - 6x + 9} & 3)\underline{9x^3 + 6x^2 - 51x + 36} & \text{Reserve 3.} \\
2x^3 + 5x^2 - 2x + 3 & 3x^3 + 2x^2 - 17x + 12 & \\
11 & 2 & \\
\hline
22x^3 + 55x^2 - 22x + 33 & 6x^3 + 4x^2 - 34x + 24 & 3 \\
22x^3 + 56x^2 - 30x & 6x^3 + 15x^2 - 6x + 9 & \\
\hline
-x^2 + 8x + 33 & -11x^2 - 28x + 15 & -2x \\
-x^2 - 3x & -11x^2 + 88x + 363 & 11 \\
\hline
11x + 33 & -116)\underline{-116x - 348} & \\
11x + 33 & x + 3 & -x + 11
\end{array}$$

$\therefore$ H.C.F. $= 3(x + 3)$.

16.

$$\begin{array}{r|r|l}
4x^3 - x^2y - xy^2 - 5y^3 & 7x^3 + 4x^2y + 4xy^2 - 3y^3 & \\
4x^3 + 4x^2y + 4xy^2 & 4 & \\
\hline
-5x^2y - 5xy^2 - 5y^3 & 28x^3 + 16x^2y + 16xy^2 - 12y^3 & 7 \\
-5x^2y - 5xy^2 - 5y^3 & 28x^3 - 7x^2y - 7xy^2 - 35y^3 & \\
\hline
 & 23y)\underline{23x^2y + 23xy^2 + 23y^3} & \\
 & x^2 + xy + y^2 & 4x - 5y
\end{array}$$

$\therefore$ H.C.F. $= x^2 + xy + y^2$.

17.

$$\begin{array}{r|r|l}
2a^3 - 2a^2 - 3a - 2 & 3a^3 - a^2 - 2a - 16 & \\
2 & 2 & \\
\hline
4a^3 - 4a^2 - 6a - 4 & 6a^3 - 2a^2 - 4a - 32 & 3 \\
4a^3 + 5a^2 - 26a & 6a^3 - 6a^2 - 9a - 6 & \\
\hline
-9a^2 + 20a - 4 & 4a^2 + 5a - 26 & a \\
4 & 4a^2 - 8a & -9 \\
\hline
-36a^2 + 80a - 16 & 13a - 26 & \\
-36a^2 - 45a + 234 & 13a - 26 & 4a + 13 \\
\hline
125)\underline{125a - 250} & & \\
a - 2 & &
\end{array}$$

$\therefore$ H.C.F. $= a - 2$.

18.

$2)\underline{12y^3 + 2y^2 - 94y - 60}$	$2)\underline{48y^3 - 24y^2 - 348y + 30}$	Reserve 2.
$6y^3 + y^2 - 47y - 30$	$24y^3 - 12y^2 - 174y + 15$	4
8	$\underline{24y^3 + 4y^2 - 188y - 120}$	
$48y^3 + 8y^2 - 376y - 240$	$-16y^2 + 14y + 135$	$-3y - 25$
$\underline{48y^3 - 42y^2 - 405y}$	$\underline{-16y^2 - 40y}$	
$50y^2 + 29y - 240$	$54y + 135$	$-8y + 27$
8	$\underline{54y + 135}$	
$400y^2 + 232y - 1920$		
$\underline{400y^2 - 350y - 3375}$		
$291)\underline{582y + 1455}$		
$2y + 5$		

$\therefore$ H.C.F. $= 2(2y + 5)$.

19.

$9x(2x^4 - 6x^3 - x^2 + 15x - 10)$	$6x^2(4x^4 + 6x^3 - 4x^2 - 15x - 15)$	Reserve $3x$.
$2x^4 - 6x^3 - x^2 + 15x - 10$	$4x^4 + 6x^3 - 4x^2 - 15x - 15$	2
9	$\underline{4x^4 - 12x^3 - 2x^2 + 30x - 20}$	
$18x^4 - 54x^3 - 9x^2 + 135x - 90$	$18x^3 - 2x^2 - 45x + 5$	$x - 13$
$\underline{18x^4 - 2x^3 - 45x^2 + 5x}$	$\underline{18x^3 - 45x}$	$9x - 1$
$2)\underline{-52x^3 + 36x^2 + 130x - 90}$	$-2x^2 + 5$	
$-26x^3 + 18x^2 + 65x - 45$	$\underline{-2x^2 + 5}$	
9		
$-234x^3 + 162x^2 + 585x - 405$		
$\underline{-234x^3 + 26x^2 + 585x - 65}$		
$68)\underline{136x^2 - 340}$		
$2x^2 - 5$		

$\therefore$ H.C.F. $= 3x(2x^2 - 5)$

20.

$15x^4 + 2x^3 - 75x^2 + 5x + 2$	$35x^4 + x^3 - 175x^2 + 30x + 1$	
$\underline{15x^4 - 75x^2 + 15x}$	3	
$2x^3 - 10x + 2$	$105x^4 + 3x^3 - 525x^2 + 90x + 3$	7
$\underline{2x^3 - 10x + 2}$	$\underline{105x^4 + 14x^3 - 525x^2 + 35x + 14}$	
$\therefore$ H.C.F. $= x^3 - 5x + 1$.	$-11)\underline{-11x^3 + 55x - 11}$	$15x + 2$
	$x^3 - 5x + 1$	

21.

$21x^3 - 32x^2 - 54x - 7$	$21x^4 - 4x^3 - 15x^2 - 2x$	x
5	$\underline{21x^4 - 32x^3 - 54x^2 - 7x}$	
$105x^3 - 160x^2 - 270x - 35$	$28x^3 + 39x^2 + 5x$	
$\underline{105x^3 + 99x^2 + 12x}$	3	
$-259x^2 - 282x - 35$	$84x^3 + 117x^2 + 15x$	4
5	$\underline{84x^3 - 128x^2 - 216x - 28}$	
$-1295x^2 - 1410x - 175$	$7)\underline{245x^2 + 231x + 28}$	
$\underline{-1295x^2 - 1221x - 148}$	$35x^2 + 33x + 4$	$3x - 37$
$-27)\underline{-189x - 27}$	$\underline{35x^2 + 5x}$	
$7x + 1$	$28x + 4$	$5x + 4$
	$\underline{28x + 4}$	

$\therefore$ H.C.F. $= 7x + 1$.

22.

$$\begin{array}{l}
y)\underline{9x^4y-22x^2y^3-3xy^4+10y^5}\\
\quad 9x^4-22x^2y^2-3xy^3+10y^4\\
\quad \underline{2\qquad\qquad\qquad\qquad\qquad}\\
\quad 18x^4-44x^2y^2-\ \ 6xy^3+20y^4\\
\quad \underline{18x^4-\ \ 9x^2y^2+105xy^3-69x^3y}\\
y)\underline{69x^3y-35x^2y^2-111xy^3+20y^4}\\
\quad 69x^3-35x^2y-111xy^2+20y^3\\
\quad \underline{2\qquad\qquad\qquad\qquad\qquad}\\
\quad 138x^3-\ \ 70x^2y-222xy^2+\ \ 40y^3\\
\quad \underline{138x^3-529x^2y-\ \ 69xy^2+805y^3}\\
\quad 153y)\underline{459x^2y-153xy^2-765y^3}\\
\qquad\qquad 3x^2-\quad xy-\quad 5y^2
\end{array}$$

$$\begin{array}{l|l}
xy)\underline{9x^5y-6x^4y^2+x^3y^3-25xy^5} & \text{Res. } y.\\
\quad 9x^4-6x^3y+x^2y^2\qquad -25y^4 & 1\\
\quad \underline{9x^4\qquad -22x^2y^2-3xy^3+10y^4} & \\
-y)\underline{-6x^3y+23x^2y^2+3xy^3-35y^4} & \\
\qquad 6x^3-23x^2y-3xy^2+35y^3 & 3x\\
\qquad \underline{6x^3-\ \ 2x^2y-10xy^2\qquad\quad} & 23\\
\quad -7y)\underline{-21x^2y+7xy^2+35y^3} & 2x\\
\qquad\qquad 3x^2-xy-5y^2 & 1\\
\qquad\qquad \underline{3x^2-xy-5y^2} &
\end{array}$$

$\therefore$ H.C.F. $=y(3x^2-xy-5y^2)$.

23.

$$\begin{array}{r}
4x^4+2x^3-18x^2+3x-5\\
\underline{4x^4-8x^3+\ \ 2x^2-2x\quad}\\
10x^3-20x^2+5x-5\\
\underline{10x^3-20x^2+5x-5}
\end{array}$$

$$\begin{array}{r|l}
6x^5-4x^4-11x^3-\ \ 3x^2-\ \ 3x-1 & \\
\underline{2\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & \\
12x^5-8x^4-22x^3-\ \ 6x^2-\ \ 6x-2 & 3x\\
\underline{12x^5+6x^4-54x^3+\ \ 9x^2-15x\quad} & \\
-14x^4+32x^3-15x^2+\ \ 9x-2 & \\
\underline{2\qquad\qquad\qquad\qquad\qquad\quad} & \\
-28x^4+64x^3-\ \ 30x^2+18x-\ \ 4 & -7\\
\underline{-28x^4-14x^3+126x^2-21x+35} & \\
39)\underline{78x^3-156x^2+39x-39} & \\
2x^3-\ \ 4x^2+\ \ x-\ 1 & 2x+5
\end{array}$$

$\therefore$ H.C.F. $=2x^3-4x^2+x-1$.

24.

$$\begin{array}{r}
3x^3-7ax^2+3a^2x-2a^3\\
\underline{2\qquad\qquad\qquad\qquad\qquad}\\
6x^3-14ax^2+\ \ 6a^2x-4a^3\\
\underline{6x^3-\ \ 9ax^2-\ \ 6a^2x\qquad}\\
-a)\underline{-5ax^2+12a^2x-4a^3}\\
5x^2-12ax\ +4a^2\\
\underline{2\qquad\qquad\qquad\qquad}\\
10x^2-24ax+\ \ 8a^2\\
\underline{10x\ -15ax-10a^2}\\
-9a)\underline{-9ax+18a^2}\\
x-\ \ 2a
\end{array}$$

$$\begin{array}{r|l}
x^4-\ \ ax^3-\ \ a^2x^2-\ \ a^3x-2a^4 & \\
\underline{3\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & \\
3x^4-3ax^3-3a^2x^2-3a^3x-6a^4 & x\\
\underline{3x^4-7ax^3+3a^2x^2-2a^3x\qquad} & \\
4ax^3-6a^2x^2-\ \ a^3x-6a^4 & \\
\underline{3\qquad\qquad\qquad\qquad\qquad\qquad} & \\
12ax^3-18a^2x^2-\ \ 3a^3x-18a^4 & 4a\\
\underline{12ax^3-28a^2x^2+12a^3x-\ \ 8a^4} & \\
5a^2)\underline{10a^2x^2-15a^3x-10a^4} & \\
2x^2-\ \ 3ax-\ \ 2a^2 & 3x\\
\underline{2x^2-\ \ 4ax\qquad\quad} & 5\\
ax-\ \ 2a^2 & 2x+a\\
\underline{ax-\ \ 2a^2} &
\end{array}$$

$\therefore$ H.C.F. $=x-2a$.

Exercise XLIX.

1. $2x^2+x-1=(x+1)(2x-1),$
$x^2+5x+4=(x+1)(x+4),$
$x^3+1=(x+1)(x^2-x+1).$
$\therefore$ H.C.F. $= x+1.$

2. $y^3-y^2-y+1=y^2(y-1)-(y-1)$
$=(y^2-1)(y-1),$
$3y^2-2y-1=(y-1)(3y+1),$
$y^3-y^2+y-1=y^2(y-1)+(y-1)$
$=(y^2+1)(y-1).$
$\therefore$ H.C.F. $= y-1.$

3.
$$
\begin{array}{r|r|l}
x^3-4x^2+9x-10 & x^3+2x^2-3x+20 & 1 \\
\underline{x^3-2x^2+5x} & \underline{x^3-4x^2+9x-10} & \\
-2x^2+4x-10 & 6)\,6x^2-12x+30 & \\
\underline{-2x^2+4x-10} & x^2-2x+5 & x-2 \\
x^2-2x+5 & x^3+5x^2-9x+35 & x+7 \\
 & \underline{x^3-2x^2+5x} & \\
 & 7x^2-14x+35 & \\
 & \underline{7x^2-14x+35} &
\end{array}
$$
$\therefore$ H.C.F. $= x^2-2x+5.$

4.
$$
\begin{array}{r|r|l}
x^3-7x^2+16x-12 & 3x^3-14x^2+16x & 3 \\
\underline{7} & \underline{3x^3-21x^2+48x-36} & \\
7x^3-49x^2+112x-84 & 7x^2-32x+36 & x-17 \\
\underline{7x^3-32x^2+36x} & \underline{7x^2-14x} & \\
-17x^2+76x-84 & -18x+36 & 7x-18 \\
\underline{7} & \underline{-18x+36} & \\
-119x^2+532x-588 & & \\
\underline{-119x^2+544x-612} & & \\
-12)\,-12x+24 & & \\
x-2 & 5x^3-10x^2+7x-14 & 5x^2+7 \\
 & \underline{5x^3-10x^2} & \\
 & 7x-14 & \\
 & \underline{7x-14} &
\end{array}
$$
$\therefore$ H.C.F. $= x-2.$

5.
$$
\begin{array}{r|r|l}
y^3-5y^2+11y-15 & y^3-y^2+3y+5 & 1 \\
\underline{y^3-y^2+3y+5} & \underline{y^3-2y^2+5y} & y+1 \\
-4)\,-4y^2+8y-20 & y^2-2y+5 & \\
y^2-2y+5 & \underline{y^2-2y+5} & \\
y^2-2y+5 & 2y^3-7y^2+16y-15 & 2y-3 \\
 & \underline{2y^3-4y^2+10y} & \\
 & -3y^2+6y-15 & \\
 & \underline{-3y^2+6y-15} &
\end{array}
$$
$\therefore$ H.C.F. $= y^2-2y+5.$

6. $$\begin{aligned} 2x^2+3x-5&=(2x+5)(x-1).\\ 3x^2-x-2&=(3x+2)(x-1),\\ 2x^2+x-3&=(2x+3)(x-1).\end{aligned}$$
$$\therefore\ \text{H.C.F.}=x-1.$$

7.
$$\begin{array}{rr|l|l}
x^3-1 & & x^3-x^2-x-2 & 1\\
\underline{x^3+x^2+x} & & \underline{x^3\qquad\quad -1} & \\
-x^2-x-1 & & -x^2-x-1 & -x+1\\
\underline{-x^2-x-1} & & & \\
 & x^2+x+1 & 2x^3-\ \ x^2-\ \ x-3 & 2x-3\\
 & & \underline{2x^3+2x^2+2x} & \\
 & & -3x^2-3x-3 & \\
 & & \underline{-3x^2-3x-3} &
\end{array}$$
$$\therefore\ \text{H. C. F.}=x^2+x+1$$

8.
$$\begin{array}{r|r|l}
x^3-3x\qquad -2 & 2x^3+3x^2\qquad -1 & 2\\
\underline{x^3+2x^2+\ \ x} & \underline{2x^3\qquad -6x-4} & \\
-2x^2-4x-2 & 3)\,3x^2+6x+3 & \\
\underline{-2x^2-4x-2} & x^2+2x+1 & x-2\\
 & & \\
x^2+2x+1 & x^3+1 & x-2\\
\underline{x^2+\ \ x} & \underline{x^3+2x^2+\ \ x} & \\
x+1 & -2x^2-\ \ x+1 & \\
\underline{x+1} & \underline{-2x^2-4x-2} & \\
 & 3)\,3x+3 & \\
 & x+1 & x+1
\end{array}$$
$$\therefore\ \text{H.C.F.}=x+1$$

9. $$\begin{aligned} 12(x^4-y^4)&=12(x^2+y^2)(x^2-y^2)\\ &=12(x^2+y^2)(x+y)(x-y);\\ 10(x^6-y^6)&=10(x^3+y^3)(x^3-y^3)\\ &=10(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)\\ 8(x^4y+xy^4)&=8xy(x^3+y^3)\\ &=8xy(x+y)(x^2-xy+y^2).\end{aligned}$$
$$\therefore\ \text{H. C. F.}=2(x+y).$$

10. $$\begin{aligned} x^4+xy^3&=x(x^3+y^3)\\ &=x(x+y)(x^2-xy+y^2);\\ x^3y+y^4&=y(x^3+y^3)\\ &=y(x+y)(x^2-xy+y^2);\\ x^4+x^2y^2+y^4&=(x^4+2x^2y^2+y^4)-x^2y^2\\ &=(x^2+y^2)^2-x^2y^2\\ &=(x^2+xy+y^2)(x^2-xy+y^2).\end{aligned}$$
$$\therefore\ \text{H.C.F.}=x^2-xy+y^2.$$

11. $2(x^2y - xy^2) = 2xy(x - y),$
$3(x^3y - xy^3) = 3xy(x + y)(x - y),$
$4(x^4y - xy^4) = 4xy(x - y)(x^2 + xy + y^2),$
$5(x^5y - xy^5) = 5xy(x + y)(x - y)(x^2 + y^2).$
$\therefore$ H.C.F. $= xy(x - y).$

Exercise L.

1. $4a^3x = 2^2 \times a^3 \times x,$
$6a^2x^2 = 3 \times 2 \times a^2 \times x^2,$
$2ax^2 = 2 \times a \times x^2.$
$\therefore$ L.C.M. $= 12a^3x^2.$

2. $18ax^2 = 3^2 \times 2 \times a \times x^2,$
$72ay^2 = 3^2 \times 2^3 \times a \times y^2,$
$12xy = 3 \times 2^2 \times x \times y.$
$\therefore$ L.C.M. $= 72ax^2y^2.$

3. $x^2 = x \times x,$
$ax + x^2 = x(a + x).$
$\therefore$ L.C.M. $= x^2(a + x).$

4. $x^2 - 1 = (x + 1)(x - 1),$
$(x^2 - x) = x(x - 1).$
$\therefore$ L.C.M. $= x(x + 1)(x - 1).$

5. $a^2 - b^2 = (a + b)(a - b),$
$a^2 + ab = a(a + b).$
$\therefore$ L.C.M. $= a(a+b)(a-b).$

6. $2x - 1 = 2x - 1,$
$4x^2 - 1 = (2x + 1)(2x - 1).$
$\therefore$ L.C.M. $= (2x+1)(2x-1).$

7. $a + b = a + b,$
$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$
$\therefore$ L.C.M. $= (a+b)(a^2-ab+b^2).$

8. $x^2 - 1 = (x + 1)(x - 1),$
$x^2 + 1 = x^2 + 1,$
$x^4 - 1 = (x^2 + 1)(x + 1)(x - 1).$
$\therefore$ L.C.M. $= (x^2+1)(x+1)(x-1).$

9. $x^2 - x = x(x - 1),$
$x^3 - 1 = (x - 1)(x^2 + x + 1),$
$x^3 + 1 = (x + 1)(x^2 - x + 1).$
$\therefore$ L.C.M. $= x(x^3 + 1)(x^3 - 1).$

10. $x^2 - 1 = (x + 1)(x - 1),$
$x^2 - x = x(x + 1),$
$x^3 - 1 = (x - 1)(x^2 + x + 1).$
$\therefore$ L.C.M. $= x(x + 1)(x^3 - 1).$

11. $2a + 1 = 2a + 1,$
$4a^2 - 1 = (2a + 1)(2a - 1),$
$8a^3 + 1 = (2a+1)(4a^2-2a+1).$
$\therefore$ L.C.M. $= (8a^3+1)(2a-1).$

12. $(a + b)^2 = (a + b)^2,$
$(a^2 - b^2) = (a + b)(a - b).$
$\therefore$ L.C.M. $= (a + b)^2(a - b).$

13. $4(1 + x) = 4(1 + x),$
$4(1 - x) = 4(1 - x),$
$2(1 - x^2) = 2(1 + x)(1 - x).$
$\therefore$ L.C.M. $= 4(1 + x)(1 - x).$

14. $x - 1 = x - 1,$
$x^2 + x + 1 = x^2 + x + 1,$
$x^3 - 1 = (x - 1)(x^2 + x + 1).$
$\therefore$ L.C.M. $= x^3 - 1.$

15. $x^2 - y^2 = (x + y)(x - y),$
$(x + y)^2 = (x + y)^2,$
$(x - y)^2 = (x - y)^2.$
$\therefore$ L.C.M. $= (x + y)^2(x - y)^2.$

16. $x^2 - y^2 = (x + y)(x - y),$
$3(x - y)^2 = 3(x - y)^2,$
$12(x^3+y^3) = 12(x+y)(x^2-xy+y^2).$
$\therefore$ L.C.M. $= 12(x^3+y^3)(x - y)^2.$

17. $6(x^2 + xy) = 6x(x + y),$
$8(xy - y^2) = 8y(x - y),$
$10(x^2 - y^2) = 10(x + y)(x - y).$
$\therefore$ L.C.M. $= 120xy(x + y)(x - y)$

18. $x^2+5x+6=(x+3)(x+2),$
$x^2+6x+8=(x+2)(x+4).$

$\therefore$ L.C.M. $=(x+2)(x+3)(x+4).$

19. $a^2-a-20=(a-5)(a+4),$
$a^2+a-12=(a+4)(a-3).$

$\therefore$ L.C.M. $=(a-3)(a+4)(a-5).$

20. $x^2+11x+30=(x+6)(x+5),$
$x^2+12x+35=(x+5)(x+7).$

$\therefore$ L.C.M. $=(x+5)(x+6)(x+7).$

21. $x^2-9x-22=(x+2)(x-11),$
$x^2-13x+22=(x-2)(x-11).$

$\therefore$ L.C.M. $=(x+2)(x-2)(x-11).$

22. $4ab(a^2-3ab+2b^2)=4ab(a-2b)(a-b),$
$5a^2(a^2+ab-6b^2)=5a^2(a+3b)(a-2b).$

$\therefore$ L.C.M. $=20a^2b(a-b)(a-2b)(a+3b).$

23. $20(x^2-1)=20(x+1)(x-1),$
$24(x^2-x-2)=24(x-2)(x+1),$
$16(x^2+x-2)=16(x+2)(x-1).$

$\therefore$ L.C.M. $=240(x+1)(x-1)(x+2)(x-2).$

24. $12xy(x^2-y^2)=12xy(x+y)(x-y),$
$2x^2(x+y)^2=2x^2(x+y)(x+y),$
$3y^2(x-y)^2=3y^2(x-y)(x-y).$

$\therefore$ L.C.M. $=12x^2y^2(x-y)^2(x+y)^2.$

25. $(a-b)(b-c)=(a-b)(b-c),$
$(b-c)(c-a)=-(a-c)(b-c),$
$(c-a)(a-b)=-(a-b)(a-c).$

$\therefore$ L.C.M. $=(a-b)(b-c)(c-a).$

26. $(a-b)(a-c)=(a-b)(a-c),$
$(b-a)(b-c)=-(a-b)(b-c),$
$(c-a)(c-b)=(a-c)(b-c).$

$\therefore$ L.C.M. $=(a-b)(a-c)(b-c).$

27. $x^3-4x^2+3x=x(x^2-4x+3)$
$=x(x-3)(x-1),$
$x^4+x^3-12x^2=x^2(x^2+x-12)$
$=x^2(x+4)(x-3),$
$x^5+3x^4-4x^3=x^3(x^2+3x-4)$
$=x^3(x-1)(x+4).$

$\therefore$ L.C.M. $=x^3(x-1)(x-3)(x+4).$

28. $x^2y-xy^2=xy(x-y),$
$3x(x-y)^2=3x(x-y)^2,$
$4y(x-y)^3=4y(x-y)^3.$

$\therefore$ L.C.M. $=12xy(x-y)^3.$

29. $(a+b)^2-(c+d)^2=(a+b+c+d)(a+b-c-d),$
$(a+c)^2-(b+d)^2=(a+b+c+d)(a-b+c-d),$
$(a+d)^2-(b+c)^2=(a+b+c+d)(a-b-c+d).$

$\therefore$ L.C.M. $=(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d).$

30. $(2x-4)(3x-6)=2(x-2)\times 3(x-2),$
$(x-3)(4x-8)=(x-3)\times 4(x-2),$
$(2x-6)(5x-10)=2(x-3)\times 5(x-2).$

$\therefore$ L.C.M. $=60(x-2)^2(x-3).$

Exercise LI.

1. $6x^2 - x - 2 = (3x - 2)(2x + 1),$
$21x^2 - 17x + 2 = (3x - 2)(7x - 1),$
$14x^2 + 5x - 1 = (2x + 1)(7x - 1).$
$\therefore$ L.C.M. $= (3x - 2)(2x + 1)(7x - 1).$

2. $x^2 - 1 = (x + 1)(x - 1),$
$x^2 + 2x - 3 = (x + 3)(x - 1),$
$6x^2 - x - 2 = (3x - 2)(2x + 1).$
$\therefore$ L.C.M. $= (2x + 1)(3x - 2)(x - 1)(x + 1)(x + 3).$

3. $x^3 - 27 = (x - 3)(x^2 + 3x + 9),$
$x^2 - 15x + 36 = (x - 3)(x - 12),$
$x^3 - 3x^2 - 2x + 6 = x^2(x - 3) - 2(x - 3)$
$= (x^2 - 2)(x - 3).$
$\therefore$ L.C.M. $= (x - 3)(x - 12)(x^2 - 2)(x^2 + 3x + 9).$

4. $5x^2 + 19x - 4 = (5x - 1)(x + 4),$
$10x^2 + 13x - 3 = (5x - 1)(2x + 3).$
$\therefore$ L.C.M. $= (5x - 1)(x + 4)(2x + 3).$

5. $12x^2 + xy - 6y^2 = (4x + 3y)(3x - 2y),$
$18x^2 + 18xy - 20y^2 = 2(3x - 2y)(3x + 5y).$
$\therefore$ L. C. M. $= 2(3x - 2y)(3x + 5y)(4x + 3y).$

6.
$$\begin{array}{r|l}
x & x^4 - 2x^3 + x \\
\hline
 & x^3 - 2x^2 + 1 \\
 & x^3 - 2x - 1 \\
\hline
-2 & -2x^2 + 2x + 2 \\
\hline
 & x^2 - x - 1
\end{array}
\qquad
\begin{array}{r|r|l}
2 & 2x^4 - 2x^3 - 2x - 2 & \\
\hline
 & x^4 - x^3 - x - 1 & x \\
 & x^4 - 2x^3 + x & \\
\hline
 & x^3 - 2x - 1 & 1 \\
 & x^3 - x^2 - x & x + 1 \\
\hline
 & x^2 - x - 1 & \\
 & x^2 - x - 1 &
\end{array}$$

Hence, $x^4 - 2x^3 + x = (x^2 - x - 1) \times x(x - 1),$
and $2x^4 - 2x^3 - 2x - 2 = (x^2 - x - 1) \times 2(x^2 + 1).$
$\therefore$ L.C.M. $= 2x(x^2 - x - 1)(x^2 + 1)(x - 1).$

7. $12x^2 + 2x - 4 = (6x + 4)(2x - 1) = 2(3x + 2)(2x - 1),$
$12x^2 - 42x - 24 = (6x + 3)(2x - 8) = 6(2x + 1)(x - 4),$
$12x^2 - 28x - 24 = (6x + 4)(2x - 6) = 4(3x + 2)(x - 3).$
$\therefore$ L.C.M. $= 12(3x + 2)(2x - 1)(2x + 1)(x - 4)(x - 3).$

$$\begin{array}{r|r|l}
\textbf{8.}\quad x^3-6x^2+11x-6 & x^3-9x^2+26x-24 & 1\\
\underline{x^3-5x^2+\ \ 6x} & \underline{x^3-6x^2+11x-\ \ 6} & \\
-\ \ x^2+\ \ 5x-6 & -3)\underline{-3x^2+15x-18} & \\
\underline{-\ \ x^2+\ \ 5x-6} & x^2-\ \ 5x+\ \ 6 & x-1\\
 & & \\
x^2-5x+6 & x^3-8x^2+19x-12 & x-3\\
\underline{x^2-3x} & \underline{x^3-5x^2+\ \ 6x} & \\
-2x+6 & -3x^2+13x-12 & \\
\underline{-2x+6} & \underline{-3x^2+15x-18} & \\
 & -2)\underline{-\ \ 2x+\ \ 6} & \\
 & x-\ \ 3 & x-2
\end{array}$$

Hence,
$$\begin{aligned}
x^3-6x^2+11x-\ \ 6 &= (x-1)(x-2)(x-3),\\
x^3-9x^2+26x-24 &= (x-2)(x-3)(x-4),\\
x^3-8x^2+19x-12 &= (x-1)(x-3)(x-4).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = (x-1)(x-2)(x-3)(x-4).$$

9.
$$\begin{aligned}
x^2-4a^2 &= (x+2a)(x-2a),\\
x^3+2ax^2+4a^2x+8a^3 &= x^2(x+2a)+4a^2(x+2a)\\
&= (x^2+4a^2)(x+2a),\\
x^3-2ax^2+4a^2x-8a^3 &= x^2(x-2a)+4a^2(x-2a)\\
&= (x^2+4a^2)(x-2a).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = (x+2a)(x-2a)(x^2+4a^2).$$

10.
$$\begin{aligned}
x^3+2x^2y-xy^2-2y^3 &= x^2(x+2y)-y^2(x+2y)\\
&= (x^2-y^2)(x+2y),\\
x^3-2x^2y-xy^2+2y^3 &= x^2(x-2y)-y^2(x-2y)\\
&= (x^2-y^2)(x-2y).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = (x^2-y^2)(x+2y)(x-2y).$$

11.
$$\begin{aligned}
1+p+p^2 &= 1+p+p^2,\\
1-p+p^2 &= 1-p+p^2,\\
1+p^2+p^4 &= (1+p+p^2)(1-p+p^2).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = 1+p^2+p^4.$$

12.
$$\begin{aligned}
1-a &= 1-a,\\
(1-a)^2 &= (1-a)(1-a),\\
(1-a)^3 &= (1-a)(1-a)(1-a).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = (1-a)^3.$$

13.
$$\begin{aligned}
(a+c)^2-b^2 &= (a+b+c)(a-b+c),\\
(a+b)^2-c^2 &= (a+b+c)(a+b-c),\\
(b+c)^2-a^2 &= (a+b+c)(-a+b+c).
\end{aligned}$$

$$\therefore \text{ L.C.M.} = (a+b+c)(a+b-c)(a-b+c)(-a+b+c)$$

14.
$$\begin{array}{r|r|l}
c)\underline{4c^3 - c^2y - 3cy^2} & 3c^3 - 3c^2y + cy^2 - y^3 & \\
4c^2 - cy - 3y^2 & 4 & \\
9 & 12c^3 - 12c^2y + 4cy^2 - 4y^3 & 3c \\
36c^2 - 9cy - 27y^2 & 12c^3 - 3c^2y - 9cy^2 & \\
36c^2 - 52cy + 16y^2 & -y)\underline{-9c^2y + 13cy^2 - 4y^3} & 4 \\
43y)\underline{43cy - 43y^2} & 9c^2 - 13cy + 4y^2 & 9c - 4y \\
c - y & 9c^2 - 9cy & \\
 & -4cy + 4y^2 & \\
 & -4cy + 4y^2 &
\end{array}$$

Hence $\quad 4c^3 - c^2y - 3cy^2 = (c-y)(4c^2 + 3cy),$
$3c^3 - 3c^2y + cy^2 - y^3 = (c-y)(3c^2 + y^2).$

$\therefore$ L.C.M. $= c(c-y)(4c+3y)(3c^2+y^2).$

15. $m^3 - 8m + 3 = (m+3)(m^2 - 3m + 1),$
$m^6 + 3m^5 + m + 3 = m^5(m+3) + (m+3)$
$= (m^5 + 1)(m+3).$

$\therefore$ L.C.M. $= (m+3)(m^2 - 3m + 1)(m^5 + 1).$

16. $20n^4 + n^2 - 1 = (5n^2 - 1)(4n^2 + 1),$
$25n^4 + 5n^3 - n - 1 = (5n^2 - 1)(5n^2 + n + 1)$

$\therefore$ L.C.M. $= (5n^2 - 1)(4n^2 + 1)(5n^2 + n + 1).$

17.
$$\begin{array}{r|r|l}
4b^3 - 12b^2 + 9b - 1 & b^4 - 2b^3 + b^2 - 8b + 8 & \\
7 & 4 & \\
28b^3 - 84b^2 + 63b - 7 & 4b^4 - 8b^3 + 4b^2 - 32b + 32 & b + 1 \\
28b^3 - 160b^2 + 132b & 4b^4 - 12b^3 + 9b^2 - b & \\
76b^2 - 69b - 7 & 4b^3 - 5b^2 - 31b + 32 & \\
7 & 4b^3 - 12b^2 + 9b - 1 & \\
532b^2 - 483b - 49 & 7b^2 - 40b + 33 & 4b + 76 \\
532b^2 - 3040b + 2508 & 7b^2 - 7b & \\
2557)\underline{2557b - 2557} & -33b + 33 & 7b - 33 \\
b - 1 & -33b + 33 &
\end{array}$$

Hence, $\quad 4b^3 - 12b^2 + 9b - 1 = (b-1)(4b^2 - 8b + 1),$
$b^4 - 2b^3 + b^2 - 8b + 8 = (b-1)(b^3 - b^2 - 8).$

$\therefore$ L.C.M. $= (b-1)(4b^2 - 8b + 1)(b^3 - b^2 - 8).$

18.
$$\begin{array}{r|r|l}
2r)\underline{2r^5 - 8r^4 + 12r^3 - 8r^2 + 2r} & 3r)\underline{3r^5 - 6r^3 + 3r} & \text{Reserve } r. \\
r^4 - 4r^3 + 6r^2 - 4r + 1 & r^4 - 2r^2 + 1 & 1 \\
r^4 - 2r^2 + 1 & r^4 - 2r^3 + r^2 & r^2 + 2r + 1 \\
-4r)\underline{-4r^3 + 8r^2 - 4r} & 2r^3 - 3r^2 + 1 & \\
r^2 - 2r + 1 & 2r^3 - 4r^2 + 2r & \\
 & r^2 - 2r + 1 & \\
 & r^2 - 2r + 1 &
\end{array}$$

Hence, $2r^5 - 8r^4 + 12r^3 - 8r^2 + 2r = 2r(r-1)^4,$
$3r^5 - 6r^3 + 3r = 3r(r^2 - 1)^2.$

$\therefore$ L.C.M. $= 6r(r-1)^4(r+1)^2.$

Exercise LII.

1. $\dfrac{x^2-1}{4x(x+1)}$
$=\dfrac{(x+1)(x-1)}{4x(x+1)}$
$=\dfrac{x-1}{4x}.$

2. $\dfrac{x^2-9x+20}{x^2-7x+12}$
$=\dfrac{(x-5)(x-4)}{(x-3)(x-4)}$
$=\dfrac{x-5}{x-3}.$

3. $\dfrac{x^2-2x-3}{x^2-10x+21}$
$=\dfrac{(x-3)(x+1)}{(x-7)(x-3)}$
$=\dfrac{x+1}{x-7}.$

4. $\dfrac{x^4+x^2+1}{x^2+x+1}$
$=\dfrac{(x^2+x+1)(x^2-x+1)}{x^2+x+1}$
$=x^2-x+1.$

5. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}$
$=\dfrac{(x^3+y^3)(x^3+y^3)}{(x^3+y^3)(x^3-y^3)}$
$=\dfrac{x^3+y^3}{x^3-y^3}.$

6. $\dfrac{a^3+1}{a^3+2a^2+2a+1}$
$=\dfrac{(a+1)(a^2-a+1)}{(a+1)(a^2+a+1)}$
$=\dfrac{a^2-a+1}{a^2+a+1}.$

7. $\dfrac{a^2-a-20}{a^2+a-12}$
$=\dfrac{(a-5)(a+4)}{(a-3)(a+4)}$
$=\dfrac{a-5}{a-3}.$

8.
$$
\begin{array}{r|r|l}
x^3-4x^2+9x-10 & x^3+2x^2-3x+20 & 1 \\
x^3+2x^2-3x+20 & x^3-2x^2+5x & x \\
\hline
-6)\,-6x^2+12x-30 & 4x^2-8x+20 & \\
x^2-2x+5 & 4x^2-8x+20 & 4
\end{array}
$$

$\therefore$ H. C. F. $=x^2-2x+5.$

$$\therefore\ \frac{x^3-4x^2+9x-10}{x^3+2x^2-3x+20}=\frac{(x^2-2x+5)(x-2)}{(x^2-2x+5)(x+4)}=\frac{x-2}{x+4}.$$

9.
$$\begin{array}{l|l|l}
x^3 - 5x^2 + 11x - 15 & x^3 - x^2 + 3x + 5 & 1 \\
\underline{x^3 - 2x^2 + 5x} & \underline{x^3 - 5x^2 + 11x - 15} & \\
-3x^2 + 6x - 15 & 4)\underline{4x^2 - 8x + 20} & \\
\underline{-3x^2 + 6x - 15} & x^2 - 2x + 5 & x - 3
\end{array}$$

$\therefore$ H.C.F. $= x^2 - 2x + 5.$

$$\therefore \frac{x^3 - 5x^2 + 11x - 15}{x^3 - x^2 + 3x + 5} = \frac{(x-3)(x^2 - 2x + 5)}{(x+1)(x^2 - 2x + 5)} = \frac{x-3}{x+1}.$$

10.
$$\begin{array}{l|l|l}
x^4 - x^3y - xy^3 - y^4 & x^4 + x^3y + xy^3 - y^4 & 1 \\
\underline{x^4 + x^2y^2} & \underline{x^4 - x^3y - xy^3 - y^4} & \\
-x^3y - x^2y^2 - xy^3 - y^4 & 2xy)\underline{2x^3y + 2xy^3} & \\
\underline{-x^3y \qquad\quad - xy^3} & x^2 + y^2 & x^2 \\
-x^2y^2 \qquad\quad - y^4 & & -xy \\
\underline{-x^2y^2 \qquad\quad - y^4} & & -y^2
\end{array}$$

$\therefore$ H.C.F. $= x^2 + y^2.$

$$\therefore \frac{x^4 + x^3y + xy^3 - y^4}{x^4 - x^3y - xy^3 - y^4} = \frac{(x^2 + y^2)(x^2 + xy - y^2)}{(x^2 + y^2)(x^2 - xy - y^2)} = \frac{x^2 + xy - y^2}{x^2 - xy - y^2}.$$

11.
$$\begin{array}{l|l|l}
a^3 - 3a + 2 & a^3 + 4a^2 - 5 & 1 \\
\underline{4} & \underline{a^3 - 3a + 2} & \\
4a^3 \qquad - 12a + 8 & 4a^2 + 3a - 7 & a \\
\underline{4a^3 + 3a^2 - 7a} & \underline{-3} & \\
-3a^2 - 5a + 8 & -12a^2 - 9a + 21 & 4 \\
\underline{-3a^2 + 3a} & \underline{-12a^2 - 20a + 32} & \\
-8a + 8 & 11)\underline{11a - 11} & \\
\underline{-8a + 8} & a - 1 & -3a \\
 & & -8
\end{array}$$

$\therefore$ H.C.F. $= a - 1.$

$$\therefore \frac{a^3 + 4a^2 - 5}{a^3 - 3a + 2} = \frac{(a-1)(a^2 + 5a + 5)}{(a-1)(a^2 + a - 2)} = \frac{a^2 + 5a + 5}{a^2 + a - 2}.$$

12.
$$\begin{array}{l|l|l}
x^3 + x^2 - x - 1 & 3x^2 + 2x - 1 & x \\
\underline{3} & \underline{3x^2 - 6x - 9} & 3 \\
3x^3 + 3x^2 - 3x - 3 & 8)\underline{8x + 8} & \\
\underline{3x^3 + 2x^2 - x} & x + 1 & x - 3 \\
x^2 - 2x - 3 & & \\
\underline{x^2 + x} & & \\
-3x - 3 & & \\
\underline{-3x - 3} & &
\end{array}$$

$\therefore$ H.C.F. $= x + 1.$

$$\therefore \frac{3x^2 + 2x - 1}{x^3 + x^2 - x - 1} = \frac{(x+1)(3x-1)}{(x+1)(x^2-1)} = \frac{3x-1}{x^2-1}.$$

13.
$$\begin{array}{r|r|l}
x^3 - x^2 - 2x + 2 & x^3 - 3x^2 + 4x - 2 & 1 \\
x^3 - 3x^2 + 2x & x^3 - x^2 - 2x + 2 & \\
2x^2 - 4x + 2 & -2)\,-2x^2 + 6x - 4 & x \\
2x^2 - 6x + 4 & x^2 - 3x + 2 & 2 \\
2)\,2x - 2 & x^2 - x & \\
x - 1 & -2x + 2 & \\
 & -2x + 2 & x - 2
\end{array}$$

$\therefore$ H. C. F. $= x - 1$.

$$\therefore \frac{x^3 - 3x^2 + 4x - 2}{x^3 - x^2 - 2x + 2} = \frac{(x-1)(x^2 - 2x + 2)}{(x-1)(x^2 - 2)} = \frac{x^2 - 2x + 2}{x^2 - 2}.$$

14.
$$\frac{4x^2 - 12ax + 9a^2}{8x^3 - 27a^3} = \frac{(2x - 3a)(2x - 3a)}{(2x - 3a)(4x^2 + 6ax + 9a^2)} = \frac{2x - 3a}{4x^2 + 6ax + 9a^2}.$$

16.
$$\frac{a^2 - b^2 - 2bc - c^2}{a^2 + 2ab + b^2 - c^2} = \frac{a^2 - (b^2 + 2bc + c^2)}{(a^2 + 2ab + b^2) - c^2} = \frac{(a + b + c)(a - b - c)}{(a + b + c)(a + b - c)} = \frac{a - b - c}{a + b - c}.$$

15.
$$\frac{15a^2 + ab - 2b^2}{9a^2 + 3ab - 2b^2} = \frac{(5a + 2b)(3a - b)}{(3a + 2b)(3a - b)} = \frac{5a + 2b}{3a + 2b}.$$

17.
$$\frac{x^4 - x^2 - 2x + 2}{2x^3 - x - 1} = \frac{(x-1)(x^3 + x^2 - 2)}{(x-1)(2x^2 + 2x + 1)} = \frac{x^3 + x^2 - 2}{2x^2 + 2x + 1}.$$

18.
$$\begin{array}{r|r|l}
x^3 - 2x^2 - x + 2 & x^3 - 6x^2 + 11x - 6 & 1 \\
x^2 - 3x^2 + 2x & x^3 - 2x^2 - x + 2 & \\
x^2 - 3x + 2 & -4)\,-4x^2 + 12x - 8 & \\
x^2 - 3x + 2 & x^2 - 3x + 2 & x + 1
\end{array}$$

$\therefore$ H. C. F. $= x^2 - 3x + 2$.

$$\therefore \frac{x^3 - 6x^2 + 11x - 6}{x^3 - 2x^2 - x + 2} = \frac{(x^2 - 3x + 2)(x - 3)}{(x^2 - 3x + 2)(x + 1)} = \frac{x - 3}{x + 1}.$$

19.
$$\begin{array}{r|r|l}
6x^3 - 17x^2 + 11x - 2 & 6x^3 - 23x^2 + 16x - 3 & 1 \\
6x^3 - 5x^2 + x & 6x^3 - 17x^2 + 11x - 2 & \\
-12x^2 + 10x - 2 & -1)\,-6x^2 + 5x - 1 & x - 2 \\
-12x^2 + 10x - 2 & 6x^2 - 5x + 1 &
\end{array}$$

$$\therefore \frac{6x^3 - 23x^2 + 16x - 3}{6x^3 - 17x^2 + 11x - 2} = \frac{(6x^2 - 5x + 1)(x - 3)}{(6x^2 - 5x + 1)(x - 2)} = \frac{x - 3}{x - 2}.$$

20. $$\frac{x^4 - x^3 - x + 1}{x^4 - 2x^3 - x^2 - 2x + 1}$$
$$= \frac{x^3(x-1) - (x-1)}{(x^2 + x + 1)(x^2 - 3x + 1)}$$
$$= \frac{(x^3 - 1)(x - 1)}{(x^2 + x + 1)(x^2 - 3x + 1)}$$
$$= \frac{(x-1)(x^2 + x + 1)(x - 1)}{(x^2 - 3x + 1)(x^2 + x + 1)}$$
$$= \frac{(x-1)^2}{x^2 - 3x + 1}.$$

21. $$\begin{array}{r|r|l} a)\underline{a^4 - a^3b - a^2b^2 + ab^3} & a^5 - a^4b - ab^4 + b^5 & a^2 + b^2 \\ a^3 - a^2b - ab^2 + b^3 & \underline{a^5 - a^4b - a^3b^2 + a^2b^3} & \\ & a^3b^2 - a^2b^3 - ab^4 + b^5 & \\ & \underline{a^3b^2 - a^2b^3 - ab^4 + b^5} & \end{array}$$

$\therefore$ H.C.F. $= a^3 - a^2b - ab^2 + b^3.$

$$\therefore \frac{a^5 - a^4b - ab^4 + b^5}{a^4 - a^3b - a^2b^2 + ab^3} = \frac{(a^2 + b^2)(a^3 - a^2b - ab^2 + b^3)}{a(a^3 - a^2b - ab^2 + b^3)} = \frac{a^2 + b^2}{a}.$$

22. $$\frac{(a+b)^2}{a^2 - ab - 2b^2}$$
$$= \frac{(a+b)(a+b)}{(a-2b)(a+b)}$$
$$= \frac{a+b}{a-2b}.$$

23. $$\frac{3ab(a^2 - b^2)}{4(a^2b - ab^2)^2}$$
$$= \frac{3ab(a+b)(a-b)}{4a^2b^2(a-b)(a-b)}$$
$$= \frac{3(a+b)}{4ab(a-b)}.$$

24. $$\frac{a^2 + 2ab + b^2 - c^2}{a^2 + ab - ac}$$
$$= \frac{(a^2 + 2ab + b^2) - c^2}{a^2 + ab - ac}$$
$$= \frac{(a+b+c)(a+b-c)}{a(a+b-c)}$$
$$= \frac{a+b+c}{a}.$$

25. $$\frac{6x^3 - 11x^2y + 3xy^2}{6x^2y - 5xy^2 - 6y^3}$$
$$= \frac{x(6x^2 - 11xy + 3y^2)}{y(6x^2 - 5xy - 6y^2)}$$
$$= \frac{x(2x - 3y)(3x - y)}{y(2x - 3y)(3x + 2y)}$$
$$= \frac{x(3x - y)}{y(3x + 2y)}.$$

26. $$\frac{a^2 - (b + c + d)^2}{(a-b)^2 - (c+d)^2}$$
$$= \frac{(a+b+c+d)(a-b-c-d)}{(a-b+c+d)(a-b-c-d)}$$
$$= \frac{a+b+c+d}{a-b+c+d}.$$

27. $$\frac{6x^2 - 5x - 6}{8x^2 - 2x - 15}$$
$$= \frac{(3x+2)(2x-3)}{(4x+5)(2x-3)}$$
$$= \frac{3x+2}{4x+5}.$$

28. $$\frac{x^4+x^2y^2+y^4}{(x-y)(x^3-y^3)}$$
$$=\frac{(x^2+xy+y^2)(x^2-xy+y^2)}{(x-y)(x-y)(x^2+xy+y^2)}$$
$$=\frac{x^2-xy+y^2}{(x-y)^2}.$$

29. $$\frac{x^6+y^6}{x^4-x^2y^2+y^4}$$
$$=\frac{(x^2+y^2)(x^4-x^2y^2+y^4)}{x^4-x^2y^2+y^4}$$
$$=x^2+y^2.$$

30. $$\frac{(a^3+b^3)(a^2+ab+b^2)}{(a^3-b^3)(a^2-ab+b^2)}$$
$$=\frac{(a+b)(a^2-ab+b^2)(a^2+ab+b^2)}{(a-b)(a^2+ab+b^2)(a^2-ab+b^2)}$$
$$=\frac{a+b}{a-b}.$$

Exercise LIII.

1. $$\frac{x^2-2x+1}{x-1}=x-1.$$

2. $$\begin{array}{r|l} 3x^2+\ \ 2x+\ \ 1 & x+4 \\ \underline{3x^2+12x\qquad} & 3x-10 \\ -10x+\ \ 1 & \\ \underline{-10x-40} & \\ +41 & \end{array}$$
$$\therefore\ \frac{3x^2+2x+1}{x+4}=3x-10+\frac{41}{x+4}.$$

3. $$\begin{array}{r|l} 3x^2+\ \ 6x+\ \ 5 & x+4 \\ \underline{3x^2+12x\qquad} & 3x-6 \\ -6x+\ \ 5 & \\ \underline{-6x-24} & \\ +29 & \end{array}$$
$$\therefore\ \frac{3x^2+6x+5}{x+4}=3x-6+\frac{29}{x+4}.$$

4. $$\begin{array}{r|l} a^2-\ \ ax+\ \ x^2 & a+x \\ \underline{a^2+\ \ ax\qquad} & a-2x \\ -2ax+\ \ x^2 & \\ \underline{-2ax-2x^2} & \\ +3x^2 & \end{array}$$
$$\therefore\ \frac{a^2-ax+x^2}{a+x}=a-2x+\frac{3x^2}{a+x}.$$

5. $$\begin{array}{r|l} 2x^2\qquad\quad+\ \ 5 & x-3 \\ \underline{2x^2-6x\qquad} & 2x+6 \\ 6x+\ \ 5 & \\ \underline{6x-18} & \\ +23 & \end{array}$$
$$\therefore\ \frac{2x^2+5}{x-3}=2x+6+\frac{23}{x-3}.$$

6. $$\begin{array}{r|l} 10a^2-17ax+10x^2 & 5a-x \\ \underline{10a^2-\ \ 2ax\qquad\quad} & 2a-3x \\ -15ax+10x^2 & \\ \underline{-15ax+\ \ 3x^2} & \\ +7x^2 & \end{array}$$
$$\therefore\ \frac{10a^2-17ax+10x^2}{5a-x}=2a-3x+\frac{7x^2}{5a-x}.$$

7. $$\begin{array}{r|l} 48x^2\qquad\quad+16 & 4x-1 \\ \underline{48x^2-12x\qquad} & 12x+3 \\ 12x+16 & \\ \underline{12x-\ \ 3} & \\ +19 & \end{array}$$
$$\therefore\ \frac{48x^2+16}{4x-1}=12x+3+\frac{19}{4x-1}.$$

8.
$$\begin{array}{l|l}
2x^2-5x-\ \ 2 & x-4 \\
\underline{2x^2-8x\qquad\ \ } & 2x+3 \\
\qquad\ \ 3x-\ \ 2 & \\
\qquad\ \ \underline{3x-12} & \\
\qquad\quad\ +10 &
\end{array}$$

$$\therefore \frac{2x^2-5x-2}{x-4}=2x+3+\frac{10}{x-4}.$$

9.
$$\begin{array}{l|l}
a^2\qquad\quad +b^2 & a-b \\
\underline{a^2\ -\ ab\qquad} & a+b \\
\qquad ab+b^2 & \\
\qquad \underline{ab-b^2} & \\
\qquad\ \ +2b^2 &
\end{array}$$

$$\therefore \frac{a^2+b^2}{a-b}=a+b+\frac{2b^2}{a-b}.$$

10.
$$\begin{array}{l|l}
5x^3-\ x^2\qquad\ +5 & 5x^2+4x-1 \\
\underline{5x^3+4x^2-\ x\quad\ } & x-1 \\
\quad\ -5x^2+\ x+5 & \\
\quad\ \underline{-5x^2-4x+1} & \\
\qquad\qquad +5x+4 &
\end{array}$$

$$\therefore \frac{5x^3-x^2+5}{5x^2+4x-1}=x-1+\frac{5x+4}{5x^2+4x-1}.$$

Exercise LIV.

1. $1-\dfrac{x-y}{x+y}$

$$=\frac{x+y-(x-y)}{x+y}$$

$$=\frac{x+y-x+y}{x+y}$$

$$=\frac{2y}{x+y}.$$

2. $1+\dfrac{x-y}{x+y}$

$$=\frac{x+y+(x-y)}{x+y}$$

$$=\frac{x+y+x-y}{x+y}$$

$$=\frac{2x}{x+y}.$$

3. $3x-\dfrac{1+2x^2}{x}$

$$=\frac{3x^2-(1+2x^2)}{x}$$

$$=\frac{3x^2-1-2x^2}{x}$$

$$=\frac{x^2-1}{x}.$$

4. $a-x+\dfrac{a^2+x^2}{a-x}$

$$=\frac{a^2-2ax+x^2+(a^2+x^2)}{a-x}$$

$$=\frac{2(a^2-ax+x^2)}{a-x}.$$

5. $5a-2b-\dfrac{3a^2-4b^2}{5a-6b}$

$$=\frac{25a^2-40ab+12b^2-(3a^2-4b^2)}{5a-6b}$$

$$=\frac{22a^2-40ab+16b^2}{5a-6b}.$$

6. $a+b-\dfrac{a^2+b^2}{a+b}$

$$=\frac{a^2+2ab+b^2-(a^2+b^2)}{a+b}$$

$$=\frac{2ab}{a+b}.$$

7. $7a-\dfrac{2-3a+4a^2}{5-6a}$

$$=\frac{35a-42a^2-(2-3a+4a^2)}{5-6a}$$

$$=\frac{38a-46a^2-2}{5-6a}.$$

8. $3x - \dfrac{5ax-3}{2a}$
$= \dfrac{6ax-(5ax-3)}{2a}$
$= \dfrac{ax+3}{2a}.$

9. $\dfrac{a+b}{a-b}+1$
$= \dfrac{a+b+(a-b)}{a-b}$
$= \dfrac{2a}{a-b}.$

10. $\dfrac{a-b}{a+b}-1$
$= \dfrac{a-b-(a+b)}{a+b}$
$= \dfrac{-2b}{a+b}.$

11. $\dfrac{2x^2}{x+y}-(x+y)$
$= \dfrac{2x^2-(x^2+2xy+y^2)}{x+y}$
$= \dfrac{x^2-2xy-y^2}{x+y}.$

12. $\dfrac{5a-12x}{4}+6a+3x$
$= \dfrac{5a-12x+24a+12x}{4}$
$= \dfrac{29a}{4}.$

13. $a-1+\dfrac{1}{a+1}$
$= \dfrac{a^2-1+1}{a+1}$
$= \dfrac{a^2}{a+1}.$

14. $x+5-\dfrac{2x-15}{x-3}$
$= \dfrac{x^2+2x-15-2x+15}{x-3}$
$= \dfrac{x^2}{x-3}.$

15. $2a-b-\dfrac{2ab}{a+b}$
$= \dfrac{2a^2+ab-b^2-2ab}{a+b}$
$= \dfrac{2a^2-ab-b^2}{a+b}.$

16. $3x-10+\dfrac{41}{x+4}$
$= \dfrac{3x^2+2x-40+41}{x+4}$
$= \dfrac{3x^2+2x+1}{x+4}.$

17. $x^2+x+1+\dfrac{2}{x-1}$
$= \dfrac{x^3-1+2}{x-1}$
$= \dfrac{x^3+1}{x-1}.$

18. $x^3-3x-\dfrac{3x(3-x)}{x-2}$
$= \dfrac{x^4-2x^3-3x^2+6x-9x+3x^2}{x-2}$
$= \dfrac{x^4-2x^3-3x}{x-2}$
$= \dfrac{x(x^3-2x^2-3)}{x-2}.$

19. $a^2-2ax+4x^2-\dfrac{6x^3}{a+2x}$
$= \dfrac{a^3+8x^3-6x^3}{a+2x}$
$= \dfrac{a^3+2x^3}{a+2x}.$

20. $x - a + y + \dfrac{a^2 - ay + y^2}{x + a}$

$$= \frac{x^2 - a^2 + xy + ay + a^2 - ay + y^2}{x + a}$$

$$= \frac{x^2 + xy + y^2}{x + a}.$$

Exercise LV.

1. $\dfrac{3x-7}{6}$, $\dfrac{4x-9}{18}$.

L.C.D. $= 18$.

The multipliers are 3 and 1 respectively.

$$\frac{3x-7}{6} = \frac{9x-21}{18};$$

$$\frac{4x-9}{18} = \frac{4x-9}{18}.$$

2 $\dfrac{2x-4y}{5x^2}$, $\dfrac{3x-8y}{10x}$.

L.C.D. $= 10x^2$.

The multipliers are 2 and x respectively.

$$\frac{2x-4y}{5x^2} = \frac{4x-8y}{10x^2};$$

$$\frac{3x-8y}{10x} = \frac{3x^2-8xy}{10x^2}.$$

3. $\dfrac{4a-5c}{5ac}$, $\dfrac{3a-2c}{12a^2c}$.

L.C.D. $= 60a^2c$.

The multipliers are $12a$ and 5 respectively.

$$\frac{4a-5c}{5ac} = \frac{48a^2-60ac}{60a^2c};$$

$$\frac{3a-2c}{12a^2c} = \frac{15a-10c}{60a^2c}.$$

4. $\dfrac{5}{1-x}$, $\dfrac{6}{1-x^2}$.

L.C.D. $= 1 - x^2$.

The multipliers are $1+x$ and 1

$$\frac{5}{1-x} = \frac{5+5x}{1-x^2};$$

$$\frac{6}{1-x^2} = \frac{6}{1-x^2}.$$

5. $\dfrac{1}{(a-b)(b-c)}$, $\dfrac{1}{(a-b)(a-c)}$.

L.C.D. $= (a-b)(a-c)(b-c)$.

The multipliers are $a - c$ and $b - c$.

$$\frac{1}{(a-b)(b-c)} = \frac{a-c}{(a-b)(a-c)(b-c)};$$

$$\frac{1}{(a-b)(a-c)} = \frac{b-c}{(a-b)(a-c)(b-c)}.$$

6. $\dfrac{4x^2}{3(a+b)}, \dfrac{xy}{6(a^2-b^2)}.$

L. C. D. $= 6(a^2-b^2)$.

The multipliers are $2(a-b)$ and 1.

$$\frac{4x^2}{3(a+b)} = \frac{8x^2(a-b)}{6(a^2-b^2)};$$
$$\frac{xy}{6(a^2-b^2)} = \frac{xy}{6(a^2-b^2)}.$$

7. $\dfrac{8x+2}{x-2}, \dfrac{2x-1}{3x-6}, \dfrac{3x+2}{5x-10}.$

L. C. D. $= 15(x-2)$.

The multipliers are 15, 5 and 3.

$$\frac{8x+2}{x-2} = \frac{30(4x+1)}{15(x-2)};$$
$$\frac{2x-1}{3x-6} = \frac{5(2x-1)}{15(x-2)};$$
$$\frac{3x+2}{5x-10} = \frac{3(3x+2)}{15(x-2)}.$$

8. $\dfrac{a-bm}{mx}, 1, \dfrac{c-bn}{nx}.$

L. C. D. $= mnx$.

The multipliers are n, mnx, and m.

$$\frac{a-bm}{mx} = \frac{an-bmn}{mnx};$$
$$1 = \frac{mnx}{mnx};$$
$$\frac{c-bn}{nx} = \frac{cm-bmn}{mnx}.$$

EXERCISE LVI.

1. $\dfrac{3x-2y}{5x} + \dfrac{5x-7y}{10x} + \dfrac{8x+2y}{25}.$

L. C. D. $= 50x$.

The multipliers are 10, 5, and $2x$.

$$\begin{array}{lll} 30x & -20y & = \text{first numerator,} \\ 25x & -35y & = \text{second numerator,} \\ 16x^2 \quad +4xy & & = \text{third numerator.} \\ \hline 16x^2+55x+4xy & -55y & = \text{sum of numerators.} \end{array}$$

$$\therefore \text{ Sum of fractions} = \frac{16x^2+55x+4xy-55y}{50x}.$$

2. $\frac{4x^2-7y^2}{3x^2}+\frac{3x-8y}{6x}+\frac{5-2y}{12}.$

L. C. D. $=12x^2$.

The multipliers are 4, $2x$, and x^2.

$$\begin{array}{ll}
16x^2 \qquad\qquad\qquad\quad -28y^2 & = \text{first numerator,} \\
6x^2 \qquad -16xy & = \text{second numerator,} \\
\underline{5x^3-2x^2y \qquad\qquad\qquad\quad} & = \text{third numerator.} \\
27x^2-2x^2y-16xy-28y^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{27x^2-2x^2y-16xy-28y^2}{12x^2}.$

3. $\frac{4a^2+5b^2}{2b^2}+\frac{3a+2b}{5b}+\frac{7-2a}{9}.$

L.C.D. $=90b^2$.

The multipliers are 45, $18b$, and $10b^2$.

$$\begin{array}{ll}
180a^2+225b^2 & = \text{first numerator,} \\
\qquad 36b^2+54ab & = \text{second numerator,} \\
\underline{\qquad 70b^2 \qquad\quad -20ab^2} & = \text{third numerator.} \\
180a^2+331b^2+54ab-20ab^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{180a^2+54ab-20ab^2+331b^2}{90b^2}.$

4. $\frac{4x+5}{3}-\frac{3x-7}{5x}+\frac{9}{12x^2}.$

L. C. D. $=60x^2$.

The multipliers are $20x^2$, $12x$, and 5.

$$\begin{array}{ll}
80x^3+100x^2 & = \text{first numerator,} \\
\qquad -\ 36x^2+84x & = \text{second numerator,} \\
\underline{\qquad\qquad\qquad\qquad\quad 45} & = \text{third numerator.} \\
80x^3+\ 64x^2+84x+45 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{80x^3+64x^2+84x+45}{60x^2}.$

5. $\frac{4x-3y}{7}+\frac{3x+7y}{14}-\frac{5x-2y}{21}+\frac{9x+2y}{42}.$

L.C.D. $=42$.

The multipliers are 6, 3, 2, and 1.

$$\begin{array}{rl}
24x-18y & = \text{first numerator,} \\
9x+21y & = \text{second numerator,} \\
-10x+\ 4y & = \text{third numerator,} \\
\underline{9x+\ 2y} & = \text{fourth numerator.} \\
32x+\ 9y & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{32x+9y}{42}.$

6. $\dfrac{3xy-4}{x^2y^2}-\dfrac{5y^2+7}{xy^3}-\dfrac{6x^2-11}{x^3y}.$

L. C. D. $= x^3y^3$.

$$\begin{array}{rl}
3x^2y^2-4xy \qquad\qquad\qquad & = \text{first numerator,}\\
-5x^2y^2 \qquad\qquad\qquad\qquad -7x^2 & = \text{second numerator,}\\
\underline{-6x^2y^2 \qquad\qquad +11y^2 \qquad\qquad} & = \text{third numerator.}\\
-8x^2y^2-4xy+11y^2-7x^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{11y^2-4xy-8x^2y^2-7x^2}{x^3y^3}.$

7. $\dfrac{a^2-2ac+c^2}{a^2c^2}-\dfrac{b^2-2bc+c^2}{b^2c^2}.$

L. C. D. $= a^2b^2c^2$.

$$\begin{array}{rl}
a^2b^2-2ab^2c \qquad\qquad +b^2c^2 \qquad & = \text{first numerator,}\\
\underline{-a^2b^2 \qquad\qquad +2a^2bc \qquad -a^2c^2} & = \text{second numerator,}\\
-2ab^2c+2a^2bc+b^2c^2-a^2c^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{b^2c^2-2ab^2c+2a^2bc-a^2c^2}{a^2b^2c^2}.$

8. $\dfrac{5a^3-2}{8a^2}-\dfrac{3a^2-a}{8}.$

L. C. D. $= 8a^2$.

$$\begin{array}{rl}
5a^3 \qquad\qquad -2 & = \text{first numerator,}\\
\underline{a^3-3a^4 \qquad\quad} & = \text{second numerator.}\\
6a^3-3a^4-2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{6a^3-3a^4-2}{8a^2}.$

9. $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{ab^2+bc^2+ca^2}{abc}.$

L. C. D. $= abc$.

$$\begin{array}{rl}
a^2b \qquad\qquad\qquad -ab^2 \qquad\qquad\qquad & = \text{first numerator,}\\
b^2c \qquad\qquad\qquad -bc^2 \qquad\qquad & = \text{second numerator,}\\
ac^2 \qquad\qquad\qquad -a^2c & = \text{third numerator,}\\
\underline{ab^2+bc^2+a^2c} & = \text{fourth numerator.}\\
a^2b+b^2c+ac^2 \qquad\qquad\qquad\qquad & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{a^2b+b^2c+ac^2}{abc}$

10. $\frac{1}{2x^2y}-\frac{1}{6y^2z}-\frac{1}{2xz^2}+\frac{2x-z}{4x^2z^2}+\frac{y-2z}{4x^2yz}.$

L. C. D. $=12x^2y^2z^2$.

$$\begin{array}{rl}
6yz^2 \qquad\qquad\qquad\qquad & = \text{first numerator,}\\
-2x^2z \qquad\qquad & = \text{second numerator,}\\
-6xy^2 \qquad & = \text{third numerator,}\\
6xy^2-3y^2z & = \text{fourth numerator,}\\
\underline{-6yz^2 \qquad\qquad\qquad +3y^2z} & = \text{fifth numerator.}\\
-2x^2z \qquad\qquad & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=-\frac{2x^2z}{12x^2y^2z^2}=-\frac{1}{6y^2z}.$

Exercise LVII.

1. $\frac{1}{x-6}+\frac{1}{x+5}.$

L. C. D. $=(x-6)(x+5)$.

The multipliers are $x+5$ and $x-6$ respectively.

$$\begin{array}{rl}
x+5 & = \text{first numerator,}\\
\underline{x-6} & = \text{second numerator.}\\
2x-1 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{2x-1}{x^2-x-30}.$

2. $\frac{1}{x-7}-\frac{1}{x-3}.$

L. C. D. $=(x-7)(x-3)$.

The multipliers are $x-3$ and $x-7$ respectively.

$$\begin{array}{rl}
x-3 & = \text{first numerator,}\\
\underline{-x+7} & = \text{second numerator.}\\
4 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{4}{x^2-10x+21}.$

3. $\frac{1}{1+x}+\frac{1}{1-x}.$

L. C. D $=1-x^2$.

The multipliers are $1-x$ and $1+x$ respectively.

$$\begin{array}{rl}
1-x & = \text{first numerator,}\\
\underline{1+x} & = \text{second numerator.}\\
2 \quad & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{2}{1-x^2}.$

4. $\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.

L.C.D. $=1-x^2$.

The multipliers are $1+x$ and 1.

$$\begin{array}{ll} 1+x & = \text{first numerator,} \\ \underline{-2\quad\;\;} & = \text{second numerator.} \\ -1+x & = \text{sum of numerators.} \\ & = -(1-x). \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{-(1-x)}{1-x^2}=-\dfrac{1}{1+x}$.

5. $\dfrac{1}{x-y}+\dfrac{x}{(x-y)^2}$.

L.C.D. $=(x-y)^2$.

The multipliers are $x-y$ and 1.

$$\begin{array}{ll} x-y & = \text{first numerator,} \\ \underline{\;\;x\quad\;} & = \text{second numerator.} \\ 2x-y & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2x-y}{(x-y)^2}$.

6. $\dfrac{1}{2a(a+x)}+\dfrac{1}{2a(a-x)}$.

L.C.D. $=2a(a+x)(a-x)$.

The multipliers are $a-x$ and $a+x$.

$$\begin{array}{ll} a-x & = \text{first numerator,} \\ \underline{a+x} & = \text{second numerator.} \\ 2a & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2a}{2a(a+x)(a-x)}=\dfrac{1}{a^2-x^2}$.

7. $\dfrac{a}{(a+b)b}-\dfrac{b}{(a-b)a}$.

L.C.D. $=ab(a^2-b^2)$.

The multipliers are $a(a-b)$ and $b(a+b)$

$$\begin{array}{ll} a^3-a^2b & = \text{first numerator,} \\ \underline{\qquad\qquad -ab^2-b^3} & = \text{second numerator.} \\ a^3-a^2b-ab^2-b^3 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{a^3-a^2b-ab^2-b^3}{ab(a^2-b^2)}$.

8. $\dfrac{5}{2x(x-1)} - \dfrac{3}{4x(x-2)}$.

L. C. D. $= 4x(x-1)(x-2)$.

The multipliers are $2(x-2)$ and $(x-1)$.

$$\begin{array}{rl} 10x - 20 & = \text{first numerator,} \\ -3x + 3 & = \text{second numerator.} \\ \hline 7x - 17 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{7x-17}{4x(x^2-3x+2)}$.

9. $\dfrac{1+x}{1+x+x^2} - \dfrac{1-x}{1-x+x^2}$.

L. C. D. $= 1 + x^2 + x^4$.

The multipliers are $1-x+x^2$ and $1+x+x^2$.

$$\begin{array}{rl} 1 + x^3 & = \text{first numerator,} \\ -1 + x^3 & = \text{second numerator.} \\ \hline 2x^3 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{2x^3}{1+x^2+x^4}$.

10. $\dfrac{2ax-3by}{2xy(x-y)} - \dfrac{2ax+3by}{2xy(x+y)}$.

L. C. D. $= 2xy(x^2-y^2)$.

The multipliers are $x+y$ and $x-y$.

$$\begin{array}{rl} 2ax^2 + 2axy - 3bxy - 3by^2 & = \text{first numerator,} \\ -2ax^2 + 2axy - 3bxy + 3by^2 & = \text{second numerator.} \\ \hline 4axy - 6bxy & = \text{sum of numerators.} \\ \text{or } 2xy(2a-3b) & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{2a-3b}{x^2-y^2}$.

Exercise LVIII.

1. $\dfrac{1}{1+a} + \dfrac{1}{1-a} + \dfrac{2a}{1-a^2}$.

L. C. D. $= 1 - a^2$.

The multipliers are $1-a$, $1+a$, and 1.

$$\begin{array}{rl} 1 - a & = \text{first numerator,} \\ 1 + a & = \text{second numerator,} \\ 2a & = \text{third numerator.} \\ \hline 2 + 2a & = 2(1+a) = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{2(1+a)}{(1+a)(1-a)} = \dfrac{2}{1-a}$.

2. $\dfrac{1}{1-x}-\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}.$

L. C. D. $=(1-x)(1+x)(1+x^2).$

$$\begin{array}{rl}
1+x+x^2+\ x^3 & =\text{first numerator,}\\
-1+x-x^2+\ x^3 & =\text{second numerator,}\\
\underline{\quad 2x \qquad -2x^3} & =\text{third numerator.}\\
4x \qquad\qquad & =\text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{4x}{1-x^4}.$

3. $\dfrac{x}{1-x}-\dfrac{x^2}{1-x}+\dfrac{x}{1+x^2}.$

L. C. D. $=(1-x)(1+x^2).$

$$\begin{array}{rl}
x \qquad +x^3 \qquad & =\text{first numerator,}\\
-x^2 \qquad -x^4 & =\text{second numerator,}\\
\underline{x \quad -x^2 \qquad\qquad} & =\text{third numerator.}\\
2x-2x^2+x^3-x^4 & =\text{sum of numerators.}\\
 & =2x(1-x)+x^3(1-x).
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{(2x+x^3)(1-x)}{(1+x^2)(1-x)}=\dfrac{2x+x^3}{1+x^2}.$

4. $\dfrac{x}{y}+\dfrac{y}{x+y}+\dfrac{x^2}{x^2+xy}.$

L. C. D. $=xy(x+y).$

$$\begin{array}{rl}
x^3 \quad +x^2y \qquad & =\text{first numerator,}\\
+xy^2 & =\text{second numerator,}\\
\underline{\quad +x^2y \qquad\quad} & =\text{third numerator.}\\
x^3+2x^2y+xy^2 & =\text{sum of numerators.}\\
 & =x(x+y)^2.
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{x(x+y)^2}{xy(x+y)}=\dfrac{x+y}{y}.$

5. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}.$

L. C. D. $=(x-2)(x-3)(x-4).$

$$\begin{array}{rl}
x^3-\ 8x^2+19x-12 & =\text{first numerator,}\\
x^3-\ 8x^2+20x-16 & =\text{second numerator,}\\
\underline{x^3-\ 8x^2+21x-18} & =\text{third numerator.}\\
3x^3-24x^2+60x-46 & =\text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{3x^3-24x^2+60x-46}{x^3-9x^2+26x-24}.$

6. $\frac{3}{x-a}+\frac{4a}{(x-a)^2}-\frac{5a^2}{(x-a)^3}.$

L.C.D. $=(x-a)^3$.

$$\begin{array}{r}
3x^2-6ax+3a^2 = \text{first numerator,}\\
4ax-4a^2 = \text{second numerator,}\\
-5a^2 = \text{third numerator.}\\
\hline
3x^2-2ax-6a^2 = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{3x^2-2ax-6a^2}{(x-a)^3}.$

7. $\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+1)(x+2)}.$

L.C.D. $=(x-1)(x+1)(x+2)$.

$$\begin{array}{r}
x^2+3x+2 = \text{first numerator,}\\
-x^2 \qquad +1 = \text{second numerator,}\\
-3x+3 = \text{third numerator.}\\
\hline
6 = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\frac{6}{(x^2-1)(x+2)}.$

8. $\frac{a-b}{(b+c)(c+a)}+\frac{b-c}{(c+a)(a+b)}+\frac{c-a}{(a+b)(b+c)}.$

L.C.D. $=(b+c)(a+b)(c+a)$.

$$\begin{array}{r}
a^2-b^2 \qquad = \text{first numerator,}\\
+b^2-c^2 = \text{second numerator,}\\
-a^2 \qquad +c^2 = \text{third numerator.}\\
\hline
0 = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=0$.

9. $\frac{x-a}{x-b}+\frac{x-b}{x-a}-\frac{(a-b)^2}{(x-a)(x-b)}.$

L.C.D. $=(x-a)(x-b)$.

$$\begin{array}{lr}
x^2-2ax+a^2 & = \text{first numerator,}\\
x^2-2bx+b^2 & = \text{second numerator,}\\
-a^2+2ab-b^2 & = \text{third numerator.}\\
\hline
2x^2-2bx+2ab-2ax & = \text{sum of numerators.}\\
 & =2(x-a)(x-b).
\end{array}$$

$\therefore$ Sum of fractions $=\frac{2(x-a)(x-b)}{(x-a)(x-b)}=2.$

10. $\dfrac{x+y}{y} - \dfrac{2x}{x+y} + \dfrac{x^2y - x^3}{y(x^2-y^2)}$.

L.C.D. $= y(x^2 - y^2)$.

$$\begin{array}{rl} x^3 - xy^2 + x^2y - y^3 &= \text{first numerator,} \\ 2xy^2 - 2x^2y &= \text{second numerator,} \\ -x^3 + x^2y &= \text{third numerator.} \\ \hline xy^2 - y^3 &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{y^2(x-y)}{y(x^2-y^2)} = \dfrac{y}{x+y}$.

11. $\dfrac{a+b}{(b-c)(c-a)} + \dfrac{b+c}{(c-a)(a-b)} + \dfrac{c+a}{(a-b)(b-c)}$.

L.C.D. $= (b-c)(c-a)(a-b)$.

$$\begin{array}{rl} a^2 - b^2 &= \text{first numerator,} \\ + b^2 - c^2 &= \text{second numerator,} \\ -a^2 + c^2 &= \text{third numerator.} \\ \hline 0 &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= 0$.

12. $\dfrac{a^2 - bc}{(a+b)(a+c)} + \dfrac{b^2 - ac}{(b+a)(b+c)} + \dfrac{c^2 + ab}{(c+b)(c+a)}$.

L.C.D. $= (a+b)(b+c)(a+c)$.

$$\begin{array}{rl} a^2b - b^2c + a^2c - bc^2 &= \text{first numerator,} \\ ab^2 + b^2c - a^2c - ac^2 &= \text{second numerator,} \\ a^2b + ac^2 + bc^2 + ab^2 &= \text{third numerator.} \\ \hline 2a^2b + 2ab^2 &= \text{sum of numerators.} \\ &= 2ab(a+b). \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{2ab(a+b)}{(a+b)(b+c)(a+c)} = \dfrac{2ab}{(b+c)(a+c)}$.

13. $\dfrac{a}{a-x} - \dfrac{x}{a+2x} - \dfrac{a^2 + x^2}{(a-x)(a+2x)}$.

L.C.D. $= (a-x)(a+2x)$.

$$\begin{array}{rl} a^2 + 2ax &= \text{first numerator,} \\ - ax + x^2 &= \text{second numerator,} \\ -a^2 - x^2 &= \text{third numerator.} \\ \hline ax &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= \dfrac{ax}{(a-x)(a+2x)}$.

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(a-b)(a-c)}+\dfrac{6}{(a-c)(b-c)}.$

L.C.D. $=(a-b)(a-c)(b-c)$.

$$\begin{array}{rl} 3a \qquad\quad -3c &= \text{first numerator,} \\ -4b+4c &= \text{second numerator,} \\ \underline{6a \quad -6b \qquad\quad} &= \text{third numerator.} \\ 9a-10b \quad +c &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{9a-10b+c}{(a-b)(a-c)(b-c)}.$

15. $\dfrac{x-2y}{x(x-y)}-\dfrac{2x+y}{y(x+y)}-\dfrac{2x}{x^2-y^2}.$

L.C.D. $=xy(x^2-y^2)$.

$$\begin{array}{rl} x^2y-xy^2-2y^3 &= \text{first numerator,} \\ -2x^3+x^2y+xy^2 \qquad &= \text{second numerator,} \\ \underline{-2x^2y \qquad\qquad} &= \text{third numerator.} \\ -2x^3 \qquad\qquad -2y^3 &= \text{sum of numerators.} \\ &= -2(x+y)(x^2-xy+y^2). \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{-2(x+y)(x^2-xy+y^2)}{xy(x+y)(x-y)}$

$=-\dfrac{2(x^2-xy+y^2)}{xy(x-y)}.$

16. $\dfrac{a-b}{x(a+b)}-\dfrac{a-b}{y(a+b)}-\dfrac{(a-b)(x+y)}{xy(a+b)}.$

L.C.D. $=xy(a+b)$.

$$\begin{array}{rl} ay-by \qquad\qquad\quad &= \text{first numerator,} \\ -ax+bx &= \text{second numerator,} \\ \underline{-ay+by-ax+bx} &= \text{third numerator.} \\ -2ax+2bx &= \text{sum of numerators.} \\ &= 2x(b-a). \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2x(b-a)}{xy(a+b)}=\dfrac{2(b-a)}{y(a+b)}.$

17. $\dfrac{3x}{(x+y)^2}-\dfrac{x+2y}{x^2-y^2}+\dfrac{3y}{(x-y)^2}.$

L.C.D. $=(x+y)^2(x-y)^2$.

$$\begin{array}{rl} 3x^3-6x^2y+3xy^2 \qquad\quad &= \text{first numerator,} \\ -x^3-2x^2y+xy^2+2y^3 &= \text{second numerator,} \\ \underline{+3x^2y+6xy^2+3y^3} &= \text{third numerator.} \\ 2x^3-5x^2y+10xy^2+5y^3 &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2x^3-5x^2y+10xy^2+5y^3}{(x+y)^2(x-y)^2}.$

18. $\dfrac{a-c}{(a+b)^2-c^2}-\dfrac{a-b}{(a+c)^2-b^2}$.

L.C.D. $=(a+b+c)(a+b-c)(a-b+c)$.

$$\begin{array}{ll} a^2 \qquad\qquad\qquad -c^2 & = \text{first numerator,} \\ \underline{-a^2+ac+b^2-ab \qquad\;\;} & = \text{second numerator.} \\ \qquad ac+b^2-ab-c^2 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{ac-ab+b^2-c^2}{(a+b+c)(a+b-c)(a+c-b)}$.

19. $\dfrac{a+b}{ax+by}-\dfrac{a-b}{ax-by}+\dfrac{ab(x-y)}{a^2x^2-b^2y^2}$.

L.C.D. $=(ax+by)(ax-by)$.

$$\begin{array}{ll} a^2x+abx-aby-b^2y & = \text{first numerator,} \\ -a^2x+abx-aby+b^2y & = \text{second numerator,} \\ \underline{\qquad +abx-aby \qquad\qquad} & = \text{third numerator.} \\ \quad +3abx-3aby & = \text{sum of numerators;} \end{array}$$

or, $3ab(x-y)=$ sum of numerators.

$\therefore$ Sum of fractions $=\dfrac{3ab(x-y)}{a^2x^2-b^2y^2}$.

Exercise LIX.

1. $\dfrac{x}{x-y}+\dfrac{x-y}{y-x}=\dfrac{x}{x-y}-\dfrac{x-y}{x-y}$.

L.C.D. $=x-y$.

$$\begin{array}{ll} \quad x & = \text{first numerator,} \\ \underline{y-x} & = \text{second numerator.} \\ y & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{y}{x-y}$.

2. $\dfrac{3+2x}{2-x}+\dfrac{3x-2}{2+x}+\dfrac{16x-x^2}{x^2-4}$

$=\dfrac{3+2x}{2-x}+\dfrac{3x-2}{2+x}-\dfrac{16x-x^2}{4-x^2}$.

L.C.D. $=4-x^2$.

$$\begin{array}{ll} 6+\;\;7x+2x^2 & = \text{first numerator,} \\ -4+\;\;8x-3x^2 & = \text{second numerator,} \\ \underline{\quad -16x+\;\;x^2} & = \text{third numerator.} \\ 2 \quad -x & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2-x}{4-x^2}=\dfrac{1}{2+x}$.

3. $\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}-\dfrac{x}{1-x}=\dfrac{x^2}{x^2-1}+\dfrac{x}{x+1}+\dfrac{x}{x-1}.$

L. C. D. $= x^2-1.$

$$\begin{array}{ll} x^2 & = \text{first numerator,} \\ x^2-x & = \text{second numerator,} \\ \underline{x^2+x} & = \text{third numerator.} \\ 3x^2 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{3x^2}{x^2-1}.$

4. $\dfrac{4}{3-3y^2}+\dfrac{1}{2-2y}+\dfrac{1}{6y+6}$

$=\dfrac{4}{3(1+y)(1-y)}+\dfrac{1}{2(1-y)}+\dfrac{1}{6(1+y)}.$

L. C. D. $=6(1+y)(1-y).$

$$\begin{array}{rl} 8 & = \text{first numerator,} \\ 3y+\ 3 & = \text{second numerator,} \\ \underline{-y+\ 1} & = \text{third numerator.} \\ 2y+12 & = \text{sum of numerators;} \end{array}$$

or, $2(6+y)=$ sum of numerators.

$\therefore$ Sum of fractions $=\dfrac{6+y}{3(1-y^2)}.$

5. $\dfrac{1}{(2-m)(3-m)}-\dfrac{2}{(m-1)(m-3)}+\dfrac{1}{(m-1)(m-2)}$

$=\dfrac{1}{(2-m)(3-m)}-\dfrac{2}{(1-m)(3-m)}+\dfrac{1}{(1-m)(2-m)}.$

L. C. D. $=(1-m)(2-m)(3-m).$

$$\begin{array}{rl} 1-\ \ m & = \text{first numerator,} \\ -4+2m & = \text{second numerator,} \\ \underline{3-\ \ m} & = \text{third numerator.} \\ 0 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=0.$

6. $\dfrac{1}{(b-a)(x+a)}+\dfrac{1}{(a-b)(x+b)}$

$=\dfrac{1}{(b-a)(x+a)}-\dfrac{1}{(b-a)(x+b)}.$

L. C. D. $=(b-a)(x+a)(x+b).$

$$\begin{array}{rl} x+b\qquad & = \text{first numerator,} \\ \underline{-x\qquad -a} & = \text{second numerator.} \\ b-a & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{b-a}{(b-a)(x+a)(x+b)}=\dfrac{1}{(x+a)(x+b)}.$

7. $\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{2ab^2}{b^3-a^3}+\dfrac{2a^2b}{a^3+b^3}=\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{2ab^2}{a^3-b^3}+\dfrac{2a^2b}{a^3+b^3}.$

L.C.D. $=(a^3-b^3)(a^3+b^3)$.

$$\begin{array}{llll}
a^6 & +2a^4b^2+2a^2b^4 & +b^6 & = \text{first numerator,}\\
 & -2a^4b^2 \quad -2ab^5 & & = \text{second numerator,}\\
+2a^5b & -2a^2b^4 & & = \text{third numerator.}\\
\hline
a^6+2a^5b & & -2ab^5+b^6 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{a^6+2a^5b-2ab^5+b^6}{(a^3-b^3)(a^3+b^3)}.$

8. $\dfrac{b-a}{x-b}-\dfrac{a-2b}{b+x}-\dfrac{3x(a-b)}{b^2-x^2}=\dfrac{b-a}{x-b}-\dfrac{a-2b}{x+b}+\dfrac{3x(a-b)}{x^2-b^2}.$

L.C.D. $=x^2-b^2$.

$$\begin{array}{rl}
-ab-ax+bx+b^2 & = \text{first numerator,}\\
ab-ax+2bx-2b^2 & = \text{second numerator,}\\
3ax-3bx \qquad & = \text{third numerator.}\\
\hline
ax \qquad\qquad -b^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{ax-b^2}{x^2-b^2}.$

9. $\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}+\dfrac{16x-x^2}{x^2-4}=\dfrac{3+2x}{2-x}-\dfrac{2-3x}{2+x}-\dfrac{16x-x^2}{4-x^2}$

L.C.D. $=4-x^2$.

$$\begin{array}{rl}
6+7x+2x^2 & = \text{first numerator,}\\
-4+8x-3x^2 & = \text{second numerator,}\\
-16x+x^2 & = \text{third numerator.}\\
\hline
2 \quad -x \qquad & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{2-x}{4-x^2}=\dfrac{1}{2+x}.$

10. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}=\dfrac{3}{1-2x}-\dfrac{7}{1+2x}+\dfrac{4-20x}{1-4x^2}.$

L.C.D. $=1-4x^2$.

$$\begin{array}{rl}
3+6x & = \text{first numerator,}\\
-7+14x & = \text{second numerator,}\\
4-20x & = \text{third numerator.}\\
\hline
0 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=0$.

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(b-a)(a-c)}+\dfrac{c+a}{(a-b)(b-c)}$

$=-\dfrac{a+b}{(b-c)(a-c)}-\dfrac{b+c}{(a-b)(a-c)}+\dfrac{a+c}{(a-b)(b-c)}.$

L. C. D. $=(a-b)(a-c)(b-c)$.

$$\begin{array}{lll} -a^2+b^2 & & = \text{first numerator,} \\ & -b^2+c^2 & = \text{second numerator,} \\ \underline{a^2} & \underline{\quad -c^2} & = \text{third numerator.} \\ & 0 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=0$.

12. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}$

$=\dfrac{a^2-bc}{(a-b)(a-c)}-\dfrac{ac+b^2}{(b+c)(a-b)}-\dfrac{ab+c^2}{(a-c)(b+c)}.$

L. C. D. $=(a-b)(a-c)(b+c)$.

$$\begin{array}{ll} a^2b-b^2c+a^2c-bc^2 & = \text{first numerator,} \\ \quad b^2c-a^2c \quad +ac^2-ab^2 & = \text{second numerator,} \\ \underline{-a^2b \qquad\quad +bc^2-ac^2+ab^2} & = \text{third numerator.} \\ \qquad\qquad\qquad\qquad\quad 0 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=0$.

13. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}$

$=\dfrac{y+z}{(x-y)(x-z)}-\dfrac{x+z}{(x-y)(y-z)}+\dfrac{x+y}{(x-z)(y-z)}.$

L. C. D. $=(x-y)(y-z)(x-z)$.

$$\begin{array}{lll} & y^2-z^2 & = \text{first numerator,} \\ -x^2 & \quad +z^2 & = \text{second numerator,} \\ \underline{x^2-y^2} & & = \text{third numerator.} \\ & 0 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=0$.

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(b-a)(c-a)}-\dfrac{6}{(a-c)(c-b)}$

$=\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(a-b)(a-c)}+\dfrac{6}{(a-c)(b-c)}.$

L. C. D. $=(a-b)(a-c)(b-c)$.

$$\begin{array}{lllll} 3a & & -3c & = \text{first numerator,} \\ & -\;4b & +4c & = \text{second numerator,} \\ \underline{6a} & \underline{-\;6b} & & = \text{third numerator.} \\ 9a & -10b & +\;c & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{9a-10b+c}{(a-b)(a-c)(b-c)}.$

15. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}-\dfrac{1}{xyz}$

$=\dfrac{1}{x(x-y)(x-z)}-\dfrac{1}{y(x-y)(y-z)}-\dfrac{1}{xyz}.$

L. C. D. $= xyz(x-y)(x-z)(y-z)$.

$$\begin{array}{ll} y^2z \qquad\qquad -yz^2 & = \text{first numerator,} \\ \qquad -x^2z+xz^2 & = \text{second numerator} \\ \underline{-x^2y+xy^2-y^2z+x^2z-xz^2+yz^2} & = \text{third numerator.} \\ -x^2y+xy^2 & = \text{sum of numerators;} \\ \text{or,} \qquad -xy(x-y) & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{-xy(x-y)}{xyz(x-y)(x-z)(y-z)}$

$=-\dfrac{1}{z(x-z)(y-z)}.$

Exercise LX.

1. $\dfrac{a}{bx}\times\dfrac{cx}{d}.$

Cancelling common factor x,

$=\dfrac{ac}{bd}.$

2. $\dfrac{2x}{a}\times\dfrac{3ab}{c}\times\dfrac{3ac}{2b}.$

Cancelling $2abc$,

$=9ax.$

3. $\dfrac{3p}{2p-2}\div\dfrac{2p}{p-1}$

$=\dfrac{3p}{2(p-1)}\times\dfrac{p-1}{2p}.$

Cancelling p and $p-1$,

$=\frac{3}{4}.$

4. $\dfrac{8x^4y}{15ab^3}\div\dfrac{2x^3}{3ab^2}$

$=\dfrac{8x^4y}{15ab^3}\times\dfrac{3ab^2}{2x^3}.$

Cancelling $2x^3$ and $3ab^2$,

$=\dfrac{4xy}{5b}.$

5. $\dfrac{8a^2b^3}{45x^2y}\times\dfrac{15xy^2}{24a^3b^2}.$

Cancelling 8, 15, a^2, b^2, x, and y,

$=\dfrac{by}{9ax}.$

6. $\dfrac{9x^2y^2z}{10a^2b^2c}\times-\dfrac{20a^3b^2c}{18xy^2z}.$

Cancelling $9xy^2z$, $10a^2b^2c$, and 2,

$=-ax.$

7. $\dfrac{3x^2y}{4xz^2}\times\dfrac{5y^2z}{6xy}\times-\dfrac{12x^2}{2xy^3}.$

Cancelling 2, 6, x^3, y^3, and z,

$=-\dfrac{15x}{4z}.$

8. $\dfrac{9m^2n^2}{8p^3q^3}\times\dfrac{5p^2q}{2xy}\times\dfrac{24x^2y^2}{90mn}.$

Cancelling 9, 5, 8, mn, p^2q, and xy,

$=\dfrac{3mnxy}{4pq^2}.$

9. $\dfrac{25k^3m^2}{14n^2q^2}\times\dfrac{70n^3q}{75p^2m}\times\dfrac{3pm}{4k^2n}.$

Cancelling $25k^2m$, $14n^3q$, and $3p$,

$=\dfrac{5km^2}{4pq}$

10. $\dfrac{a-b}{a^2+ab}\times\dfrac{a^2-b^2}{a^2-ab}$

$=\dfrac{a-b}{a(a+b)}\times\dfrac{(a+b)(a-b)}{a(a-b)}.$

Cancelling $a-b$ and $a+b$,

$=\dfrac{a-b}{a^2}.$

11. $\dfrac{a^2+b^2}{a^2-b^2}\div\dfrac{a-b}{a+b}$

$=\dfrac{a^2+b^2}{(a-b)(a+b)}\times\dfrac{a+b}{a-b}.$

Cancelling $a+b$,

$=\dfrac{a^2+b^2}{(a-b)^2}.$

12. $\dfrac{x^2+x-2}{x^2-7x}\times\dfrac{x^2-13x+42}{x^2+2x}$

$=\dfrac{(x+2)(x-1)}{x(x-7)}\times\dfrac{(x-6)(x-7)}{x(x+2)}.$

Cancelling $x-7$ and $x+2$

$=\dfrac{(x-1)(x-6)}{x^2}.$

13. $\dfrac{x^2-11x+30}{x^2-6x+9}\times\dfrac{x^2-3x}{x^2-5x}$

$=\dfrac{(x-5)(x-6)}{(x-3)(x-3)}\times\dfrac{x(x-3)}{x(x-5)}$

$=\dfrac{x-6}{x-3}.$

14. $\dfrac{a^3-x^3}{a^3+x^3}\times\dfrac{(a+x)^2}{(a-x)^2}$

$=\dfrac{(a-x)(a^2+ax+x^2)}{(a+x)(a^2-ax+x^2)}\times\dfrac{(a+x)^2}{(a-x)^2}$

$=\dfrac{(a+x)(a^2+ax+x^2)}{(a-x)(a^2-ax+x^2)}.$

15. $\dfrac{2a(x^2-y^2)^2}{cx}\times\dfrac{x^3}{(x-y)(x+y)^2}$

$=\dfrac{2a(x+y)^2(x-y)^2}{cx}\times\dfrac{x^3}{(x-y)(x+y)^2}$

$=\dfrac{2ax^2(x-y)}{c}.$

16. $\dfrac{a^2+2ab}{a^2+4b^2}\times\dfrac{ab-2b^2}{a^2-4b^2}$

$=\dfrac{a(a+2b)}{a^2+4b^2}\times\dfrac{b(a-2b)}{(a-2b)(a+2b)}$

$=\dfrac{ab}{a^2+4b^2}.$

17. $\dfrac{x^2-4}{x^2+5x}\times\dfrac{x^2-25}{x^2+2x}$

$=\dfrac{(x+2)(x-2)}{x(x+5)}\times\dfrac{(x+5)(x-5)}{x(x+2)}$

$=\dfrac{(x-2)(x-5)}{x^2}.$

18. $\dfrac{x^2+xy}{x-y}\times\dfrac{(x-y)^2}{x^4-y^4}$

$=\dfrac{x(x+y)}{x-y}\times\dfrac{(x-y)(x-y)}{(x^2+y^2)(x+y)(x-y)}$

$=\dfrac{x}{x^2+y^2}.$

19. $\dfrac{m^2-n^2}{c^3+d^3}\div\dfrac{n-m}{c+d}$

$=\dfrac{(m+n)(m-n)}{(c+d)(c^2-cd+d^2)}\times-\dfrac{c+d}{m-n}$

$=-\dfrac{m+n}{c^2-cd+d^2}.$

20. $\dfrac{a^2-4a+3}{a^2-5a+4}\times\dfrac{a^2-9a+20}{a^2-10a+21}\times\dfrac{a^2-7a}{a^2-5a}$

$=\dfrac{(a-3)(a-1)}{(a-4)(a-1)}\times\dfrac{(a-5)(a-4)}{(a-7)(a-3)}\times\dfrac{a(a-7)}{a(a-5)}$

$=1.$

21. $\dfrac{b^2-7b+6}{b^2+3b-4}\times\dfrac{b^2+10b+24}{b^2-14b+48}\div\dfrac{b^2+6b}{b^3-8b^2}$

$=\dfrac{(b-6)(b-1)}{(b+4)(b-1)}\times\dfrac{(b+6)(b+4)}{(b-8)(b-6)}\times\dfrac{b^2(b-8)}{b(b+6)}$

$=b.$

22. $\dfrac{x^2-y^2}{x^2-3xy+2y^2}\times\dfrac{xy-2y^2}{x^2+xy}\times\dfrac{x^2-xy}{(x-y)^2}$

$=\dfrac{(x+y)(x-y)}{(x-2y)(x-y)}\times\dfrac{y(x-2y)}{x(x+y)}\times\dfrac{x(x-y)}{(x-y)(x-y)}$

$=\dfrac{y}{x-y}.$

23. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \dfrac{2ab-2b^2}{3} \times \dfrac{a^2+ab}{a-b}$

$= \dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \times \dfrac{3}{2ab-2b^2} \times \dfrac{a^2+ab}{a-b}$

$= \dfrac{(a-b)(a-b)(a-b)}{(a+b)(a-b)} \times \dfrac{3}{2b(a-b)} \times \dfrac{a(a+b)}{a-b}$

$= \dfrac{3a}{2b}.$

24. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2} \div \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}$

$= \dfrac{(a+b+c)(a+b-c)(c-a+b)(c+a-b)}{(a-b+c)(a+b-c)(c-a-b)(c+a+b)}$

$= \dfrac{c-a+b}{c-a-b}.$

25. $\dfrac{(x-a)^2-b^2}{(x-b)^2-a^2} \times \dfrac{x^2-(b-a)^2}{x^2-(a-b)^2}$

$= \dfrac{(x-a+b)(x-a-b)(x+b-a)(x-b+a)}{(x-b+a)(x-b-a)(x+a-b)(x-a+b)}$

$= \dfrac{x-a+b}{x+a-b}.$

26. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \div \dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}$

$= \dfrac{(a+b+c+d)(a+b-c-d)(a-b+d-c)(a-b-d+c)}{(a+c+b+d)(a+c-b-d)(a-c+d-b)(a-c-d+b)}$

$= 1.$

27. $\dfrac{x^2-2xy+y^2-z^2}{x^2+2xy+y^2-z^2} \times \dfrac{x+y-z}{x-y+z}$

$= \dfrac{(x-y)^2-z^2}{(x+y)^2-z^2} \times \dfrac{x+y-z}{x-y+z}$

$= \dfrac{(x-y+z)(x-y-z)(x+y-z)}{(x+y+z)(x+y-z)(x-y+z)}$

$= \dfrac{x-y-z}{x+y+z}.$

Exercise LXI.

1. $$\frac{\frac{3x}{2}+\frac{x-1}{3}}{\frac{13}{6}(x+1)-\frac{x}{3}-2\frac{1}{2}}$$

Multiply both terms by 6,

$$=\frac{9x+2x-2}{13x+13-2x-15}$$

$$=\frac{11x-2}{11x-2}=1.$$

2. $$\frac{x-1+\frac{6}{x-6}}{x-2+\frac{3}{x-6}}.$$

Multiply both terms by $x-6$,

$$=\frac{x^2-7x+12}{x^2-8x+15}$$

$$=\frac{(x-4)(x-3)}{(x-5)(x-3)}$$

$$=\frac{x-4}{x-5}.$$

3. $$\frac{3}{x+1}-\frac{2x-1}{x^2+\frac{x}{2}-\frac{1}{2}}.$$

Multiply both terms of second fraction by 2,

$$=\frac{3}{x+1}-\frac{4x-2}{2x^2+x-1}$$

$$=\frac{3}{x+1}-\frac{2(2x-1)}{(2x-1)(x+1)}$$

$$=\frac{3}{x+1}-\frac{2}{x+1}=\frac{1}{x+1}.$$

4. $$\frac{x-a}{x-\frac{(x-b)(x-c)}{x+a}}$$

$$=\frac{x-a}{\frac{x^2+ax-x^2+bx+cx-bc}{x+a}}$$

$$=\frac{(x-a)(x+a)}{ax+bx+cx-bc}$$

5. $$\frac{\left(\frac{a}{x}-\frac{x}{a}\right)\left(\frac{a}{x}+\frac{x}{a}\right)}{1-\frac{x-a}{x+a}}$$

$$=\frac{\left(\frac{a^2-x^2}{ax}\right)\left(\frac{a^2+x^2}{ax}\right)}{\frac{2a}{a+x}}$$

$$=\frac{(a^2-x^2)(a^2+x^2)(a+x)}{2a^3x^2}.$$

6. $$\frac{\frac{1}{x-y}-\frac{x}{x^2-y^2}}{\frac{x}{xy+y^2}-\frac{y}{x^2+xy}}$$

$$=\frac{\frac{x+y-x}{x^2-y^2}}{\frac{x^2-y^2}{xy(x+y)}}$$

$$=\frac{y}{x^2-y^2}\times\frac{xy(x+y)}{x^2-y^2}$$

$$=\frac{xy^2}{(x+y)(x-y)^2}.$$

7. $$\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$$

$$=\frac{\dfrac{(x+1)^2}{x^2-1}+\dfrac{(x-1)^2}{x^2-1}}{\dfrac{(x+1)^2}{x^2-1}-\dfrac{(x-1)^2}{x^2-1}}$$

$$=\frac{(x^2+2x+1)+(x^2-2x+1)}{(x^2+2x+1)-(x^2-2x+1)}$$

$$=\frac{2x^2+2}{4x}=\frac{x^2+1}{2x}.$$

8. $$1-\frac{1}{1+\dfrac{1}{x}}$$

$$=1-\frac{x}{x+1}$$

$$=\frac{x+1-x}{x+1}$$

$$=\frac{1}{x+1}.$$

9. $$1+\frac{x}{1+x+\dfrac{2x^2}{1-x}}$$

$$=1+\frac{x(1-x)}{(1+x)(1-x)+2x^2}$$

$$=1+\frac{x(1-x)}{1+x^2}$$

$$=\frac{1+x^2+x-x^2}{1+x^2}$$

$$=\frac{1+x}{1+x^2}.$$

10. $$\frac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$$

$$=\frac{1}{1-\dfrac{x}{x+1}}$$

$$=\frac{x+1}{x+1-x}$$

$$=\frac{x+1}{1}$$

$$=x+1.$$

11. $$\frac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$$

$$=\frac{1}{1+\dfrac{x}{\dfrac{1+x^2}{1-x}}}$$

$$=\frac{1}{1+\dfrac{x-x^2}{1+x^2}}$$

$$=\frac{1+x^2}{1+x}.$$

12. $$\frac{\left(\dfrac{a}{x}+\dfrac{x}{a}-2\right)\left(\dfrac{a}{x}+\dfrac{x}{a}+2\right)}{\left(\dfrac{a}{x}-\dfrac{x}{a}\right)^2}$$

$$=\frac{\left(\dfrac{a^2-2ax+x^2}{ax}\right)\left(\dfrac{a^2+2ax+x^2}{ax}\right)}{\left(\dfrac{a^2-x^2}{ax}\right)\left(\dfrac{a^2-x^2}{ax}\right)}$$

$$=\frac{(a-x)(a-x)(a+x)(a+x)}{(a+x)(a-x)(a+x)(a-x)}$$

$$=1.$$

13. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2x}{x+y}\left\{\dfrac{xy-x^2}{(x-y)^2}+\dfrac{x+y}{x-y}\right\}}{x-y}$$

$$=\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2x}{x+y}\left\{-\dfrac{x}{x-y}+\dfrac{x+y}{x-y}\right\}}{x-y}$$

$$=\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2x}{x+y}\left\{\dfrac{y}{x-y}\right\}}{x-y}$$

$$=\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2xy}{x^2-y^2}}{x-y}$$

$$=\frac{\dfrac{x^2+2xy+y^2}{x^2-y^2}}{x-y}$$

$$=\frac{(x+y)(x+y)}{(x-y)(x+y)}\times\frac{1}{x-y}=\frac{x+y}{x^2-2xy+y^2}.$$

14. $$\frac{\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2}}{\dfrac{xy}{x+y}}$$

$$=\frac{(x+y)(x-y)2x(x-y)(x+y)}{4(x-y)(x-y)xy}$$

$$=\frac{(x+y)^2}{2y}.$$

15. $$\frac{\dfrac{ab}{x^2+(a+b)x+ab}-\dfrac{ac}{x^2+(a+c)x+ac}}{\dfrac{b-c}{x^2+(b+c)x+bc}}$$

$$=\frac{\dfrac{(abx+abc)-(acx+abc)}{(x+a)(x+b)(x+c)}}{\dfrac{(b-c)}{(x+b)(x+c)}}$$

$$=\frac{ax(b-c)(x+b)(x+c)}{(x+a)(x+b)(x+c)(b-c)}=\frac{ax}{x+a}.$$

16. $$\frac{x}{1+\frac{1}{x}}+1-\frac{1}{x+1}$$

$$=\frac{x^2}{x+1}+1-\frac{1}{x+1}$$

$$=\frac{x^2+x}{x+1}$$

$$=x.$$

17. $$\frac{\frac{a+b}{b}+\frac{b}{a+b}}{\frac{1}{a}+\frac{1}{b}}$$

$$=\frac{\frac{(a+b)^2+b^2}{b(a+b)}}{\frac{a+b}{ab}}$$

$$=\frac{(a+b)^2+b^2}{b(a+b)}\times\frac{ab}{a+b}$$

$$=\frac{\{(a+b)^2+b^2\}a}{(a+b)^2}$$

$$=\frac{(a^2+2ab+2b^2)a}{(a+b)^2}$$

18. $$\frac{2m-3+\frac{1}{m}}{\frac{2m-1}{m}}$$

$$=\frac{\frac{2m^2-3m+1}{m}}{\frac{2m-1}{m}}$$

$$=\frac{(2m-1)(m-1)}{2m-1}$$

$$=m-1.$$

19. $$\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}{\frac{a^2-(b+c)^2}{ab}}$$

Multiply the terms of the numerator by abc, and factor the denominator,

$$=\frac{\frac{c+b+a}{abc}}{\frac{(a+b+c)(a-b-c)}{ab}}$$

$$=\frac{c+b+a}{abc}\times\frac{ab}{(a+b+c)(a-b-c)}$$

$$=\frac{1}{c(a-b-c)}.$$

20. $$\frac{3}{1+\frac{3}{1+\frac{3}{1-x}}}$$

$$=\frac{3}{1+\frac{3(1-x)}{1-x+3}}$$

$$=\frac{3}{\frac{7-4x}{4-x}}$$

$$=\frac{3(4-x)}{7-4x}.$$

Exercise LXII.

1. $\dfrac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$

$=\dfrac{(x-8)(x^3-x^2-x+1)}{(x+8)(x^3-x^2-x+1)}$

$=\dfrac{x-8}{x+8}.$

2. $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$

$=\dfrac{16+\frac{1}{4}-1+4}{16-\frac{1}{4}-1+1}$

$=\dfrac{19\frac{1}{4}}{15\frac{3}{4}}.$

Multiply both terms by 4,

$=\frac{77}{63}=1\frac{2}{9}.$

3. $3a^2+\dfrac{2ab^2}{c}-\dfrac{c^3}{b^2}$

$=3\times4\times4+\dfrac{2\times4\times\frac{1}{2}\times\frac{1}{2}}{1}-\dfrac{1}{\frac{1}{4}}$

$=48+2-4=46.$

4. $\dfrac{2}{(x^2-1)^2}-\dfrac{1}{2x^2-4x+2}-\dfrac{1}{1-x^2}$

$=\dfrac{2}{(x^2-1)^2}-\dfrac{1}{2(x-1)^2}+\dfrac{1}{x^2-1}.$

L. C. D. $=2(x^2-1)^2$.

$$\begin{array}{rl}
4 & = \text{first numerator,}\\
-x^2-2x-1 & = \text{second numerator,}\\
\underline{2x^2\qquad\ -2} & = \text{third numerator.}\\
x^2-2x+1 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $=\dfrac{x^2-2x+1}{2(x^2-2x+1)(x+1)^2}=\dfrac{1}{2(x+1)^2}.$

5. $\left(\dfrac{x}{1+\frac{1}{x}}+1-\dfrac{1}{x+1}\right)\div\left(\dfrac{x}{1-\frac{1}{x}}-x-\dfrac{1}{x-1}\right)$

$=\left(\dfrac{x^2}{x+1}+1-\dfrac{1}{x+1}\right)\div\left(\dfrac{x^2}{x-1}-x-\dfrac{1}{x-1}\right)$

$=\left(\dfrac{x^2}{x+1}+\dfrac{x+1}{x+1}-\dfrac{1}{x+1}\right)\div\left(\dfrac{x^2}{x-1}-\dfrac{x^2-x}{x-1}-\dfrac{1}{x-1}\right)$

$=\dfrac{x(x+1)}{x+1}\times\dfrac{x-1}{x-1}$

$=x.$

6. $\left(\frac{x-a}{x-b}\right)^3 - \left(\frac{x-2a+b}{x+a-2b}\right)$

$$=\left(\frac{\frac{a+b}{2}-a}{\frac{a+b}{2}-b}\right)^3-\left(\frac{\frac{a+b}{2}-2a+b}{\frac{a+b}{2}+a-2b}\right)$$

$$=\left(\frac{a+b-2a}{a+b-2b}\right)^3-\left(\frac{a+b-4a+2b}{a+b+2a-4b}\right)$$

$$=\left(\frac{b-a}{a-b}\right)^3-\left(\frac{3b-3a}{3a-3b}\right)$$

$$=(-1)^3-(-1)=0.$$

7. $\left(\frac{a+b}{2(a-b)}-\frac{a-b}{2(a+b)}+\frac{2b^2}{a^2-b^2}\right)\frac{a-b}{2b}.$

L. C. D. of fractions in brackets $= 2(a^2-b^2)$.

$$\begin{aligned} a^2+2ab+\;\; b^2 &= \text{first numerator,}\\ -a^2+2ab-\;\; b^2 &= \text{second numerator,}\\ 4b^2 &= \text{third numerator.}\\ \hline 4ab+4b^2 &= \text{sum of numerators ;}\end{aligned}$$

or, $\quad 4b(a+b) =$ sum of numerators.

$\therefore$ Sum of fractions in brackets $=\frac{4b(a+b)}{2(a^2-b^2)}=\frac{2b}{a-b}.$

$$\frac{2b}{a-b}\times\frac{a-b}{2b}=1.$$

8. $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$

$$=\left(\frac{x^4+2x^2y^2+y^4}{x^4-y^4}-\frac{x^4-2x^2y^2+y^4}{x^4-y^4}\right)\div\left(\frac{x^2+2xy+y^2}{x^2-y^2}-\frac{x^2-2xy+y^2}{x^2-y^2}\right)$$

$$=\frac{4x^2y^2}{x^4-y^4}\div\frac{4xy}{x^2-y^2}$$

$$=\frac{4x^2y^2}{x^4-y^4}\times\frac{x^2-y^2}{4xy}$$

$$=\frac{xy}{x^2+y^2}.$$

9. $\left(\dfrac{x^2}{y^2}-1\right)\left(\dfrac{x}{x-y}-1\right)+\left(\dfrac{x^3}{y^3}-1\right)\left(\dfrac{x^2+xy}{x^2+xy+y^2}-1\right)$

$$=\left(\frac{x^2-y^2}{y^2}\right)\left(\frac{y}{x-y}\right)+\left(\frac{x^3-y^3}{y^3}\right)\left(\frac{-y^2}{x^2+xy+y^2}\right)$$

$$=\frac{x^2-y^2}{y^2}\times\frac{y}{x-y}+\frac{x^3-y^3}{y^3}\times\frac{-y^2}{x^2+xy+y^2}$$

$$=\frac{x+y}{y}-\frac{x-y}{y}$$

$$=2.$$

10. $\left(\dfrac{a^2-ab}{a^3-b^3}\right)\left(\dfrac{a^2+ab+b^2}{a+b}\right)+\left(\dfrac{2a^3}{a^3+b^3}-1\right)\left(1-\dfrac{2ab}{a^2+ab+b^2}\right)$

$$=\left(\frac{a(a-b)}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right)+\left(\frac{a^3-b^3}{a^3+b^3}\right)\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)$$

$$=\frac{a}{a+b}+\frac{a-b}{a+b}$$

$$=\frac{2a-b}{a+b}.$$

11. $\dfrac{1+\dfrac{a-x}{a+x}}{1-\dfrac{a-x}{a+x}}\div\dfrac{1+\dfrac{a^2-x^2}{a^2+x^2}}{1-\dfrac{a^2-x^2}{a^2+x^2}}.$

Multiply both terms of first fraction by $a+x$, and both terms of the second by a^2+x^2,

$$=\frac{a+x+a-x}{a+x-a+x}\div\frac{a^2+x^2+a^2-x^2}{a^2+x^2-a^2+x^2}$$

$$=\frac{2a}{2x}\times\frac{2x^2}{2a^2}=\frac{x}{a}.$$

12. $x^3+\dfrac{1}{x^3}-3\left(\dfrac{1}{x^2}-x^2\right)+4\left(x+\dfrac{1}{x}\right)\div\left(x+\dfrac{1}{x}\right)$

$$=\left(x^3+\frac{1}{x^3}\right)+3\left(x^2-\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)\div\left(x+\frac{1}{x}\right)$$

$$=\left(x^2-1+\frac{1}{x^2}\right)+3\left(x-\frac{1}{x}\right)+4$$

$$=x^2+3x+3-\frac{3}{x}+\frac{1}{x^2}.$$

13. $$\frac{1-\frac{2xy}{(x+y)^2}}{1+\frac{2xy}{(x-y)^2}} \div \left\{\frac{1-\frac{y}{x}}{1+\frac{y}{x}}\right\}^2$$

$$=\frac{\frac{x^2+y^2}{(x+y)^2}}{\frac{x^2+y^2}{(x-y)^2}} \div \frac{\left(\frac{x-y}{x}\right)^2}{\left(\frac{x+y}{x}\right)^2}$$

$$=\frac{(x^2+y^2)(x-y)^2}{(x^2+y^2)(x+y)^2} \times \frac{(x+y)^2}{(x-y)^2}$$

$$=1.$$

14. $$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$$

$$=\frac{\frac{ab}{a+b}+2a}{2b-\frac{ab}{a+b}}+\frac{\frac{ab}{a+b}-2a}{2b+\frac{ab}{a+b}}-\frac{4ab}{4b^2-\left(\frac{ab}{a+b}\right)^2}$$

$$=\frac{3ab+2a^2}{ab+2b^2}-\frac{2a^2+ab}{3ab+2b^2}-\frac{4ab(a+b)^2}{b^2(3a^2+8ab+4b^2)}$$

$$=\frac{a(3b+2a)}{b(a+2b)}-\frac{a(2a+b)}{b(3a+2b)}-\frac{4ab(a+b)^2}{b^2(3a+2b)(a+2b)}.$$

L.C.D. $= b^2(3a+2b)(a+2b)$.

$$\begin{array}{rl} 6a^3b+13a^2b^2+6ab^3 &= \text{first numerator,}\\ -2a^3b-\ \ 5a^2b^2-2ab^3 &= \text{second numerator,}\\ \underline{-4a^3b-\ \ 8a^2b^2-4ab^3} &= \text{third numerator.}\\ 0 &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $= 0$.

15. $$\frac{x+y-1}{x-y+1}$$

$$=\frac{\frac{a+1}{ab+1}+\frac{ab+a}{ab+1}-1}{\frac{a+1}{ab+1}-\frac{ab+a}{ab+1}+1}$$

$$=\frac{\frac{2a}{ab+1}}{\frac{2}{ab+1}}$$

$$=a.$$

16. $$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$$

$$=\frac{1}{a(a-b)(a-c)}-\frac{1}{b(b-c)(a-b)}+\frac{1}{c(a-c)(b-c)}.$$

L. C. D. $= abc(a-b)(a-c)(b-c)$.

$$\begin{array}{ll} b^2c-bc^2 & = \text{first numerator,} \\ \quad -a^2c+ac^2 & = \text{second numerator,} \\ \qquad +a^2b-ab^2 & = \text{third numerator.} \\ \hline b^2c-bc^2-a^2c+ac^2+a^2b-ab^2 & = \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{b^2c-bc^2-a^2c+ac^2+a^2b-ab^2}{abc(b^2c-bc^2-a^2c+ac^2+a^2b-ab^2)}$

$$=\frac{1}{abc}.$$

17. $$\frac{3abc}{bc+ca-ab}-\frac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}.$$

Multiply both terms of the second fraction by abc,

$$=\frac{3abc}{bc+ca-ab}-\frac{abc-bc+abc-ac+abc-ab}{bc+ca-ab}$$

$$=\frac{3abc}{bc+ca-ab}-\frac{3abc-bc-ac-ab}{bc+ca-ab}$$

$$=\frac{bc+ac+ab}{bc+ac-ab}.$$

18. $$\frac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\times\frac{m^2-n^2}{m^3+n^3}$$

$$=\frac{m^2-mn+n^2}{n}\times\frac{mn}{m-n}\times\frac{(m+n)(m-n)}{(m+n)(m^2-mn+n^2)}$$

$$=m.$$

19. $$\frac{\frac{1}{a}+\frac{1}{b+c}}{\frac{1}{a}-\frac{1}{b+c}}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{(b+c+a)(2bc+b^2+c^2-a^2)}{(b+c-a)2bc}$$

$$=\frac{(b+c+a)\{(b+c)^2-a^2\}}{(b+c-a)2bc}$$

$$=\frac{(b+c+a)(b+c+a)(b+c-a)}{(b+c-a)2bc}$$

$$=\frac{(b+c+a)^2}{2bc}.$$

20. $$3a-[b+\{2a-(b-c)\}]+\tfrac{1}{2}+\frac{2c^2-\frac{1}{2}}{2c+1}$$

$$=3a-[b+2a-b+c]+\tfrac{1}{2}+\frac{2c^2-\frac{1}{2}}{2c+1}$$

$$=3a-b-2a+b-c+\tfrac{1}{2}+\frac{2c^2-\frac{1}{2}}{2c+1}$$

$$=a-c+\tfrac{1}{2}+\frac{2c^2-\frac{1}{2}}{2c+1}$$

$$=a-c+\tfrac{1}{2}+\frac{4c^2-1}{2(2c+1)}$$

$$=a-c+\tfrac{1}{2}+\frac{2c-1}{2}$$

$$=a-c+\tfrac{1}{2}+c-\tfrac{1}{2}$$

$$=a.$$

21. $$\frac{\frac{1}{a-x}-\frac{1}{a-y}+\frac{x}{(a-x)^2}-\frac{y}{(a-y)^2}}{\frac{1}{(a-y)(a-x)^2}-\frac{1}{(a-x)(a-y)^2}}$$

$$=\frac{\frac{(a-x)(a-y)^2-(a-y)(a-x)^2+x(a-y)^2-y(a-x)^2}{(a-x)^2(a-y)^2}}{\frac{x-y}{(a-x)^2(a-y)^2}}$$

$$=\frac{a(2a-x-y)(x-y)}{(a-x)^2(a-y)^2}\times\frac{(a-x)^2(a-y)^2}{x-y}$$

$$=a(2a-x-y).$$

22. $$\frac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$$

$$=\frac{1}{x+\dfrac{3-x}{3+1}}$$

$$=\frac{4}{4x+3-x}$$

$$=\frac{4}{3(x+1)}.$$

23. $$\frac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2+\dfrac{xy}{x+y}}$$

$$=\frac{(x+y)(x-y)(x-y)2x}{\dfrac{4(x-y)(x-y)(x+y)}{xy}}$$

$$=\frac{x^2y}{2}.$$

24. $$\left(\frac{c-b}{c+b}-\frac{c^3-b^3}{c^3+b^3}\right)\div\left(\frac{c+b}{c-b}+\frac{c^2+b^2}{c^2-b^2}\right).$$

L. C. D. 1st expression $= c^3+b^3$. L. C. D. 2d expression $= c^2-b^2$.

$$\begin{array}{rl} c^3-2c^2b+2cb^2-b^3 &= \text{1st num.}\\ -c^3 \qquad\qquad\quad +b^3 &= \text{2d num.}\\ \hline -2c^2b+2cb^2 \quad &= \text{sum of nums.}\\ \text{or,}\quad -2cb(c-b) &= \text{sum of nums.} \end{array} \qquad \begin{array}{rl} c^2+2cb+\ b^2 &= \text{1st num.}\\ c^2 \qquad +\ b^2 &= \text{2d num.}\\ \hline 2c^2+2cb+2b^2 &= \text{sum of nums.}\\ \text{or,}\ 2(c^2+cb+b^2) &= \text{sum of nums.} \end{array}$$

$$=\frac{-2cb(c-b)}{(c+b)(c^2-cb+b^2)}\times\frac{(c+b)(c-b)}{2(c^2+cb+b^2)}$$

$$=\frac{-cb(c-b)^2}{c^4+c^2b^2+b^4}$$

$$=\frac{-bc(b-c)^2}{b^4+b^2c^2+c^4}.$$

25. $$\frac{y}{(x-y)(x-z)}+\frac{x}{(y-x)(y-z)}+\frac{x+y}{(z-x)(z-y)}$$

$$=\frac{y}{(x-y)(x-z)}-\frac{x}{(x-y)(y-z)}+\frac{x+y}{(x-z)(y-z)}.$$

L. C. D. $=(x-y)(x-z)(y-z)$.

$$\begin{array}{rl} y^2 \qquad -yz &= \text{first numerator,}\\ -x^2 \qquad +xz \qquad &= \text{second numerator,}\\ x^2-y^2 \qquad\qquad &= \text{third numerator.}\\ \hline xz-yz &= \text{sum of numerators;}\\ \text{or,}\qquad z(x-y) &= \text{sum of numerators.} \end{array}$$

$\therefore$ Sum of fractions $=\dfrac{z(x-y)}{(x-y)(y-z)(x-z)}=\dfrac{z}{(x-z)(y-z)}.$

26. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}-\dfrac{1}{abc}$

$=\dfrac{1}{a(a-b)(a-c)}-\dfrac{1}{b(a-b)(b-c)}-\dfrac{1}{abc}$

L.C.D. $= abc(a-b)(b-c)(a-c)$.

$$\begin{array}{lr}
-bc^2 \qquad\qquad + b^2c & = \text{first numerator,}\\
-a^2c+ac^2 & = \text{second numerator,}\\
\underline{-bc^2-a^2b+ab^2-b^2c+a^2c-ac^2} & = \text{third numerator.}\\
-a^2b+ab^2 & = \text{sum of numerators.}
\end{array}$$

$\therefore$ Sum of fractions $= -\dfrac{ab(a-b)}{abc(a-b)(b-c)(a-c)}$

$= -\dfrac{1}{c(b-c)(a-c)}.$

27. $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{6}{x-1}}\times\dfrac{1-\dfrac{x+5}{x^2-1}}{(x-1)(x-2)}$

$=\dfrac{\dfrac{x^2-3x+2}{x+1}}{\dfrac{x^2-x-6}{x-1}}\times\dfrac{\dfrac{x^2-x-6}{x^2-1}}{(x-1)(x-2)}$

$=\dfrac{(x-1)(x-2)(x-1)}{(x+1)(x-3)(x+2)}\times\dfrac{(x-3)(x+2)}{(x+1)(x-1)(x-1)(x-2)}$

$=\dfrac{1}{(x+1)^2}.$

Exercise LXIII.

1. $5x-\dfrac{x+2}{2}=71.$

Multiply by 2; then

$10x-x-2=142,$
$9x=144,$
$x=16.$

2. $x-\dfrac{3-x}{3}=\dfrac{17}{3}.$

Multiply by 3; then

$3x-3+x=17,$
$4x=20,$
$x=5.$

3. $\frac{5-2x}{4}+2=x-\frac{6x-8}{2}.$

Multiply by 4; then

$5-2x+8=4x-12x+16,$

$6x=3,$

$x=\frac{1}{2}.$

5. $2x-\frac{5x-4}{6}=7-\frac{1-2x}{5}.$

Multiply by 30; then

$60x-25x+20$

$=210-6+12x,$

$23x=184,$

$x=8.$

4. $\frac{5x}{2}-\frac{5x}{4}=\frac{9}{4}-\frac{3-x}{2}.$

Multiply by 4; then

$10x-5x=9-6+2x,$

$3x=3,$

$x=1.$

6. $\frac{x+2}{2}=\frac{14}{9}-\frac{3+5x}{4}.$

Multiply by 36; then

$18x+36=56-27-45x,$

$63x=-7,$

$x=-\frac{1}{9}.$

7. $\frac{5x+3}{8}-\frac{3-4x}{3}+\frac{x}{2}=\frac{31}{2}-\frac{9-5x}{6}.$

Multiply by 24; then

$15x+9-24+32x+12x=372-36+20x,$

$39x=351,$

$x=9.$

8. $\frac{10x+3}{3}-\frac{6x-7}{2}=10(x-1).$

Multiply by 6; then

$20x+6-18x+21$

$=60x-60,$

$58x=87,$

$x=1\frac{1}{2}.$

10. $\frac{7x+5}{6}-\frac{5x-6}{4}=\frac{8-5x}{12}.$

Multiply by 12; then

$14x+10-15x+18=8-5x,$

$4x=-20,$

$x=-5.$

9. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14.$

Multiply by 6; then

$15x-21-4x-14=18x-84,$

$7x=49,$

$x=7.$

11. $\frac{x+4}{3}-\frac{x-4}{5}=2+\frac{3x-1}{15}.$

Multiply by 15; then

$5x+20-3x+12$

$=30+3x-1,$

$-x=-3,$

$x=3.$

12. $\frac{3x+5}{7}-\frac{2x+7}{3}+10-\frac{3x}{5}=0.$

Multiply by 105; then

$45x+75-70x-245+1050-63x=0,$

$-88x=-880,$

$x=10.$

13. $\frac{1}{7}(3x-4)+\frac{1}{3}(5x+3)=43-5x.$

Multiply by 21; then

$$\begin{aligned}9x-12+35x+21&\\ &=903-105x,\\ 149x&=894,\\ x&=6.\end{aligned}$$

14. $\frac{1}{2}(27-2x)=\frac{9}{2}-\frac{1}{10}(7x-54).$

Multiply by 10; then

$$\begin{aligned}135-10x&=45-7x+54,\\ -3x&=-36,\\ x&=12.\end{aligned}$$

15. $5x-\{8x-3[16-6x-(4-5x)]\}=6,$

$$\begin{aligned}&5x-\{8x-3[16-6x-4+5x]\}=6,\\ &5x-\{8x-48+18x+12-15x\}=6,\\ &5x-8x+48-18x-12+15x=6,\\ &-6x=-30,\\ &x=5.\end{aligned}$$

16. $\frac{5x-3}{7}-\frac{9-x}{3}=\frac{5x}{2}+\frac{19}{6}(x-4).$

Multiply by 42; then

$$\begin{aligned}30x-18-126+14x&=105x+133x-532,\\ -194x&=-388,\\ x&=2.\end{aligned}$$

17. $\frac{2x+7}{7}-\frac{9x-8}{11}=\frac{x-11}{2}.$

Multiply by 154; then

$$\begin{aligned}44x+154-126x+112&\\ &=77x-847,\\ -159x&=-1113,\\ x&=7.\end{aligned}$$

18. $\frac{8x-15}{3}-\frac{11x-1}{7}=\frac{7x+2}{13}.$

Multiply by 273; then

$$\begin{aligned}728x-1365-429x+39&\\ &=147x+42,\\ 152x&=1368,\\ x&=9.\end{aligned}$$

19. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}.$

Multiply by 56; then

$$\begin{aligned}49x+63-24x-8&=126x-182-996+36x,\\ -137x&=-1233,\\ x&=9.\end{aligned}$$

Exercise LXIV.

1. $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}.$

Multiply by 36; then

$$9x+20=\frac{144(x-3)}{5x-4}+9x,$$

$$\frac{144(x-3)}{5x-4}=20,$$

$$144x-432=100x-80,$$
$$44x=352,$$
$$x=8.$$

2. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

Multiply by 14; then

$$18x-27+\frac{154x-14}{3x+1}$$
$$=18x+22,$$

$$\frac{154x-14}{3x+1}=49.$$

Divide by 7,

$$\frac{22x-2}{3x+1}=7,$$

$$22x-2=21x+7,$$
$$x=9.$$

3. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

Multiply by 18; then

$$10x+17-\frac{216x+36}{13x-16}$$
$$=10x-8,$$

$$\frac{216x+36}{13x-16}=25,$$

$$325x-400=216x+36,$$
$$109x=436,$$
$$x=4.$$

4. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$

Multiply by 15; then

$$6x+13-\frac{45x+75}{5x-25}=6x,$$

$$\frac{45x+75}{5x-25}=13,$$

$$45x+75=65x-325,$$
$$-20x=-400,$$
$$x=20.$$

5. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=4\tfrac{5}{12}-\frac{101-64x}{24}.$

Reduce the mixed number to an improper fraction,

$$\frac{18x-22}{3(13-2x)}+2x+\frac{1+16x}{24}=\frac{53}{12}-\frac{101-64x}{24}.$$

Multiply by 24; then

$$\frac{8(18x-22)}{13-2x}+48x+1+16x=106-101+64x,$$

$$\frac{144x-176}{13-2x}=4,$$

$$144x-176=52-8x,$$
$$152x=228,$$
$$x=1\tfrac{1}{2}.$$

6. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}.$

Multiply by 210; then

$$84-70x-\frac{105-30x^2}{x-1}=10+30x-70x+77+2,$$
$$-\frac{105-30x^2}{x-1}=30x+5,$$
$$-105+30x^2=30x^2-25x-5,$$
$$25x=100,$$
$$x=4.$$

7. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{41}{56}.$

Multiply by 56; then

$$36x+20+\frac{224x-196}{3x+1}$$
$$=36x+15+41,$$
$$\frac{224x-196}{3x+1}=36,$$
$$224x-196=108x+36,$$
$$116x=232,$$
$$x=2.$$

8. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}.$

Multiply by 15; then

$$6x+7-\frac{30x-30}{7x-6}=6x+3,$$
$$-\frac{30x-30}{7x-6}=-4,$$
$$-30x+30=-28x+24,$$
$$-2x=-6,$$
$$x=3.$$

9. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

Multiply by 15; then

$$6x+1-\frac{30x-60}{7x-16}=6x-3,$$
$$-\frac{30x-60}{7x-16}=-4,$$
$$-30x+60=-28x+64,$$
$$-2x=4,$$
$$x=-2.$$

10. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}.$

Multiply by 35; then

$$7x-6-\frac{35x-175}{6x-101}=7x.$$

Transpose, and clear of fractions,

$$-35x+175=36x-606,$$
$$-71x=-781,$$
$$x=11.$$

Exercise LXV.

1. $ax+bc=bx+ac,$
$ax-bx=ac-bc,$
$x(a-b)=c(a-b),$
$x=c.$

2. $2a-cx=3c-5bx,$
$5bx-cx=3c-2a,$
$x(5b-c)=3c-2a,$
$x=\frac{3c-2a}{5b-c}.$

3. $a^2x + bx - c = b^2x + cx - d,$
$a^2x - b^2x + bx - cx = c - d,$
$x(a^2 - b^2 + b - c) = c - d,$
$x = \frac{c-d}{a^2 - b^2 + b - c}.$

4. $-ac^2 + b^2c + abcx = abc + cmx - ac^2x + b^2c - mc,$
$abcx - cmx + ac^2x = abc + b^2c - mc + ac^2 - b^2c,$
$x(abc - cm + ac^2) = abc - mc + ac^2,$
$x = 1.$

5. $(a + x + b)(a + b - x) = (a + x)(b - x) - ab,$
$-x^2 + a^2 + 2ab + b^2 = ab + bx - ax - x^2 - ab,$
$ax - bx = -a^2 - 2ab - b^2,$
$ax - bx = -(a^2 + 2ab + b^2),$
$x = -\frac{(a+b)^2}{a-b}.$

6. $(a^2 + x)^2 = x^2 + 4a^2 + a^4,$
$a^4 + 2a^2x + x^2 = x^2 + 4a^2 + a^4,$
$2a^2x = 4a^2,$
$x = 2.$

7. $(a^2-x)(a^2+x) = (a^4+2ax-x^2),$
$a^4 - x^2 = a^4 + 2ax - x^2,$
$-2ax = 0,$
$x = 0.$

8. $\frac{ax-b}{c} + a = \frac{x+ac}{c}.$
Multiply by c; then
$ax - b + ac = x + ac.$
$ax - x = b,$
$x(a-1) = b,$
$x = \frac{b}{a-1}.$

9. $\frac{a(b^2x+x^3)}{bx} = acx + \frac{ax^2}{b}.$
Divide by a; then
$\frac{b^2x + x^3}{bx} = cx + \frac{x^2}{b}.$
Multiply by bx,
$b^2x + x^3 = bcx^2 + x^3,$
$b^2x = bcx^2,$
$x = 0$ or $\frac{b}{c}.$

10. $ax - \frac{3a - bx}{2} = \frac{1}{2}.$
Multiply by 2,
$2ax - 3a + bx = 1,$
$2ax + bx = 3a + 1,$
$x(2a + b) = 3a + 1,$
$x = \frac{3a+1}{2a+b}.$

11. $6a - \frac{4ax - 2b}{3} = x.$

$$18a - 4ax + 2b = 3x,$$
$$-3x - 4ax = -18a - 2b,$$
$$x(3 + 4a) = 2(9a + b),$$
$$x = \frac{2(9a + b)}{3 + 4a}.$$

12. $\frac{x^2 - a}{bx} - \frac{a - x}{b} = \frac{2x}{b} - \frac{a}{x}.$

$$x^2 - a - ax + x^2 = 2x^2 - ab,$$
$$-ax = -ab + a,$$
$$x = b - 1.$$

13. $\frac{3}{c} - \frac{ab - x^2}{bx} = \frac{4x - ac}{cx}.$

$$3bx - abc + cx^2 = 4bx - abc,$$
$$cx^2 = bx,$$
$$x = 0 \text{ or } \frac{b}{c}.$$

14. $am - b - \frac{ax}{b} + \frac{x}{m} = 0.$

$$abm^2 - b^2m - amx + bx = 0,$$
$$bx - amx = b^2m - abm^2,$$
$$x = \frac{b^2m - abm^2}{b - am}$$
$$= bm.$$

15. $\frac{3ax - 2b}{3b} - \frac{ax - a}{2b} = \frac{ax}{b} - \frac{2}{3}.$

$$6ax - 4b - 3ax + 3a = 6ax - 4b,$$
$$-3ax = -3a,$$
$$x = 1.$$

16. $\frac{ab + x}{b^2} - \frac{b^2 - x}{a^2b} = \frac{x - b}{a^2} - \frac{ab - x}{b^2}.$

$$a^3b + a^2x - b^3 + bx$$
$$= b^2x - b^3 - a^3b + a^2x,$$
$$-b^2x + bx = -a^3b - a^3b,$$
$$bx(b - 1) = 2a^3b,$$
$$x = \frac{2a^3}{b - 1}.$$

17. $ax - \frac{bx + 1}{x} = \frac{a(x^2 - 1)}{x}.$

$$ax^2 - bx - 1 = ax^2 - a,$$
$$-bx = -a + 1,$$
$$x = \frac{a - 1}{b}.$$

18. $\frac{ax^2}{b - cx} + a + \frac{ax}{c} = 0.$

$$acx^2 + abc - ac^2x + abx - acx^2$$
$$= 0.$$

Divide by a,

$$cx^2 + bc - c^2x + bx - cx^2 = 0,$$
$$bx - c^2x = -bc,$$
$$x = \frac{bc}{c^2 - b}.$$

19. $\frac{ab}{x} = bc + d + \frac{1}{x}.$

$$ab = bcx + dx + 1,$$
$$-bcx - dx = -ab + 1,$$
$$x = \frac{ab - 1}{bc + d}.$$

20. $\frac{a(d^2 + x^2)}{dx} = ac + \frac{ax}{d}.$

$$ad^2 + ax^2 = acdx + ax^2,$$
$$acdx = ad^2,$$
$$x = \frac{d}{c}.$$

Exercise LXVI.

1. $\dfrac{x-3}{4(x-1)} = \dfrac{x-5}{6(x-1)} + \dfrac{1}{9}.$

Clear of fractions,

$$9x - 27 = 6x - 30 + 4x - 4,$$
$$-x = -7,$$
$$x = 7.$$

2. $x + \dfrac{x}{x-1} = \dfrac{(x-2)(x+4)}{x+1}.$

Clear of fractions,

$$x^3 - x + x^2 + x = x^3 + x^2 - 10x + 8,$$
$$10x = 8,$$
$$x = \tfrac{4}{5}.$$

3. $\dfrac{7}{x-1} = \dfrac{6x+1}{x+1} - \dfrac{3(1+2x^2)}{x^2-1}.$

Clear of fractions,

$$7x + 7 = 6x^2 + x - 6x - 1 - 3 - 6x^2,$$
$$12x = -11,$$
$$x = -\tfrac{11}{12}.$$

4. $\dfrac{1}{2(x-3)} - \dfrac{1}{3(x-2)} = \dfrac{x-1}{(x-2)(x-3)}$

Clear of fractions,

$$3x - 6 - 2x + 6 = 6x - 6,$$
$$-5x = -6,$$
$$x = 1\tfrac{1}{5}.$$

5. $1 - \dfrac{2(2x+3)}{9(7-x)} = \dfrac{6}{7-x} - \dfrac{5x+1}{4(7-x)}.$

Clear of fractions,

$$252 - 36x - 16x - 24$$
$$= 216 - 45x - 9,$$
$$-7x = -21,$$
$$x = 3.$$

6. $\dfrac{17}{x+3} - 4 = \dfrac{5(21+2x)}{3x+9} - 10.$

Clear of fractions,

$$51 - 12x - 36$$
$$= 105 + 10x - 30x - 90,$$
$$8x = 0,$$
$$x = 0.$$

7. $\dfrac{x-7}{x+7} = \dfrac{2x-15}{2x-6} - \dfrac{1}{2(x+7)}$

Clear of fractions,

$$2x^2 - 20x + 42 = 2x^2 - x - 105 - x + 3,$$
$$-18x = -144,$$
$$x = 8.$$

8. $\dfrac{x+4}{3x+5} + 1\tfrac{1}{6} = \dfrac{3x+8}{2x+3}.$

Clear of fractions,

$$12x^2 + 66x + 72 + 36x^2 + 114x + 90 + 6x^2 + 19x + 15$$
$$= 54x^2 + 134x + 240,$$
$$-35x = 63,$$
$$x = -1\tfrac{4}{5}.$$

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9. $\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52.$

Clear of fractions,

$$\begin{aligned}132x^2-131x-1+24x^2+23x+5&\\=156x^2-104x-52&,\\-4x=-56&,\\x=14&.\end{aligned}$$

10. $\frac{2}{2x-3}+\frac{1}{x-2}=\frac{6}{3x+2}.$

Clear of fractions,

$$\begin{aligned}6x^2-8x-8+6x^2-5x-6&\\=12x^2-42x+36&,\\29x=50&,\\x=1\tfrac{21}{29}&.\end{aligned}$$

11. $\frac{3x-1}{2x-1}-\frac{4x-2}{3x-2}=\frac{1}{6}.$

Clear of fractions,

$$\begin{aligned}54x^2-54x+12-48x^2+48x-12&\\=6x^2-7x+2&,\\x=2&.\end{aligned}$$

12. $\frac{3}{x-1}-\frac{x+1}{x-1}=\frac{x^2}{1-x^2};$

or, $\frac{3}{x-1}-\frac{x+1}{x-1}=\frac{-x^2}{x^2-1}.$

Clear of fractions,

$$\begin{aligned}3x+3-x^2-2x-1=-x^2&,\\x=-2&.\end{aligned}$$

13. $\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}.$

Then

$$\frac{(x-4)(x-6)}{(x-5)(x-6)}-\frac{(x-5)(x-5)}{(x-5)(x-6)}=\frac{(x-7)(x-9)}{(x-8)(x-9)}-\frac{(x-8)(x-8)}{(x-8)(x-9)}.$$

$$\frac{-1}{(x-5)(x-6)}=\frac{-1}{(x-8)(x-9)}.$$

Clear of fractions,

$$\begin{aligned}-x^2+17x-72&=-x^2+11x-30,\\6x&=42,\\x&=7.\end{aligned}$$

14. $(x-a)(x-b)=(x-a-b)^2,$

$$\begin{aligned}x^2-ax-bx+ab&=x^2-2ax-2bx+a^2+2ab+b^2,\\ax+bx&=a^2+ab+b^2,\\x&=\frac{a^2+ab+b^2}{a+b}.\end{aligned}$$

15. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0,$

$$\begin{aligned}ax-bx-ac+bc-bx+cx+ab-ac-cx+ax+bc-ab&=0,\\ax-bx-bx+cx-cx+ax&=ac-bc-ab+ac-bc+ab,\\2ax-2bx&=2ac-2bc,\\2x(a-b)&=2c(a-b),\\x&=c.\end{aligned}$$

16. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x.$

Clear of fractions,

$$x^3+1+x^3-1=2x^3-2x,$$
$$2x=0,$$
$$x=0.$$

17. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x+6}.$

Clear of fractions,

$$4x+12+7x+14=37,$$
$$11x=11,$$
$$x=1.$$

18. $(x+1)^2=x[6-(1-x)]-2,$
$$(x+1)^2=x(6-1+x)-2,$$
$$x^2+2x+1=6x-x+x^2-2,$$
$$-3x=-3,$$
$$x=1.$$

19. $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=\dfrac{23}{x+1}+5.$

Reduce the complex to simple fractions,

$$\frac{75-x}{3(x+1)}+\frac{80x+21}{5(3x+2)}=\frac{23}{x+1}+5.$$

Clear of fractions,

$$1115x-15x^2+750+240x^2+303x+63$$
$$=1035x+690+225x^2+375x+150,$$
$$8x=27,$$
$$x=3\tfrac{3}{8}.$$

20. $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}.$

Clear of fractions,

$$3a^4bc+6a^3b^2c+3a^2b^3c+a^3b^2+2a^2b^2x+3ab^3x+b^4x$$
$$=3a^4cx+9a^3bcx+9a^2b^2cx+3ab^3cx+a^3bx+3a^2b^2x+3ab^3x+b^4x,$$
$$3a^4cx+9a^3bcx+9a^2b^2cx+3ab^3cx+a^3bx+a^2b^2x$$
$$=3a^4bc+6a^3b^2c+3a^2b^3c+a^3b^2,$$
$$ax(3a^3c+9a^2bc+9ab^2c+3b^3c+a^2b+ab^2)$$
$$=a^2b(3a^2c+6abc+3b^2c+ab),$$
$$x\{3c(a+b)^3+ab(a+b)\}=ab\{3c(a+b)^2+ab\}.$$
$$x=\frac{ab}{a+b}.$$

21. $\dfrac{4}{x-8}+\dfrac{3}{2x-16}-\dfrac{29}{24}=\dfrac{2}{3x-24}.$

Clear of fractions,

$$96+36-29x+232=16,$$
$$-29x=-348,$$
$$x=12.$$

22. $5-x\left(\dfrac{7}{2}-\dfrac{2}{x}\right)=\dfrac{x}{2}-\dfrac{3x-(4-5x)}{4}$

$$5-\frac{7x}{2}+2=\frac{x}{2}-\frac{3x-4+5x}{4}$$

Clear of fractions,

$$20-14x+8=2x-3x+4-5x,$$
$$-8x=-24,$$
$$x=3.$$

23. $\frac{1}{5}-\frac{3}{x-1}=\frac{2+\frac{x+4}{1-x}}{3}.$

Multiply both terms of right member by $1-x$; then

$$\frac{1}{5}-\frac{3}{x-1}=\frac{6-x}{3(1-x)},$$

$$\frac{1}{5}-\frac{3}{x-1}=\frac{x-6}{3(x-1)}.$$

Clear of fractions,

$$\begin{aligned}3x-3-45&=5x-30,\\ -2x&=18,\\ x&=-9.\end{aligned}$$

24. $\frac{x-\frac{3}{2}}{\frac{3}{2}(x-1)}+\frac{x-\frac{5}{2}}{\frac{5}{2}(x+1)}=1+\frac{1}{15\left(1-\frac{1}{x^2}\right)}$

Reduce the complex to simple fractions,

$$\frac{2x-3}{3x-3}+\frac{2x-5}{5x+5}=1+\frac{x^2}{15x^2-15}.$$

Clear of fractions,

$$\begin{aligned}10x^2-5x-15+6x^2-21x+15&=15x^2-15+x^2,\\ -26x&=-15,\\ x&=\tfrac{15}{26}.\end{aligned}$$

Exercise LXVII.

1. Find the number whose third and fourth parts added together make 14.

Let $x=$ the number.

Then $\frac{x}{3}=$ one-third of the number,

and $\frac{x}{4}=$ one-fourth of the number,

and $\frac{x}{3}+\frac{x}{4}=$ sum of the two parts.

But $14=$ sum of the two parts.

$\therefore \frac{x}{3}+\frac{x}{4}=14.$ Whence, $x=24.$

2. Find the number whose third part exceeds its fourth part by 14.

Let $x =$ the number.

Then $\frac{x}{3} =$ one-third of the number,

and $\frac{x}{4} =$ one-fourth of the number,

and $\frac{x}{3} - \frac{x}{4} =$ the excess.

But $14 =$ the excess.

$\therefore \frac{x}{3} - \frac{x}{4} = 14.$ Whence, $x = 168.$

3. The half, fourth, and fifth of a certain number are together equal to 76; find the number.

Let $x =$ the number.

Then $\frac{x}{2} =$ one-half of the number,

and $\frac{x}{4} =$ one-fourth of the number,

$\frac{x}{5} =$ one-fifth of the number,

$\frac{x}{2} + \frac{x}{4} + \frac{x}{5} =$ sum of the parts.

But $76 =$ sum of the parts.

$\therefore \frac{x}{2} + \frac{x}{4} + \frac{x}{5} = 76.$ Whence, $x = 80.$

4. Find the number whose double exceeds its half by 12.

Let $x =$ the number.

Then $\frac{x}{2} =$ one-half the number,

and $2x =$ double the number,

$2x - \frac{x}{2} =$ the excess.

But $12 =$ the excess.

$\therefore 2x - \frac{x}{2} = 12.$ Whence, $x = 8.$

5. Divide 60 into two such parts that a seventh of one part may be equal to an eighth of the other.

Let $\quad x =$ one part,

and $\quad 60 - x =$ the other part.

Then $\quad \frac{x}{7} =$ one-seventh of one part,

and $\quad \frac{60 - x}{8} =$ one-eighth of the other part.

$$\therefore \frac{60 - x}{8} = \frac{x}{7}.$$

Whence, $\quad x = 28,$

and $\quad 60 - x = 32.$

6. Divide 50 into two such parts that a fourth of one part increased by five-sixths of the other part may be equal to 40.

Let $\quad x =$ the smaller part.

Then $\quad 50 - x =$ the larger part,

$\frac{x}{4} + \frac{5}{6}(50 - x) = \frac{1}{4}$ of one part increased by $\frac{5}{6}$ of the other.

But $\quad 40 = \frac{1}{4}$ of one part increased by $\frac{5}{6}$ of the other.

$$\therefore \frac{x}{4} + \frac{5}{6}(50 - x) = 40.$$

Whence, $\quad x = 2\frac{6}{7},$

and $\quad 50 - x = 47\frac{1}{7}.$

7. Divide 100 into two such parts that a fourth of one part diminished by a third of the other part may be equal to 11.

Let $\quad x =$ one part.

Then $\quad 100 - x =$ the other.

$\frac{x}{4} - \frac{100 - x}{3} = \frac{1}{4}$ of one part diminished by $\frac{1}{3}$ of the other.

But $\quad 11 = \frac{1}{4}$ of one part diminished by $\frac{1}{3}$ of the other.

$$\therefore \frac{x}{4} - \frac{100 - x}{3} = 11.$$

Whence, $\quad x = 76,$

and $\quad 100 - x = 24.$

8. The sum of the fourth, fifth, and sixth parts of a certain number exceeds the half of the number by 112. What is the number?

Let $x =$ the number.

Then $\frac{x}{2} =$ one-half of the number,

$\frac{x}{4} =$ one-fourth of the number,

$\frac{x}{5} =$ one-fifth of the number,

$\frac{x}{6} =$ one-sixth of the number.

$$\therefore \frac{x}{4} + \frac{x}{5} + \frac{x}{6} = 112 + \frac{x}{2}.$$

Whence, $x = 960.$

9. The sum of two numbers is 5760, and their difference is equal to one-third of the greater. What are the numbers?

Let $x =$ the greater number.

Then $5760 - x =$ the smaller number.

$$x - (5760 - x) = \frac{x}{3}.$$

$$\therefore 3x - 17{,}280 + 3x = x.$$

Whence, $x = 3456,$

and $5760 - x = 2304.$

10. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.

Let $x =$ first number.

Then $45 - x =$ second number,

$\frac{x}{2} =$ first divided by 2,

$90 - 2x =$ second multiplied by 2.

Then $\frac{x}{2} = 90 - 2x.$

$$\therefore x = 180 - 4x.$$

Whence, $x = 36,$

and $45 - x = 9.$

11. Find a number such that the sum of its fifth and its seventh parts shall exceed the difference of its fourth and its seventh parts by 99.

Let $x =$ the number.

Then $\frac{x}{5} =$ one-fifth of the number,

$\frac{x}{4} =$ one-fourth of the number,

$\frac{x}{7} =$ one-seventh of the number,

$\frac{x}{5} + \frac{x}{7} =$ sum of $\frac{1}{5}$ and $\frac{1}{7}$ of the number,

$\frac{x}{4} - \frac{x}{7} =$ difference between $\frac{1}{4}$ and $\frac{1}{7}$ of the number.

$\left(\frac{x}{5} + \frac{x}{7}\right) - \left(\frac{x}{4} - \frac{x}{7}\right) =$ the excess of the sum of its fourth and seventh parts over the difference of its fourth and seventh parts.

But $99 =$ this excess.

$\therefore \left(\frac{x}{5} + \frac{x}{7}\right) - \left(\frac{x}{4} - \frac{x}{7}\right) = 99.$

Whence, $x = 420.$

12. In a mixture of wine and water, the wine was 25 gallons more than half of the mixture, and the water 5 gallons less than one-third of the mixture. How many gallons were there of each?

Let $x =$ number of gallons in mixture.

Then $\frac{x}{2} + 25 =$ number of gallons of wine,

$\frac{x}{3} - 5 =$ number of gallons of water,

$\frac{x}{2} + 25 + \frac{x}{3} - 5 =$ number of gallons in mixture.

$\therefore \frac{x}{2} + 25 + \frac{x}{3} - 5 = x.$

Whence, $x = 120,$

and $\frac{x}{2} + 25 = 85, \ \frac{x}{3} - 5 = 35.$

13. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than the third, and the charcoal 3 pounds less than the fourth of the weight. How many pounds were there of each?

Let $x =$ number of pounds in mixture.

Then $\frac{x}{2} + 6 =$ number of pounds of saltpetre,

$\frac{x}{3} - 5 =$ number of pounds of sulphur,

and $\frac{x}{4} - 3 =$ number of pounds of charcoal.

$$\therefore \frac{x}{2} + 6 + \frac{x}{3} - 5 + \frac{x}{4} - 3 = x.$$

Whence, $x = 24$, $\frac{x}{2} + 6 = 18$, $\frac{x}{3} - 5 = 3$, $\frac{x}{4} - 3 = 3$.

14. Divide 46 into two parts such that if one part be divided by 7, and the other by 3, the sum of the quotients shall be 10.

Let $x =$ first part.

Then $46 - x =$ second part,

and $\frac{x}{3} + \frac{46 - x}{7} = 10.$

Whence, $x = 18$, and $46 - x = 28$.

15. A house and garden cost \$850, and five times the price of the house was equal to twelve times the price of the garden. What is the price of each?

Let $x =$ number of dollars the house cost,

and $850 - x =$ number of dollars the garden cost.

Then $5x =$ five times cost of house,

$10{,}200 - 12x =$ twelve times cost of garden.

$$\therefore 5x = 10{,}200 - 12x.$$

Whence, $x = 600$, and $850 - x = 250$.

16. A man leaves the half of his property to his wife, a sixth to each of his two children, a twelfth to his brother, and the remainder, amounting to \$600, to his sister. What was the amount of his property?

Let x = number of dollars the property amounted to.

Then $\frac{x}{2}$ = number of dollars left to wife,

$\frac{x}{6}$ = number of dollars left to each child,

$\frac{x}{12}$ = number of dollars left to brother.

$\frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600$ = number of dollars in all.

But x = number of dollars in all.

$$\therefore \frac{x}{2}+\frac{x}{6}+\frac{x}{6}+\frac{x}{12}+600=x.$$

Whence, $x = 7200.$

17. The sum of two numbers is a and their difference is b; find the numbers.

Let x = the smaller number.

Then $x+b$ = the larger number,

$2x+b$ = the sum of the numbers.

But a = the sum of the numbers.

$$\therefore 2x+b=a.$$

Whence, $x=\frac{a-b}{2}$, and $x+b=\frac{a+b}{2}$.

18. Find two numbers of which the sum is 70, such that the first divided by the second gives 2 as a quotient and 1 as a remainder.

Let x = first number,

and $70-x$ = second number.

Then $\frac{x-1}{70-x}=2.$

Whence, $x=47$, and $70-x=23$.

19. Find two numbers of which the difference is 25, such that the second divided by the first gives 4 as a quotient and 4 as a remainder.

Let x = smaller number.

Then $x+25$ = larger number,

$$\frac{x+25}{x}=4+\frac{4}{x}.$$

Whence, $x=7$, and $x+25=32$.

20. Divide the number 208 into two parts such that the sum of the fourth of the greater and the third of the smaller is less by 4 than four times the difference of the two parts.

Let $x =$ the greater part.

Then $208 - x =$ the smaller part,

$\frac{x}{4} + \frac{208 - x}{3} =$ sum of $\frac{1}{4}$ the greater and $\frac{1}{3}$ the smaller.

$x - (208 - x) =$ difference of the parts.

$$\therefore \frac{x}{4} + \frac{208 - x}{3} + 4 = 4(x - 208 + x).$$

Whence, $x = 112$, and $208 - x = 96$.

21. Find four consecutive numbers whose sum is 82.

Let $x =$ first number.

Then $x + 1 =$ second number,

$x + 2 =$ third number,

$x + 3 =$ fourth number.

Then $x + x + 1 + x + 2 + x + 3 =$ sum of the numbers.

But $82 =$ sum of the numbers.

$$\therefore x + x + 1 + x + 2 + x + 3 = 82.$$

Whence, $x = 19$, $x + 1 = 20$, $x + 2 = 21$, $x + 3 = 22$.

22. A is 72 years old, and B's age is two-thirds of A's. How long is it since A was five times as old as B?

Let $x =$ number of years since A's age was five times that of B.

$\frac{2}{3}$ of $72 = 48 =$ B's age at present,

$72 - x =$ A's age x years since,

$48 - x =$ B's age x years since.

Then $72 - x = 5(48 - x)$.

Whence, $x = 42$.

23. A mother is 70 years old, her daughter is half that age. How long is it since the mother was three and one-third times as old as the daughter?

Let $x =$ number of years since.

Then $70 - x =$ mother's age x years since,

$35 - x =$ daughter's age x years since.

$$\therefore 70 - x = 3\tfrac{1}{3}(35 - x).$$

Whence, $x = 20$.

24. A father is three times as old as the son; four years ago the father was four times as old as the son then was. What is the age of each?

Let $x =$ number of years in son's age.
Then $3x =$ number of years in father's age,
$x - 4 =$ number of years in son's age 4 years since,
$3x - 4 =$ number of years in father's age 4 years since.
$\therefore 3x - 4 = 4x - 16$. Whence, $x = 12$, and $3x = 36$.

25. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

Let $x =$ number of years in B's age.
Then $2x =$ number of years in A's age,
$x - 7 =$ number of years in B's age 7 years since,
$2x - 7 =$ number of years in A's age 7 years since.
$\therefore x - 7 + 2x - 7 = 2x$. Whence, $x = 14$, and $2x = 28$.

26. The sum of the ages of a father and son is half what it will be in 25 years; the difference is one-third what the sum will be in 20 years. What is the age of each?

Let $x =$ number of years in father's age.
Then $50 - x =$ number of years in son's age,
$x - (50 - x) =$ difference of their ages.

But $\frac{(x + 20) + (50 - x) + 20}{3} =$ difference of their ages.

$$\therefore x - 50 + x = \frac{x + 20 + 50 - x + 20}{3}.$$

Whence, $x = 40$, and $50 - x = 10$.

27. A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, all working together?

Let $x =$ number of days required for A, B, and C, together.

Then $\frac{1}{x} =$ part all can do in one day.

But $\frac{1}{5} =$ part A can do in one day,

$\frac{1}{6} =$ part B can do in one day,

$\frac{2}{15} =$ part C can do in one day.

Then $\frac{1}{5} + \frac{1}{6} + \frac{2}{15} =$ what all can do in one day.

But $\frac{1}{x} =$ what all can do in one day.

$$\therefore \frac{1}{5} + \frac{1}{6} + \frac{2}{15} = \frac{1}{x}. \quad \text{Whence, } x = 2.$$

28. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{3}$ days, and C in $3\frac{3}{4}$ days; in what time will they do it, all working together?

Let x = number of days required for A, B, and C, together.

Then $\frac{1}{x}$ = part they can do in one day.

Now $\frac{1}{2\frac{1}{2}}$ = part A can do in one day,

$\frac{1}{3\frac{1}{3}}$ = part B can do in one day,

$\frac{1}{3\frac{3}{4}}$ = part C can do in one day.

Then $\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{3}}+\frac{1}{3\frac{3}{4}}$ = part all can do in one day.

But $\frac{1}{x}$ = part all can do in one day.

$$\therefore \frac{1}{x}=\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{3}}+\frac{1}{3\frac{3}{4}}.$$

Whence, $x=1\frac{1}{29}$.

29. Two men who can separately do a piece of work in 15 days and 16 days, can, with the help of another, do it in 6 days. How long would it take the third man to do it alone?

Let x = number of days required for third man.

$\frac{1}{x}+\frac{1}{15}+\frac{1}{16}$ = part all can do in one day.

But $\frac{1}{6}$ = part all can do in one day.

$\therefore \frac{1}{x}+\frac{1}{15}+\frac{1}{16}=\frac{1}{6}$. Whence, $x=26\frac{2}{3}$.

30. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time can each alone complete the work?

Let x = number of days C works.

Then $2x$ = number of days B works,

$4x$ = number of days A works.

Then $\frac{1}{x}+\frac{1}{2x}+\frac{1}{4x}$ = part all can do in one day.

But $\frac{1}{24}$ = part all can do in one day.

$\therefore \frac{1}{x}+\frac{1}{2x}+\frac{1}{4x}=\frac{1}{24}$. Whence, $x=42$, $2x=84$, and $4x=168$.

31. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they finish the work in 3 days more. How long would it have taken B alone to do the whole work?

Let $x =$ number of days required for B.

Then $\frac{1}{x} =$ part B can do in one day,

$\frac{1}{18} =$ part A can do in one day,

$\frac{4}{9} =$ part left to be finished,

$\frac{1}{3}$ of $\frac{4}{9}$ or $\frac{4}{27} =$ part both can do in one day.

But $\frac{1}{18} + \frac{1}{x} =$ part both can do in one day.

$$\therefore \frac{1}{18} + \frac{1}{x} = \frac{4}{27}.$$

Whence, $x = 10\frac{4}{5}$.

32. A and B together can reap a field in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time can B and C together reap it? In what time can A, B, and C together reap it?

$\frac{1}{12} =$ part A and B can do together in one hour,

and $\frac{1}{20} =$ part A can do in one hour.

$\therefore \frac{1}{12} - \frac{1}{20}$ or $\frac{1}{30} =$ part B can do in one hour,

$\frac{1}{16} =$ part A and C can do together in one hour.

$\therefore \frac{1}{16} - \frac{1}{20}$ or $\frac{1}{80} =$ part C can do in one hour.

Let $\frac{1}{x} =$ part A, B, and C can do together in one hour.

Then $$\frac{1}{x} = \frac{1}{20} + \frac{1}{30} + \frac{1}{80}.$$

Whence, $x = 10\frac{10}{23}$. B and C together in $21\frac{9}{11}$ hours.

33. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

Let x = number of days required working together.

$\frac{1}{12}$ = part A and B do in one day,

$\frac{1}{15}$ = part A and C do in one day,

$\frac{1}{20}$ = part B and C do in one day.

Then $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ = part all do in two days.

But $\frac{2}{x}$ = part all do in two days.

$$\therefore \frac{2}{x}=\frac{1}{12}+\frac{1}{15}+\frac{1}{20}.$$

Whence, $x = 10.$

34. A tank can be filled by two pipes in 24 minutes and 30 minutes respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three are running together?

Let x = number of minutes required for all running together,

$\frac{1}{x}$ = part filled by all in one minute,

$\frac{1}{24}$ = part filled by first in one minute,

$\frac{1}{30}$ = part filled by second in one minute,

$\frac{1}{20}$ = part emptied by third in one minute,

$\frac{1}{24}+\frac{1}{30}-\frac{1}{20}$ = part filled by all in one minute.

But $\frac{1}{x}$ = part filled by all in one minute.

$$\therefore \frac{1}{x}=\frac{1}{24}+\frac{1}{30}-\frac{1}{20}.$$

Whence, $x = 40.$

35. A tank can be filled in 15 minutes by two pipes, A and B, running together. After A has been running by itself for 5 minutes, B is also turned on, and the tank is filled in 13 minutes more. In what time may it be filled by each pipe separately?

Let $x =$ number of minutes required for A.

Then $\frac{1}{x} =$ part filled by A in one minute,

and $\frac{18}{x} =$ part filled by A in eighteen minutes,

$\frac{1}{15} - \frac{1}{x} =$ part filled by B in one minute,

$\frac{13}{15} - \frac{13}{x} =$ part filled by B in thirteen minutes.

$$\therefore \frac{18}{x} + \frac{13}{15} - \frac{13}{x} = 1.$$

Whence, $x = 37\frac{1}{2}$.

Therefore, it can be filled by A in $37\frac{1}{2}$ minutes, and by B in 25 minutes.

36. A cistern could be filled by two pipes in 6 hours and 8 hours respectively, and could be emptied by a third in 12 hours. In what time would the cistern be filled if the pipes were all running together?

Let $x =$ number of hours required for all running together,

$\frac{1}{x} =$ part all can fill in one hour,

$\frac{1}{6} =$ part filled by first pipe in one hour,

$\frac{1}{8} =$ part filled by second pipe in one hour,

$\frac{1}{12} =$ part emptied by third pipe in one hour.

Then $\frac{1}{6} + \frac{1}{8} - \frac{1}{12} =$ part filled by all in one hour.

But $\frac{1}{x} =$ part filled by all in one hour.

$$\therefore \frac{1}{x} = \frac{1}{6} + \frac{1}{8} - \frac{1}{12}.$$

Whence, $x = 4\frac{4}{5}$.

37. A tank can be filled by three pipes in 1 hour and 20 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled when all three pipes are running together?

Let $x =$ number of minutes required for all to fill it,

$$\frac{1}{80} = \text{part first will fill in one minute,}$$

$$\frac{1}{200} = \text{part second will fill in one minute,}$$

$$\frac{1}{300} = \text{part third will fill in one minute,}$$

$$\frac{1}{x} = \text{part all will fill in one minute.}$$

$$\therefore \frac{1}{x} = \frac{1}{80} + \frac{1}{200} + \frac{1}{300}.$$

Whence, $x = 48$.

38. If three pipes can fill a cistern in a, b, and c minutes, respectively, in what time will it be filled by all three running together?

Let $x =$ number of minutes required for all.

Then $\frac{1}{a} =$ part first fills in one minute,

$$\frac{1}{b} = \text{part second fills in one minute,}$$

$$\frac{1}{c} = \text{part third fills in one minute,}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \text{part all fill in one minute.}$$

But $\frac{1}{x} =$ part all fill in one minute.

$$\therefore \frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Whence, $x = \dfrac{abc}{ab + ac + bc}$.

39. The capacity of a cistern is $755\frac{1}{4}$ gallons. The cistern has three pipes, of which the first lets in 12 gallons in $3\frac{1}{4}$ minutes, the second $15\frac{1}{3}$ gallons in $2\frac{1}{2}$ minutes, the third 17 gallons in 3 minutes. In what time will the cistern be filled by the three pipes running together?

Let $\quad x =$ number of minutes required for all.

Then $\dfrac{755\frac{1}{4}}{x} =$ number of gallons let in per minute by all,

$\dfrac{12}{3\frac{1}{4}} =$ number of gallons let in per minute by first,

$\dfrac{15\frac{1}{3}}{2\frac{1}{2}} =$ number of gallons let in per minute by second,

$\dfrac{17}{3} =$ number of gallons let in per minute by third,

$\dfrac{12}{3\frac{1}{4}} + \dfrac{15\frac{1}{3}}{2\frac{1}{2}} + \dfrac{77}{3} =$ number of gallons let in per minute by all.

But $\dfrac{755\frac{1}{4}}{x} =$ number of gallons let in per minute by all.

$$\therefore \frac{755\frac{1}{4}}{x} = \frac{12}{3\frac{1}{4}} + \frac{15\frac{1}{3}}{2\frac{1}{2}} + \frac{17}{3}.$$

Whence, $\quad x = 48\frac{3}{4}$.

40. A sets out and travels at the rate of 7 miles in 5 hours. Eight hours afterwards, B sets out from the same place, and travels in the same direction at the rate of 5 miles in 3 hours. In how many hours will B overtake A?

Let $\quad x =$ number of hours A is travelling.
Then $\quad x - 8 =$ number of hours B is travelling,
$1\frac{2}{5} =$ number of miles per hour A is travelling,
$1\frac{2}{3} =$ number of miles per hour B is travelling,
$1\frac{2}{5}x =$ number of miles A travels,
$1\frac{2}{3}(x - 8) =$ number of miles B travels.

$$\therefore 1\tfrac{2}{5}x = 1\tfrac{2}{3}(x - 8).$$

Whence, $\quad x = 50,\ x - 8 = 42.$

41. A person walks to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and is out 5 hours. How far is it to the top of the mountain?

Let $\quad x =$ number of hours required to go up,
and $\quad 5 - x =$ number of hours required to go down.
Then $\quad 2\frac{1}{3}x =$ distance up the mountain,
and $\quad 3\frac{1}{2}(5 - x) =$ distance down the mountain.

$$\therefore 2\tfrac{1}{3}x = 3\tfrac{1}{2}(5 - x).$$

Whence, $\quad x = 3$, and $2\frac{1}{3}x = 7$.

42. A person has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

Let $x =$ number of miles he may go.

Then $\frac{x}{b} =$ number of hours he is riding,

and $\frac{x}{c} =$ number of hours he is walking.

$$\therefore \frac{x}{b} + \frac{x}{c} = a.$$

Whence, $x = \frac{abc}{b + c}$.

43. The distance between London and Edinburgh is 360 miles. One traveller starts from Edinburgh and travels at the rate of 10 miles an hour; another starts at the same time from London, and travels at the rate of 8 miles an hour. How far from London will they meet?

Let $x =$ number of hours both travel.
Then $10x =$ number of miles first travels,
and $8x =$ number of miles second travels.
$10x + 8x =$ number of miles both travel.
$\therefore 18x = 360.$
Whence, $x = 20$, and $8x = 160$.

44. Two persons set out from the same place in opposite directions. The rate of one of them per hour is a mile less than double that of the other, and in 4 hours they are 32 miles apart. Determine their rates.

Let $x =$ rate of second in miles.
Then $2x - 1 =$ rate of first in miles,
and $3x - 1 =$ number of miles apart in one hour.
$12x - 4 =$ number of miles apart in four hours.
$\therefore 12x - 4 = 32.$
Whence, $x = 3$, and $2x - 1 = 5$.

45. In going a certain distance, a train travelling 35 miles an hour takes 2 hours less than one travelling 25 miles an hour. Determine the distance.

Let $x =$ number of miles.

Then $\frac{x}{35} =$ number of hours first was travelling,

and $\frac{x}{25} =$ number of hours second was travelling.

$\therefore \frac{x}{35} + 2 = \frac{x}{25}$. Whence, $x = 175$.

46. At what time are the hands of a watch together:

I. Between 3 and 4?
II. Between 6 and 7?
III. Between 9 and 10?

I. Let CH and CM denote the positions of the hour and minute hands at 3 o'clock, and CB the position of both hands when together.

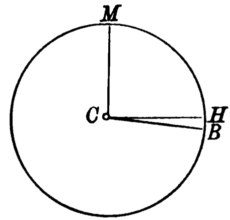
Then arc $HB = \frac{1}{12}$ of arc MHB.

Then x = number of minute-spaces in arc MB,

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 15 = number of minute-spaces in arc MH.

Now arc MB = arc MH + arc HB.



That is, $x = 15 + \frac{x}{12}$.

Whence, $x = 16\frac{4}{11}$.

II. Let CM and CH denote the positions of hour and minute hands at 6 o'clock, CB the position of both when together.

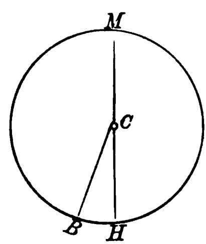
Then arc $HB = \frac{1}{12}$ of arc MHB.

Let x = number of minute-spaces in arc MHB.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 30 = number of minute-spaces in arc MH.

Now arc MHB = arc MH + arc HB.



That is, $x = 30 + \frac{x}{12}$.

Whence, $x = 32\frac{8}{11}$.

III. Let BC and BA denote the positions of the hour and minute hands at 9 o'clock, and BD the position of both hands when together.

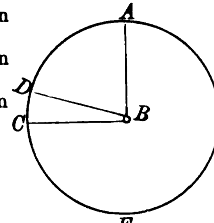
Then $CD = \frac{1}{12}$ of arc $AECD$.

Let x = number of minute-spaces in arc $AECD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc CD,

and 45 = number of minute-spaces in arc AEC.

Now arc $AECD$ = arc AEC + arc CD.



That is, $x = 45 + \frac{x}{12}$.

Whence, $x = 49\frac{1}{11}$.

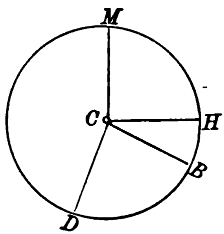
47. At what time are the hands of a watch at right angles:

I. Between 3 and 4?

II. Between 4 and 5?

III. Between 7 and 8?

I. Let CB and CD denote the positions of the hour and minute hands when at right angles.



Let x = number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 15 = number of minute-spaces in arc MH,

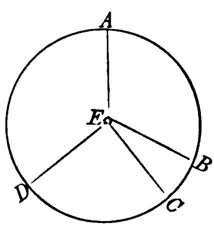
also 15 = number of minute-spaces in arc BD.

Now arc $MHBD$ = arcs $MH+HB+BD$.

That is, $x = 15 + \frac{x}{12} + 15$.

Whence, $x = 32\frac{8}{11}$.

II. Let CE and DE denote the positions of the hour and minute hands when at right angles.



Let x = number of minute-spaces in arc $ABCD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc BC,

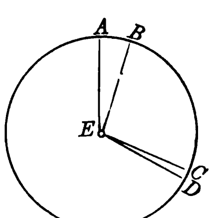
and 20 = number of minute-spaces in arc AB,

also 15 = number of minute-spaces in arc CD.

Now arc $ABCD$ = arcs $BC+AB+CD$.

That is, $x = 20 + \frac{x}{12} + 15$.

Whence, $x = 38\frac{2}{11}$.



Let x = number of minute-spaces in arc AB.

Then $\frac{x}{12}$ = number of minute-spaces in arc CD,

and 20 = number of minute-spaces in arc ABC,

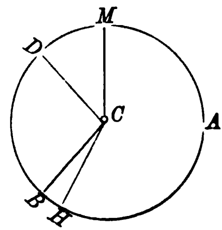
also 15 = number of minute-spaces in arc BCD.

Now arc AB = arcs $CD + AC - BD$.

That is, $x = \frac{x}{12} + 20 - 15$.

Whence, $x = 5\frac{5}{11}$.

III. Let BC and DC denote the positions of the hour and minute hands when at right angles.



Let x = number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

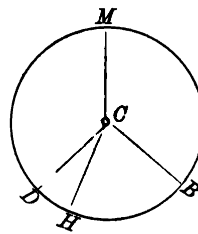
and 35 = number of minute-spaces in arc MAH,

also 15 = number of minute-spaces in arc BD.

Now arc $MHBD$ = arcs $MAH + HB + BD$.

That is, $x = 35 + \frac{x}{12} + 15.$

Whence, $x = 54\frac{6}{11}.$



Let x = number of minute-spaces in arc MB.

Then $\frac{x}{12}$ = number of minute-spaces in arc HD,

and 35 = number of minute-spaces in arc MBH,

also 15 = number of minute-spaces in arc BHD.

Now arc MB

$= \text{arcs } HBM + HD - BHD.$

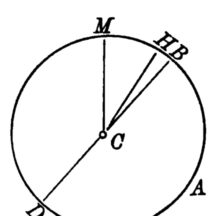
That is, $x = 35 + \frac{x}{12} - 15.$

Whence, $x = 21\frac{9}{11}.$

48. At what time are the hands of a watch opposite to each other:

I. Between 1 and 2?
II. Between 4 and 5?
III. Between 8 and 9?

I. Let CB and CD denote the positions of the hour and minute hands when opposite.



Let x = number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 5 = number of minute-spaces in arc MH,

also 30 = number of minute-spaces in arc BAD.

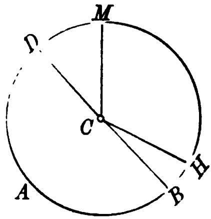
Now arc $MHBD$

$= \text{arcs } MH + HB + BAD.$

That is, $x = 5 + \frac{x}{12} + 30.$

Whence, $x = 38\frac{2}{11}.$

II. Let CB and CD denote the positions of the hour and minute hands when opposite.



Let $x =$ number of minute-spaces in arc $MHBD$.

Then $\frac{x}{12} =$ number of minute-spaces in arc HB,

and $20 =$ number of minute-spaces in arc MH,

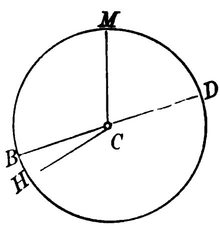
also $30 =$ number of minute-spaces in arc BAD.

Now arc $MHBD$
$= \text{arcs } MH + HB + BAD.$

That is, $x = 20 + \frac{x}{12} + 30.$

Whence, $x = 54\frac{6}{11}.$

III. Let CB and CD denote the positions of the hour and minute hands when opposite.



Let $x =$ number of minute-spaces in arc MD.

Then $\frac{x}{12} =$ number of minute-spaces in arc HB,

and $40 =$ number of minute-spaces in arc MDH,

also $30 =$ number of minute-spaces in arc DHB.

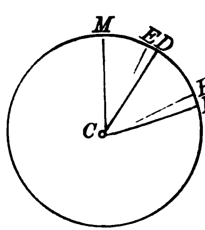
Now arc $MD = \text{arcs } MDH + HB - DHB.$

That is, $x = 40 + \frac{x}{12} - 30.$

Whence, $x = 10\frac{10}{11}.$

49. It is between 2 and 3 o'clock; but a person looking at his watch, and mistaking the hour-hand for the minute-hand, fancies that the time of day is 55 minutes earlier than it really is. What is the true time?

Let CB and CD denote the positions of the hour and minute hands and CE the 1 o'clock point.



Let $x =$ number of minute-spaces in arc MED.

Then $\frac{x}{12} =$ number of minute-spaces in arc HB,

and $10 =$ number of minute-spaces in arc $MEDH$,

also $5 =$ number of minute-spaces in arc DHB.

Now arc MED
$= \text{arcs } MEDH + HB - DHB.$

That is, $x = 10 + \frac{x}{12} - 5.$

$\therefore x = 5\frac{5}{11}.$

50. A hare takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

Let $6x =$ number of leaps taken by the hare.

Then $5x =$ number of leaps taken by the dog.

Also let $a =$ number of feet in one leap of the hare.

Then $\frac{9a}{7} =$ number of feet in one leap of the dog.

$$\therefore \left(\frac{9a}{7}\right)5x = (50 + 6x)a,$$

$$\frac{45ax}{7} = 50a + 6ax,$$

$$45ax = 350a + 42ax.$$

Divide by a, $3x = 350.$

Whence, $x = 116\frac{2}{3},$

$$6x = 700.$$

51. A greyhound makes 3 leaps while a hare makes 4; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of the greyhound's leaps. How many leaps does each take before the hare is caught?

Let $3x =$ number of leaps taken by the greyhound.

Then $4x =$ number of leaps taken by the hare.

Also let $a =$ number of feet in one leap of the hare.

Then $\frac{3a}{2} =$ number of feet in one leap of the greyhound.

That is, $3x \times \frac{3a}{2} =$ the whole distance.

But $\frac{150a}{2} + 4ax =$ the whole distance.

$$\therefore \frac{9ax}{2} = \frac{150a}{2} + 4ax.$$

Divide by a, $\frac{9x}{2} = \frac{150}{2} + 4x,$

$$9x = 150 + 8x.$$

Whence, $x = 150,$

$$3x = 450,$$

$$4x = 600.$$

52. **A greyhound makes two leaps while a hare makes 3; but 1 leap of the greyhound is equivalent to 2 of the hare's. The hare has a start of 80 of her own leaps. How many leaps will the hare take before she is caught?**

Let $2x =$ number of leaps taken by the greyhound.
Then $3x =$ number of leaps taken by the hare.
Also let $a =$ number of feet in one leap of the hare.
Then $2a =$ number of feet in one leap of the greyhound.
That is, $2x \times 2a =$ whole distance.
But $(80 + 3x)\,a =$ whole distance.
$\therefore (80 + 3x)\,a = 4ax.$
Divide by a, $80 + 3x = 4x$. Whence, $x = 80$, and $3x = 240$.

53. **A rectangle whose length is 5 feet more than its breadth would have its area increased by 22 feet if its length and breadth were each made a foot more. Find its dimensions.**

Let $x =$ number of feet in breadth.
Then $x + 5 =$ number of feet in length.
$x(x + 5) =$ number of square feet in area,
$x + 1 =$ number of feet in breadth $+ 1$,
$x + 6 =$ number of feet in length $+ 1$.
$\therefore (x + 1)(x + 6) - x(x + 5) = 22.$
Whence, $x = 8$, and $x + 5 = 13$.

54. **A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square. Find its area.**

Let $x - 3 =$ number of feet in breadth,
and $x + 5 =$ number of feet in length.
Then $(x-3)(x+5) =$ number of feet in area.
But $x^2 =$ number of feet in area.
$\therefore x^2 = x^2 + 2x - 15.$ Whence, $x = 7\frac{1}{2}$, and $x^2 = 56\frac{1}{4}$.

55. **The length of a rectangle is an inch less than double its breadth; and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches. Find the size of the rectangle at first.**

Let $x =$ number of inches in breadth.
Then $2x - 1 =$ number of inches in length,
and $6x + 12x - 6 - 36 =$ number of inches in area cut off.
But $210 =$ number of inches in area cut off.
$\therefore 6x + 12x - 6 - 36 = 210.$ Whence, $x = 14$, and $2x - 1 = 27$.

56. The length of a floor exceeds the breadth by 4 feet; if each dimension were increased by 1 foot, the area of the room would be increased by 27 square feet. Find its dimensions.

Let $\quad x =$ number of feet in breadth.
Then $\quad x + 4 =$ number of feet in length,
and $\quad x^2 + 4x =$ number of feet in area,
$x + 1 =$ number of feet in breadth $+ 1$ foot,
$x + 5 =$ number of feet in length $+ 1$ foot,
$x^2 + 6x + 5 =$ number of feet in area after addition.
But $x^2 + 4x + 27 =$ number of feet in area after addition.
$\therefore x^2 + 6x + 5 = x^2 + 4x + 27$. Whence, $x = 11$, and $x + 4 = 15$.

57. A mass of tin and lead weighing 180 pounds loses 21 pounds when weighed in water; and it is known that 37 pounds of tin lose 5 pounds, and 23 pounds of lead lose 2 pounds, when weighed in water. How many pounds of tin and of lead in the mass?

Let $\quad x =$ number of pounds of tin.
Then $\quad 180 - x =$ number of pounds of lead,
$\frac{5x}{37} =$ number of pounds x pounds of tin lose in water,
$\frac{2}{23}(180 - x) =$ number of pounds $180 - x$ pounds of lead lose in water.
But $\quad 21 =$ number of pounds tin and lead lose in water.

$$\therefore \frac{5x}{37} + \frac{2}{23}(180 - x) = 21.$$

Whence, $\quad x = 111$, and $180 - x = 69$.

58. If 19 pounds of gold lose 1 pound, and 10 pounds of silver lose 1 pound, when weighed in water, find the amount of each in a mass of gold and silver weighing 106 pounds in air and 99 pounds in water.

Let $\quad x =$ number of pounds of gold.
Then $\quad 106 - x =$ number of pounds of silver,
$\frac{x}{19} =$ number of pounds the gold loses in water,
$\frac{106 - x}{10} =$ number of pounds the silver loses in water,
$\frac{x}{19} + \frac{106 - x}{10} =$ number of pounds both lose in water.
But $\quad 7 =$ number of pounds both lose in water.

$$\therefore \frac{x}{19} + \frac{106 - x}{10} = 7.$$

Whence, $\quad x = 76$, and $106 - x = 30$.

59. Fifteen sovereigns should weigh 77 pennyweights; but a parcel of light sovereigns, having been weighed and counted, was found to contain 9 more than was supposed from the weight; and it appeared that 21 of these coins weighed the same as 20 true sovereigns. How many were there all together?

Let $x =$ number in parcel,

$\frac{77}{15} =$ number pennyweights a good sovereign weighs,

$x - 9 =$ number good sovereigns that weigh same as bad,

$\frac{77(x-9)}{15} =$ number pennyweights the good coins weigh,

$\frac{20}{21} \times \frac{77}{15}$ or $\frac{44}{9} =$ number pennyweights a bad coin weighs.

$$\therefore \frac{44x}{9} = \frac{77(x-9)}{15}.$$

Whence, $x = 189$.

60. There are two silver cups, and one cover for both. The first weighs 12 ounces, and with the cover weighs twice as much as the other without it; but the second with the cover weighs one-third more than the first without it. Find the weight of the cover.

Let $x =$ weight of cover in ounces,

$12 + x =$ weight of first cover and cup in ounces,

$2(16 - x) =$ double the weight of the second cup in ounces.

But $12 + x =$ double the weight of the second cup in ounces.

$$\therefore 12 + x = 2(16 - x).$$

Whence, $x = 6\frac{2}{3}$.

61. A man wishes to enclose a circular piece of ground with palisades, and finds that if he sets them a foot apart he will have too few by 150; but if he sets them a yard apart he will have too many by 70. What is the circuit of the piece of ground?

Let $x =$ number of feet in circuit of field.

Then $x - 150 =$ number of palisades he had.

But $\frac{x}{3} + 70 =$ number of palisades he had.

$$\therefore x - 150 = \frac{x}{3} + 70.$$

Whence, $x = 330$.

62. A horse was sold at a loss for \$200; but if it had been sold for \$250, the gain would have been three-fourths of the loss when sold for \$200. Find the value of the horse.

Let $x =$ number of dollars the horse is worth.

Then $250 - x =$ number of dollars made if sold for \$250,

$x - 200 =$ number of dollars lost if sold for \$200.

$\therefore\ 250 - x = \frac{3}{4}(x - 200).$

Whence, $x = 228\frac{4}{7}$.

63. A and B shoot by turns at a target. A puts 7 bullets out of 12, and B 9 out of 12, into the centre. Between them they put in 32 bullets. How many shots did each fire?

Let $x =$ number of shots each fired,

$\frac{7x}{12} =$ number of centres made by A,

$\frac{9x}{12} =$ number of centres made by B.

But $32 =$ number of centres made by both.

$\therefore\ \frac{7x}{12} + \frac{9x}{12} = 32.$

Whence, $x = 24$.

64. A boy buys a number of apples at the rate of 5 for 2 pence. He sells half of them at 2 a penny and the rest at 3 a penny, and clears a penny by the transaction. How many does he buy?

Let $x =$ number bought.

Then $\frac{2x}{5} =$ number of pence paid,

and $\frac{x}{2} \times \frac{1}{2}$ or $\frac{x}{4} =$ selling price of one-half.

But $\frac{x}{2} \times \frac{1}{3}$ or $\frac{x}{6} =$ selling price of the other half.

$\therefore\ \left(\frac{x}{4} + \frac{x}{6}\right) - \frac{2x}{5} = 1.$

Whence, $x = 60$.

65. A person bought a piece of land for \$6750, of which he kept $\frac{4}{9}$ for himself. At the cost of \$250 he made a road which took $\frac{1}{10}$ of the remainder, and then sold the rest at $12\frac{1}{2}$ cents a square yard more than double the price it cost him, thus clearing his outlay and \$500 besides. How much land did he buy, and what was the cost-price per yard?

Let $\qquad x =$ number of yards.

Then $\qquad \frac{4x}{9} =$ number of yards kept,

$\frac{5x}{90} =$ number of yards used for road,

$\frac{x}{2} =$ number of yards sold.

$$\therefore\ 6750 + \frac{1}{8} \times \frac{x}{2} = 7500.$$

Whence, $\qquad x = 12{,}000,$

and $\$6750.00 \div x = \$0.56\frac{1}{4}.$

66. A boy who runs at the rate of 12 yards per second starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second. How soon will the first boy be 10 yards ahead of the second?

Let $\qquad x =$ number of seconds they are running.

Then $\qquad 12x =$ number of yards first boy runs,

and $\qquad \frac{21x}{2} =$ number of yards second boy runs.

$$\therefore\ 12x - \left(10 + \frac{21x}{2}\right) = 20,$$

$$12x - \frac{20 + 21x}{2} = 20,$$

$$24x - 20 - 21x = 40,$$

$$3x = 60,$$

$$x = 20.$$

67. A merchant adds yearly to his capital one-third of it, but takes from it, at the end of each year, \$5000 for expenses. At the end of the third year, after deducting the last \$5000, he has twice his original capital. How much had he at first?

Let $\qquad x =$ number of dollars he had at first.

Then $\dfrac{4x}{3} - 5000 =$ number of dollars he had at the end of the first year,

or $\qquad \dfrac{4x - 15{,}000}{3} =$ number of dollars he had at the end of the first year,

$\dfrac{4}{3}\left(\dfrac{4x - 15{,}000}{3}\right) - 5000 =$ number of dollars he had at the end of the second year,

or $\qquad \dfrac{16x - 105{,}000}{9} =$ number of dollars he had at the end of the second year,

$\dfrac{4}{3}\left(\dfrac{16x - 105{,}000}{9}\right) - 5000 =$ number of dollars he had at the end of the third year,

or $\qquad \dfrac{64x - 555{,}000}{27} =$ number of dollars he had at the end of the third year.

But $\qquad 2x =$ number of dollars he had at the end of the third year.

$$\therefore \frac{64x - 555{,}000}{27} = 2x.$$

Whence, $\qquad x = 55{,}500.$

68. A shepherd lost a number of sheep equal to one-fourth of his flock and one-fourth of a sheep; then, he lost a number equal to one-third of what he had left and one-third of a sheep; finally, he lost a number equal to one-half of what now remained and one-half a sheep, after which he had but 25 sheep left. How many had he at first?

Let $\qquad x =$ number of sheep he had at first.

Then $\dfrac{3x-1}{4} =$ number of sheep he had left after first loss,

$\dfrac{3(x+1)}{12} =$ number of sheep he lost the second time,

$\dfrac{x-1}{2} =$ number of sheep he had left after second loss,

$\dfrac{x+1}{4} =$ number of sheep he lost the third time,

$\dfrac{x-3}{4} =$ number of sheep he had left after third loss.

But $\qquad 25 =$ number of sheep he had left after third loss.

$$\therefore \frac{x-3}{4} = 25.$$

Whence, $\quad x = 103.$

69. A trader maintained himself for three years at an expense of $250 a year; and each year increased that part of his stock which was not so expended by one-third of it. At the end of the third year his original stock was doubled. What was his original stock?

Let $x =$ number of dollars in stock at first.

Then $\frac{4}{3}(x-250)$

or $\frac{4x-1000}{3} =$ number of dollars in stock at the end of first year,

$\frac{4}{3}\left(\frac{4x-1000}{3}-250\right)$

or $\frac{16x-7000}{9} =$ number of dollars in stock at the end of second year,

$\frac{4}{3}\left(\frac{16x-7000}{9}-250\right) =$ number of dollars in stock at the end of third year.

But $2x =$ number of dollars in stock at the end of third year.

$$\therefore \frac{4}{3}\left(\frac{16x-7000}{9}-250\right)=2x.$$

Whence, $x = 3700.$

70. A cask contains 12 gallons of wine and 18 gallons of water; another cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each cask to produce a mixture containing 7 gallons of wine and 7 gallons of water?

Let $x =$ number of gallons drawn from 1st cask,

$14 - x =$ number of gallons drawn from 2d cask,

$\frac{2}{5} =$ proportion of wine to water in 1st cask,

$\frac{3}{4} =$ proportion of wine to water in 2d cask.

$$\therefore \frac{2x}{5}+\frac{3}{4}(14-x)=7.$$

Whence, $x = 10,$

and $14 - x = 4.$

71. The members of a club subscribe each as many dollars as there are members. If there had been 12 more members, the subscription from each would have been $10 less, to amount to the same sum. How many members were there?

Let $x =$ number of members of the club.

Then $x =$ number of dollars each subscribed,

$x + 12 =$ number of members $+ 12$,

and $x - 10 =$ number of dollars each would have subscribed in second case.

But $x^2 =$ number of dollars all subscribed.

$$\therefore (x + 12)(x - 10) = x^2.$$

Whence, $x = 60$.

72. A number of troops being formed into a solid square, it was found there were 60 men over; but when formed in a column with 5 men more in front than before and three men less in depth, there was lacking one man to complete it. Find the number of troops.

Let $x =$ number of men on one side.

Then $x^2 + 60 =$ number of men in the square,

$x + 5 =$ number of men on a side $+ 5$,

$x - 3 =$ number of men on a side $- 3$,

and $(x + 5)(x - 3) - 1 =$ number of men in the square.

$$\therefore (x + 5)(x - 3) - 1 = x^2 + 60.$$

Whence, $x = 38$,

and $x^2 + 60 = 1504$.

73. An officer can form the men of his regiment into a hollow square 12 deep. The number of men in the regiment is 1296. Find the number of men in the front of the hollow square.

Let $x =$ number of men in front.

Then $12x =$ number of men in twelve lines,

and $24x =$ number of men in twelve lines front and rear.

$12(x - 24) =$ number of men on a side,

$12(x - 24) \times 2 =$ number of men on both sides.

Then $24x + 12(x - 24) \times 2 =$ whole number of men.

But $1296 =$ whole number of men.

$$\therefore 24x + 12(x - 24) \times 2 = 1296.$$

Whence, $x = 39$.

74. A person starts from P and walks towards Q at the rate of 3 miles an hour; 20 minutes later another person starts from Q and walks towards P at the rate of four miles an hour. The distance from P to Q is 20 miles. How far from P will they meet?

Let $x =$ number of miles first travels.

Then $20 - x =$ number of miles second travels,

$\frac{x}{3} =$ number of hours first travels,

$\frac{20 - x}{4} =$ number of hours second travels.

$\therefore \frac{x}{3} = \frac{20 - x}{4} + \frac{1}{3}$. Whence, $x = 9\frac{1}{7}$.

75. A person engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

Let $x =$ number of days he was idle.

Then $a - x =$ number of days he worked,

and $cx =$ number of cents he forfeited,

$b(a - x) =$ number of cents he received,

$(ab - bx) - cx =$ whole amount.

But $d =$ whole amount.

$\therefore (ab - bx) - cx = d.$

Whence, $x = \frac{ab - d}{b + c}$.

76. A banker has two kinds of coins: it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

Let $x =$ number of pieces of first kind.

Then $c - x =$ number of pieces of second kind,

$\frac{1}{a} =$ the part of a dollar in one piece of first,

$\frac{1}{b} =$ the part of a dollar in one piece of second.

$\therefore \frac{x}{a} + \frac{c - x}{b} = 1.$

Whence, $x = \frac{a(b - c)}{b - a}$, and $c - x = \frac{b(c - a)}{b - a}$.

Exercise LXVIII.

1.

$$\begin{aligned} 2x + 3y &= 7 \quad (1) \\ 4x - 5y &= 3 \quad (2) \end{aligned}$$

Multiply (1) by 2,

$$\begin{aligned} 4x + 6y &= 14 \\ \text{(2) is} \quad 4x - 5y &= 3 \\ \text{Subtract,} \quad 11y &= 11 \\ \therefore y &= 1 \end{aligned}$$

Substitute value of y in (2),

$$\begin{aligned} 4x - 5 &= 3. \\ \therefore x &= 2. \end{aligned}$$

2.

$$\begin{aligned} x - 2y &= 4 \quad (1) \\ 2x - y &= 5 \quad (2) \end{aligned}$$

Multiply (1) by 2,

$$\begin{aligned} 2x - 4y &= 8 \\ \text{(2) is} \quad 2x - y &= 5 \\ \text{Subtract,} \quad -3y &= 3 \\ \therefore y &= -1. \end{aligned}$$

Substitute value of y in (2),

$$\begin{aligned} x + 2 &= 4. \\ \therefore x &= 2. \end{aligned}$$

3.

$$\begin{aligned} 7x + 2y &= 30 \quad (1) \\ -3x + y &= 2 \quad (2) \end{aligned}$$

$$\begin{aligned} \text{(1) is} \quad 7x + 2y &= 30 \\ \text{(2) by 2,} \quad -6x + 2y &= 4 \\ \text{Subtract,} \quad 13x &= 26 \\ \therefore x &= 2. \end{aligned}$$

Substitute value of x in (1),

$$\begin{aligned} 14 + 2y &= 30, \\ 2y &= 16. \\ \therefore y &= 8. \end{aligned}$$

4.

$$\begin{aligned} 3x - 5y &= 51 \quad (1) \\ 2x + 7y &= 3 \quad (2) \end{aligned}$$

Multiply (1) by 2, and (2) by 3,

$$\begin{aligned} 6x - 10y &= 102 \\ 6x + 21y &= 9 \\ \text{Subtract,} \quad -31y &= 93 \\ \therefore y &= -3. \end{aligned}$$

Substitute value of y in (1),

$$\begin{aligned} 3x + 15 &= 51, \\ 3x &= 36. \\ \therefore x &= 12. \end{aligned}$$

5.

$$\begin{aligned} 5x + 4y &= 58 \quad (1) \\ 3x + 7y &= 67 \quad (2) \end{aligned}$$

Multiply (1) by 3, and (2) by 5,

$$\begin{aligned} 15x + 12y &= 174 \\ 15x + 35y &= 335 \\ \text{Subtract,} \quad -23y &= -161 \\ \therefore y &= 7. \end{aligned}$$

Substitute value of y in (1),

$$\begin{aligned} 5x + 28 &= 58. \\ \therefore x &= 6. \end{aligned}$$

6.

$$\begin{aligned} 3x + 2y &= 39 \quad (1) \\ 3y - 2x &= 13 \quad (2) \end{aligned}$$

Multiply (1) by 3, and (2) by 2,

$$\begin{aligned} 9x + 6y &= 117 \\ -4x + 6y &= 26 \\ \text{Subtract,} \quad 13x &= 91 \\ \therefore x &= 7. \end{aligned}$$

Substitute value of x in (1),

$$\begin{aligned} 21 + 2y = 39, \quad 2y &= 18. \\ \therefore y &= 9. \end{aligned}$$

7.

$$\begin{aligned} 3x - 4y &= -5 \quad (1) \\ 4x - 5y &= 1 \quad (2) \end{aligned}$$

Multiply (1) by 4 and (2) by 3,

$$\begin{aligned} 12x - 16y &= -20 \\ 12x - 15y &= 3 \\ \text{Subtract,} \quad -y &= -23 \\ \therefore y &= 23. \end{aligned}$$

Substitute value of y in (1),

$$\begin{aligned} 3x - 92 = -5, \quad 3x &= 87. \\ \therefore x &= 29. \end{aligned}$$

8.

$$\begin{aligned} 11x + 3y &= 100 \quad (1) \\ 4x - 7y &= 4 \quad (2) \end{aligned}$$

Multiply (1) by 4 and (2) by 11,

$$\begin{aligned} 44x + 12y &= 400 \\ 44x - 77y &= 44 \\ \text{Subtract,} \quad 89y &= 356 \\ \therefore y &= 4. \end{aligned}$$

Substitute value of y in (1),

$$\begin{aligned} 11x + 12 = 100, \quad 11x &= 88. \\ \therefore x &= 8. \end{aligned}$$

9.

$$\begin{aligned} x + 49y &= 693 \quad (1)\\ 49x + y &= 357 \quad (2) \end{aligned}$$

Add, $50x + 50y = 1050 \quad (3)$

Divide by 50, $x + y = 21 \quad (4)$

Subtract (4) from (1),

$$48y = 672.$$
$$\therefore y = 14.$$

Subtract (4) from (2),

$$48x = 336.$$
$$\therefore x = 7.$$

10.

$$\begin{aligned} 17x + 3y &= 57 \quad (1)\\ -3x + 16y &= 23 \quad (2) \end{aligned}$$

Multiply (1) by 3 and (2) by 17,

$$\begin{aligned} 51x + 9y &= 171\\ -51x + 272y &= 391 \end{aligned}$$

Add, $281y = 562$

$$\therefore y = 2.$$

Substitute value of y in (2),

$$3x + 32 = 23,$$
$$-3x = -9.$$
$$\therefore x = 3.$$

11.

$$\begin{aligned} 12x + 7y &= 176 \quad (1)\\ 3y - 19x &= 3 \quad (2) \end{aligned}$$

Multiply (1) by 3 and (2) by 7,

$$\begin{aligned} 36x + 21y &= 528 \quad (3)\\ -133x + 21y &= 21 \end{aligned}$$

Subt., $169x = 507$

$$\therefore x = 3.$$

Substitute value of x in (2),

$$3y - 57 = 3,$$
$$3y = 60.$$
$$\therefore y = 20.$$

12.

$$\begin{aligned} 2x - 7y &= 8 \quad (1)\\ -9x + 4y &= 19 \quad (2) \end{aligned}$$

Multiply (1) by 4 and (2) by 7,

$$\begin{aligned} 8x - 28y &= 32 \quad (3)\\ -63x + 28y &= 133 \quad (4) \end{aligned}$$

Add, $-55x = 165$

$$\therefore x = -3.$$

Substitute value of x in (2),

$$27 + 4y = 19,$$
$$4y = -8.$$
$$\therefore y = -2.$$

13.

$$\begin{aligned} 69y - 17x &= 103 \quad (1)\\ 14x - 13y &= -41 \quad (2) \end{aligned}$$

Multiply (1) by 14, and (2) by 17,

$$\begin{aligned} -238x + 966y &= 1442 \quad (3)\\ 238x - 221y &= -697 \quad (4) \end{aligned}$$

Add $745y = 745$

$$\therefore y = 1.$$

Substitute value of y in (2),

$$14x - 13 = -41,$$
$$14x = -28.$$
$$\therefore x = -2.$$

14.

$$\begin{aligned} 17x + 30y &= 59 \quad (1)\\ 19x + 28y &= 77 \quad (2) \end{aligned}$$

Multiply (1) by 14, and (2) by 15,

$$\begin{aligned} 238x + 420y &= 826 \quad (3)\\ 285x + 420y &= 1155 \quad (4) \end{aligned}$$

$-47x = -329$

$$\therefore x = 7.$$

Substitute value of x in (2),

$$133 + 28y = 77,$$
$$28y = -56.$$
$$\therefore y = -2.$$

EXERCISE LXIX.

1.
$$3x - 4y = 2 \qquad (1)$$
$$7x - 9y = 7 \qquad (2)$$

Transpose $-4y$ in (1),
$$3x = 2 + 4y.$$
Divide by coefficient of x,
$$x = \frac{2 + 4y}{3}.$$
Substitute value of x in (2),
$$7\left(\frac{2 + 4y}{3}\right) - 9y = 7.$$
Simplify,
$$14 + 28y - 27y = 21.$$
$$\therefore y = 7.$$
Substitute value of y in (1),
$$3x - 28 = 2,$$
$$3x = 30.$$
$$\therefore x = 10.$$

2.
$$7x - 5y = 24 \qquad (1)$$
$$4x - 3y = 11 \qquad (2)$$

Transpose $5y$ in (1),
$$7x = 24 + 5y.$$
Divide by coefficient of x,
$$x = \frac{24 + 5y}{7}.$$
Substitute value of x in (2),
$$4\left(\frac{24 + 5y}{7}\right) - 3y = 11.$$
Simplify,
$$96 + 20y - 21y = 77,$$
$$-y = -19.$$
$$\therefore y = 19.$$
Substitute value of y in (1),
$$7x - 95 = 24,$$
$$7x = 119.$$
$$\therefore x = 17.$$

3.
$$3x + 2y = 32 \qquad (1)$$
$$20x - 3y = 1 \qquad (2)$$

Transpose $2y$ in (1),
$$3x = 32 - 3y.$$
Divide by coefficient of x,
$$x = \frac{32 - 2y}{3}.$$
Substitute value of x in (2),
$$20\left(\frac{32 - 2y}{3}\right) - 3y = 1,$$
$$\frac{640 - 40y}{3} - 3y = 1.$$
Simplify,
$$640 - 40y - 9y = 3,$$
$$-49y = -637.$$
$$\therefore y = 13.$$
Substitute value of y in (1),
$$3x + 26 = 32.$$
$$\therefore x = 2.$$

4.
$$11x - 7y = 37 \qquad (1)$$
$$8x + 9y = 41 \qquad (2)$$

Transpose $7y$ in (1),
$$11x = 37 + 7y.$$
Divide by coefficient of x,
$$x = \frac{37 + 7y}{11}.$$
Substitute value of x in (2),
$$8\left(\frac{37 + 7y}{11}\right) + 9y = 41.$$
Simplify,
$$296 + 56y + 99y = 451,$$
$$155y = 155.$$
$$\therefore y = 1.$$
Substitute value of y in (2),
$$8x + 9 = 41.$$
$$\therefore x = 4.$$

5.
$$7x + 5y = 60 \quad (1)$$
$$13x - 11y = 10 \quad (2)$$

Transpose $5y$ in (1),
$$7x = 60 - 5y.$$

Divide by coefficient of x,
$$x = \frac{60 - 5y}{7}.$$

Substitute value of x in (2),
$$13\left(\frac{60 - 5y}{7}\right) - 11y = 10.$$

Simplify,
$$780 - 65y - 77y = 70,$$
$$780 - 142y = 70,$$
$$142y = 710.$$
$$\therefore y = 5.$$

Substitute value of y in (1),
$$7x + 25 = 60,$$
$$7x = 35.$$
$$\therefore x = 5.$$

7.
$$10x + 9y = 290 \quad (1)$$
$$12x - 11y = 130 \quad (2)$$

Transpose $9y$ in (1),
$$10x = 290 - 9y.$$

Divide by coefficient of x,
$$x = \frac{290 - 9y}{10}.$$

Substitute value of x in (2),
$$12\left(\frac{290 - 9y}{10}\right) - 11y = 130.$$

Simplify,
$$3480 - 108y - 110y = 1300,$$
$$218y = 2180.$$
$$\therefore y = 10.$$

Substitute value of y in (1),
$$10x + 90 = 290,$$
$$10x = 200.$$
$$\therefore x = 20.$$

6.
$$6x - 7y = 42 \quad (1)$$
$$7x - 6y = 75 \quad (2)$$

Transpose $7y$ in (1),
$$6x = 42 + 7y.$$

Divide by coefficient of x,
$$x = \frac{42 + 7y}{6}.$$

Substitute value of x in (2),
$$7\left(\frac{42 + 7y}{6}\right) - 6y = 75.$$

Simplify,
$$294 + 49y - 36y = 450,$$
$$13y = 156.$$
$$\therefore y = 12.$$

Substitute value of y in (1),
$$6x - 84 = 42.$$
$$\therefore x = 21.$$

8.
$$3x - 4y = 18 \quad (1)$$
$$3x + 2y = 0 \quad (2)$$

Transpose $4y$ in (1),
$$3x = 18 + 4y.$$

Divide by coefficient of x,
$$x = \frac{18 + 4y}{3}.$$

Substitute value of x in (2),
$$3\left(\frac{18 + 4y}{3}\right) + 2y = 0.$$

Simplify,
$$54 + 12y + 6y = 0,$$
$$18y = -54.$$
$$\therefore y = -3.$$

Substitute value of y in (2),
$$3x - 6 = 0.$$
$$\therefore x = 2.$$

9. $$9x - 5y = 52 \quad (1)$$
$$8y - 3x = 8 \quad (2)$$

Transpose $5y$ in (1),
$$9x = 52 + 5y.$$

Divide by coefficient of x,
$$x = \frac{52 + 5y}{9}.$$

Substitute value of x in (2),
$$8y - 3\left(\frac{52 + 5y}{9}\right) = 8.$$

Simplify,
$$72y - 156 - 15y = 72,$$
$$57y = 228.$$
$$\therefore y = 4.$$

Substitute value of y in (1),
$$9x - 20 = 52,$$
$$9x = 72.$$
$$\therefore x = 8.$$

10. $$5x - 3y = 4 \quad (1)$$
$$12y - 7x = 10 \quad (2)$$

Transpose $-3y$ in (1),
$$5x = 4 + 3y.$$

Divide by coefficient of x,
$$x = \frac{4 + 3y}{5}.$$

Substitute value of x in (2),
$$12y - 7\left(\frac{4 + 3y}{5}\right) = 10.$$

Simplify,
$$60y - 28 - 21y = 50,$$
$$39y = 78.$$
$$\therefore y = 2.$$

Substitute value of y in (1),
$$5x - 6 = 4,$$
$$5x = 10.$$
$$\therefore x = 2.$$

11. $$9y - 7x = 13 \quad (1)$$
$$15x - 7y = 9 \quad (2)$$

Transpose $-7x$ in (1),
$$9y = 13 + 7x.$$

Divide by coefficient of y,
$$y = \frac{13 + 7x}{9}.$$

Substitute value of y in (2),
$$15x - 7\left(\frac{13 + 7x}{9}\right) = 9.$$

Simplify,
$$135x - 91 - 49x = 81,$$
$$86x = 172.$$
$$\therefore x = 2.$$

Substitute value of x in (1),
$$9y - 14 = 13.$$
$$\therefore y = 3.$$

12. $$5x - 2y = 51 \quad (1)$$
$$19x - 3y = 180 \quad (2)$$

Transpose $2y$ in (1),
$$5x = 51 + 2y.$$

Divide by coefficient of x,
$$x = \frac{51 + 2y}{5}.$$

Substitute value of x in (2),
$$19\left(\frac{51 + 2y}{5}\right) - 3y = 180.$$

Simplify,
$$969 + 38y - 15y = 900,$$
$$23y = -69.$$
$$\therefore y = -3.$$

Substitute value of y in (1),
$$5x + 6 = 51,$$
$$5x = 45.$$
$$\therefore x = 9.$$

13.
$$4x + 9y = 106 \quad (1)$$
$$8x + 17y = 198 \quad (2)$$

Transpose $9y$ in (1),
$$4x = 106 - 9y.$$

Divide by coefficient of x,
$$x = \frac{106 - 9y}{4}.$$

Substitute value of x in (2),
$$8\left(\frac{106 - 9y}{4}\right) + 17y = 198.$$

Simplify,
$$212 - 18y + 17y = 198.$$
$$\therefore y = 14.$$

Substitute value of y in (1),
$$4x + 126 = 106,$$
$$4x = -20.$$
$$\therefore x = -5.$$

14.
$$8x + 3y = 3 \quad (1)$$
$$12x + 9y = 3 \quad (2)$$

Transpose $3y$ in (1),
$$8x = 3 - 3y.$$

Divide by coefficient of x,
$$x = \frac{3 - 3y}{8}.$$

Substitute value of x in (2),
$$12\left(\frac{3 - 3y}{8}\right) + 9y = 3.$$

Simplify,
$$9 - 9y + 18y = 6,$$
$$9y = -3.$$
$$\therefore y = -\tfrac{1}{3}.$$

Substitute value of y in (1),
$$8x - 1 = 3,$$
$$8x = 4.$$
$$\therefore x = \tfrac{1}{2}.$$

Exercise LXX.

1.
$$x + 15y = 53 \quad (1)$$
$$3x + y = 27 \quad (2)$$

Transpose $15y$ in (1) and y in (2),
$$x = 53 - 15y \quad (3)$$
$$3x = 27 - y \quad (4)$$

Divide (4) by 3,
$$x = \frac{27 - y}{3} \quad (5)$$

Equate values of x,
$$53 - 15y = \frac{27 - y}{3} \quad (6)$$

Reduce,
$$159 - 45y = 27 - y,$$
$$44y = 132.$$
$$\therefore y = 3.$$

Substitute value of y in (2),
$$3x + 3 = 27,$$
$$3x = 24.$$
$$\therefore x = 8.$$

2.
$$4x + 9y = 51 \quad (1)$$
$$8x - 13y = 9 \quad (2)$$

Transpose $9y$ in (1), and $-13y$ in (2),
$$4x = 51 - 9y \quad (3)$$
$$8x = 9 + 13y \quad (4)$$

Divide (3) by 4 and (4) by 8,
$$x = \frac{51 - 9y}{4} \quad (5)$$
$$x = \frac{9 + 13y}{8} \quad (6)$$

Equate values of x,
$$\frac{51 - 9y}{4} = \frac{9 + 13y}{8} \quad (7)$$

Reduce,
$$102 - 18y = 9 + 13y.$$
$$\therefore y = 3.$$

Substitute value of y in (1),
$$4x + 27 = 51.$$
$$\therefore x = 6.$$

3.
$$4x + 3y = 48 \quad (1)$$
$$5y - 3x = 22 \quad (2)$$

Transpose $3y$ in (1) and $5y$ in (2),
$$4x = 48 - 3y \quad (3)$$
$$3x = 5y - 22 \quad (4)$$

Divide (3) by 4 and (4) by 3,
$$x = \frac{48 - 3y}{4},$$
$$x = \frac{5y - 22}{3}.$$

Equate values of x,
$$\frac{48 - 3y}{4} = \frac{5y - 22}{3}.$$

Reduce,
$$144 - 9y = 20y - 88,$$
$$-29y = -232.$$
$$\therefore y = 8.$$

Substitute value of y in (1),
$$4x + 24 = 48,$$
$$4x = 24.$$
$$\therefore x = 6.$$

4.
$$2x + 3y = 43 \quad (1)$$
$$10x - y = 7 \quad (2)$$

Transpose $3y$ in (1) and y in (2),
$$2x = 43 - 3y \quad (3)$$
$$10x = 7 + y \quad (4)$$

Divide (3) by 2 and (4) by 10,
$$x = \frac{43 - 3y}{2},$$
$$x = \frac{7 + y}{10}.$$

Equate values of x,
$$\frac{43 - 3y}{2} = \frac{7 + y}{10}.$$

Reduce,
$$215 - 15y = 7 + y,$$
$$-16y = -208.$$
$$\therefore y = 13.$$

Substitute value of y in (1),
$$2x + 39 = 43.$$
$$\therefore x = 2.$$

5.
$$5x - 7y = 33 \quad (1)$$
$$11x + 12y = 100 \quad (2)$$

Transpose $-7y$ in (1) and $12y$ in (2),
$$5x = 33 + 7y \quad (3)$$
$$11x = 100 - 12y \quad (4)$$

Divide (3) by 5 and (4) by 11,
$$x = \frac{33 + 7y}{5},$$
$$x = \frac{100 - 12y}{11}.$$

Equate values of x,
$$\frac{33 + 7y}{5} = \frac{100 - 12y}{11}.$$

Reduce,
$$363 + 77y = 500 - 60y,$$
$$137y = 137.$$
$$\therefore y = 1.$$

Substitute value of y in (1),
$$5x - 7 = 33,$$
$$5x = 40.$$
$$\therefore x = 8.$$

6.
$$5x + 7y = 43 \quad (1)$$
$$11x + 9y = 69 \quad (2)$$

Transpose $9y$ in (2) and $7y$ in (1),
$$5x = 43 - 7y \quad (3)$$
$$11x = 69 - 9y \quad (4)$$

Divide (3) by 5 and (4) by 11,
$$x = \frac{43 - 7y}{5},$$
$$x = \frac{69 - 9y}{11}.$$

Equate values of x,
$$\frac{43 - 7y}{5} = \frac{69 - 9y}{11}.$$

Reduce,
$$473 - 77y = 345 - 45y,$$
$$-32y = -128.$$
$$\therefore y = 4.$$

Substitute value of y in (1),
$$5x + 28 = 43.$$
$$\therefore x = 3.$$

7.
$$8x - 21y = 33 \quad (1)$$
$$6x + 35y = 177 \quad (2)$$

Transpose $21y$ in (1) and $35y$ in (2),
$$8x = 33 + 21y \quad (3)$$
$$6x = 177 - 35y \quad (4)$$

Divide (3) by 8, and (4) by 6,
$$x = \frac{33 + 21y}{8},$$
$$x = \frac{177 - 35y}{6}.$$

Equate values of x,
$$\frac{33 + 21y}{8} = \frac{177 - 35y}{6}.$$

Reduce,
$$99 + 63y = 708 - 140y,$$
$$203y = 609.$$
$$\therefore y = 3.$$

Substitute value of y in (1),
$$8x - 63 = 33,$$
$$8x = 96.$$
$$\therefore x = 12.$$

8.
$$3y - 7x = 4 \quad (1)$$
$$2y + 5x = 22 \quad (2)$$

Transpose $7x$ in (1) and $5x$ in (2),
$$3y = 4 + 7x \quad (3)$$
$$2y = 22 - 5x \quad (4)$$

Divide (3) by 3 and (4) by 2,
$$y = \frac{4 + 7x}{3},$$
$$y = \frac{22 - 5x}{2}.$$

Equate values of y,
$$\frac{4 + 7x}{3} = \frac{22 - 5x}{2}.$$

Reduce,
$$8 + 14x = 66 - 15x,$$
$$29x = 58.$$
$$\therefore x = 2.$$

Substitute value of x in (1),
$$3y - 14 = 4.$$
$$\therefore y = 6.$$

9.
$$21y + 20x = 165 \quad (1)$$
$$77y - 30x = 295 \quad (2)$$

Transpose $20x$ in (1) and $30x$ in (2),
$$21y = 165 - 20x \quad (3)$$
$$77y = 295 + 30x \quad (4)$$

Divide (3) by 21 and (4) by 77,
$$y = \frac{165 - 20x}{21},$$
$$y = \frac{295 + 30x}{77}.$$

Equate values of x,
$$\frac{165 - 20x}{21} = \frac{295 + 30x}{77}.$$

Reduce,
$$1815 - 220x = 885 + 90x,$$
$$-310x = -930.$$
$$\therefore x = 3.$$

Substitute value of x in (1),
$$21y + 60 = 165,$$
$$21y = 105.$$
$$\therefore y = 5.$$

10.
$$11x - 10y = 14 \quad (1)$$
$$5x + 7y = 41 \quad (2)$$

Transpose $-10y$ in (1) and $7y$ in (2),
$$11x = 10y + 14 \quad (3)$$
$$5x = 41 - 7y \quad (4)$$

Divide (3) by 11 and (4) by 5,
$$x = \frac{10y + 14}{11},$$
$$x = \frac{41 - 7y}{5}.$$

Equate values of x,
$$\frac{10y + 14}{11} = \frac{41 - 7y}{5}.$$

Reduce,
$$50y + 70 = 451 - 77y,$$
$$127y = 381.$$
$$\therefore y = 3.$$

Substitute value of y in (1),
$$11x - 30 = 14.$$
$$\therefore x = 4.$$

11.
$$7y - 3x = 139 \quad (1)$$
$$2x + 5y = 91 \quad (2)$$

Transpose $7y$ in (1) and $5y$ in (2),
$$3x = 7y - 139 \quad (3)$$
$$2x = 91 - 5y \quad (4)$$

Divide (3) by 3 and (4) by 2,
$$x = \frac{7y - 139}{3},$$
$$x = \frac{91 - 5y}{2}.$$

Equate values of x,
$$\frac{7y - 139}{3} = \frac{91 - 5y}{2}.$$

Reduce,
$$14y - 278 = 273 - 15y,$$
$$29y = 551.$$
$$\therefore y = 19.$$

Substitute value of y in (4),
$$2x = 91 - 95,$$
$$2x = -4.$$
$$\therefore x = -2.$$

12.
$$17x + 12y = 59 \quad (1)$$
$$19x - 4y = 153 \quad (2)$$

Transpose $12y$ in (1) and $4y$ in (2),
$$17x = 59 - 12y \quad (3)$$
$$19x = 153 + 4y \quad (4)$$

Divide (3) by 17 and (4) by 19,
$$x = \frac{59 - 12y}{17},$$
$$x = \frac{153 + 4y}{19}.$$

Equate values of x,
$$\frac{153 + 4y}{19} = \frac{59 - 12y}{17}.$$

Reduce,
$$2601 + 68y = 1121 - 228y,$$
$$296y = -1480.$$
$$\therefore y = -5.$$

Substitute value of y in (1),
$$17x - 60 = 59.$$
$$\therefore x = 7.$$

13.
$$24x + 7y = 27 \quad (1)$$
$$8x - 33y = 115 \quad (2)$$

Transpose $7y$ in (1) and $33y$ in (2),
$$24x = 27 - 7y \quad (3)$$
$$8x = 115 + 33y \quad (4)$$

Divide (3) by 24 and (4) by 8,
$$x = \frac{27 - 7y}{24},$$
$$x = \frac{115 + 33y}{8}.$$

Equate values of x,
$$\frac{27 - 7y}{24} = \frac{115 + 33y}{8}.$$

Reduce,
$$27 - 7y = 345 + 99y,$$
$$-106y = 318.$$
$$\therefore y = -3.$$

Substitute value of y in (3),
$$24x = 27 + 21,$$
$$24x = 48.$$
$$\therefore x = 2.$$

14.
$$x = 3y - 19 \quad (1)$$
$$y = 3x - 23 \quad (2)$$

Transpose $3x$ and y in (2),
$$3x = 23 + y \quad (3)$$

Divide (3) by 3,
$$x = \frac{23 + y}{3}.$$

Equate values of x,
$$3y - 19 = \frac{23 + y}{3}.$$

Reduce,
$$9y - 57 = 23 + y,$$
$$8y = 80.$$
$$\therefore y = 10.$$

Substitute value of y in (1),
$$x = 30 - 19.$$
$$\therefore x = 11.$$

Exercise LXXI.

1. $$x(y+7)=y(x+1) \quad (1)$$
$$2x+20=3y+1 \quad (2)$$

Simplify (1),
$$xy+7x=xy+y.$$
Transpose and combine,
$$7x-y=0 \quad (3)$$
Transpose and combine (2),
$$2x-3y=-19 \quad (4)$$
Multiply (3) by 3,
$$21x-3y=0$$
(4) is $2x-3y=-19$

Subt., $19x=19$
$$\therefore x=1.$$
Substitute value of x in (3),
$$7-y=0.$$
$$\therefore y=7.$$

2. $$2x-\frac{y-3}{5}-4=0 \quad (1)$$
$$3y+\frac{x-2}{3}-9=0 \quad (2)$$

Simplify (1),
$$10x-y+3-20=0.$$
Transpose and combine,
$$10x-y=17 \quad (3)$$
Simplify (2),
$$9y+x-2-27=0.$$
Transpose and combine,
$$x+9y=29 \quad (4)$$
Multiply (3) by 9,
$$90x-9y=153$$
(4) is $x+9y=29$

Add, $91x=182$
$$\therefore x=2.$$
Substitute value of x in (4),
$$2+9y=29.$$
$$\therefore y=3.$$

3. $$\frac{2}{x+3}=\frac{3}{y-2} \quad (1)$$
$$5(x+3)=3(y-2)+2 \quad (2)$$

Simplify (1),
$$2y-4=3x+9.$$
Transpose and combine,
$$2y-3x=13 \quad (3)$$
Simplify (2),
$$5x+15=3y-6+2.$$
Transpose and combine,
$$5x-3y=-19 \quad (4)$$
Multiply (3) by 3 and (4) by 2,
$$6y-9x=39$$
$$-6y+10x=-38$$
Add, $x=1.$

Substitute value of x in (3),
$$2y-3=13.$$
$$\therefore y=8.$$

4. $$\frac{x-4}{5}-\frac{y+2}{10}=0 \quad (1)$$
$$\frac{x}{6}+\frac{y-2}{4}=3 \quad (2)$$

Simplify (1),
$$2x-8-y-2=0,$$
$$2x-y=10 \quad (3)$$
Simplify (2),
$$2x+3y-6=36,$$
$$2x+3y=42 \quad (4)$$
Subtract (4) from (3),
$$2x-y=10$$
$$2x+3y=42$$
$$-4y=-32$$
$$\therefore y=8.$$
Substitute value of y in (3),
$$2x-8=10.$$
$$\therefore x=9.$$

5.

$$(x+1)(y+2)-(x+2)(y+1)=-1 \qquad (1)$$
$$3(x+3)-4(y+4)=-8 \qquad (2)$$

Simplify, (1), $xy+y+2x+2-xy-2y-x-2=-1.$

Combine, $x-y=-1$ (3)

Simplify (2), $3x+9-4y-16=-8.$

Transpose and unite, $3x-4y=-1$ (4)

Multiply (3) by 3, $3x-3y=-3$

Subtract, $-y=2$

$\therefore y=-2.$

Substitute value of y in (3), $x+2=-1.$

$\therefore x=-3.$

6. $$\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4} \qquad (1)$$

$$\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4} \qquad (2)$$

Simplify (1), $12x-24-200+20x=15y-150.$

Transpose and combine, $32x-15y=74$ (3)

Simplify (2), $16y+32-6x-3y=6x+78.$

Transpose and combine, $-12x+13y=46$ (4)

Multiply (3) by 3, $96x-45y=222$

Multiply (4) by 8, $-96x+104y=368$

Add, $59y=590$

$\therefore y=10.$

Substitute value of y in (3), $32x-150=74,$

$32x=224.$

$\therefore x=7.$

7. $$\frac{x+1}{3}-\frac{y+2}{4}=\frac{2(x-y)}{5} \qquad (1)$$

$$\frac{x-3}{4}-\frac{y-3}{3}=2y-x \qquad (2)$$

Simplify (1), $20x+20-15y-30=24x-24y.$

Transpose and combine, $-4x+9y=10$ (3)

Simplify (2). $3x-9-4y+12=24y-12x.$

Transpose and combine, $15x-28y=-3$ (4)

Multiply (3) by 15, $60x-135y=-150$

Multiply (4) by 4, $60x-112y=-12$

Subtract, $-23y=-138$

$\therefore y=6.$

Substitute value of y in (3), $-4x+54=10,$

$-4x=-44.$

$\therefore x=11.$

8.

$$\frac{3x-2y}{5}+\frac{5x-3y}{3}=x+1 \qquad (1)$$

$$\frac{2x-3y}{3}+\frac{4x-3y}{2}=y+1 \qquad (2)$$

Simplify (1), $9x-6y+25x-15y=15x+15.$

Transpose and combine, $19x-21y=15$ (3)

Simplify (2), $4x-6y+12x-9y=6y+6.$

Transpose and combine, $16x-21y=6$ (4)

Subtract (3) from (4),

$$\begin{array}{r} 16x-21y=6 \\ \underline{19x-21y=15} \\ -3x \qquad =-9 \end{array}$$

$\therefore x=3.$

Substitute value of x in (4), $48-21y=6,$

$-21y=-42.$

$\therefore y=2.$

9.

$$\frac{2x-y+3}{3}-\frac{x-2y+3}{4}=4 \qquad (1)$$

$$\frac{3x-4y+3}{4}+\frac{4x-2y-9}{3}=4 \qquad (2)$$

Simplify (1), $8x-4y+12-3x+6y-9=48.$

Transpose and combine, $5x+2y=45$ (3)

Simplify (2), $9x-12y+9+16x-8y-36=48.$

Transpose and combine, $25x-20y=75$ (4)

Divide (4) by 5, $5x-4y=15$

(3) is $5x+2y=45$

Subtract, $-6y=-30$

$\therefore y=5.$

Substitute value of y in (3), $5x+10=45,$

$5x=35.$

$\therefore x=7.$

10.

$1\frac{1}{2}x=1\frac{1}{3}y+4\frac{5}{12}$ (1)

$4\frac{1}{2}x=\frac{1}{3}y-21\frac{7}{12}$ (2)

Simplify (1), $18x-16y=53$ (3)

Simplify (2), $54x-4y=-259$ (4)

Multiply (3) by 3, $54x-48y=159$

(4) is $54x-4y=-259$

Subtract, $-44y=418$

$\therefore y=-9\frac{1}{2}.$

Substitute value of y in (3), $18x+152=53,$

$18x=-99.$

$\therefore x=-5\frac{1}{2}.$

11. $$\frac{13}{x+2y+3} = -\frac{3}{4x-5y+6} \quad (1)$$

$$\frac{3}{6x-5y+4} = \frac{19}{3x+2y+1} \quad (2)$$

Simplify (1), $55x - 59y = -87$ (3)

Simplify (2), $-105x + 101y = 73$ (4)

Transpose $59y$ in (3) and $101y$ in (4), and divide by 55 and 105 respectively,

$$x = \frac{59y - 87}{55},$$

$$x = \frac{101y - 73}{105}.$$

Equate values of x, $\dfrac{59y-87}{55} = \dfrac{101y-73}{105}.$

Simplify, $1239y - 1827 = 1111y - 803,$

$$128y = 1024.$$

$$\therefore y = 8.$$

Substitute value of y in (3), $55x - 472 = -87,$

$$55x = 385.$$

$$\therefore x = 7.$$

12. $$\frac{x+y}{y-x} = \frac{15}{8} \quad (1)$$

$$9x - \frac{3y+44}{7} = 100 \quad (2)$$

Simplify (1), $8x + 8y = 15y - 15x.$

Transpose and combine, $23x - 7y = 0$ (3)

Simplify (2), $63x - 3y - 44 = 700.$

Transpose and combine, $63x - 3y = 744$ (4)

Multiply (3) by 3, $69x - 21y = 0$

Multiply (4) by 7, $441x - 21y = 5208$

Subtract, $-372x = -5208$

$$\therefore x = 14.$$

Substitute value of x in (3), $322 - 7y = 0,$

$$-7y = -322.$$

$$\therefore y = 46.$$

13.

$$\frac{3x-5y}{2}+3=\frac{2x+y}{5} \quad (1)$$

$$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3} \quad (2)$$

Simplify (1), $15x-25y+30=4x+2y.$

Transpose and combine, $11x-27y=-30 \quad (3)$

Simplify (2), $96-3x+6y=6x+4y.$

Transpose and combine, $-9x+2y=-96 \quad (4)$

Multiply (3) by 9, $99x-243y=-270$

Multiply (4) by -11, $99x-22y=1056$

Subtract, $-221y=-1326$

$\therefore y=6.$

Substitute value of y in (4), $-9x+12=-96.$

$\therefore x=12.$

14. $$\frac{4x-3y-7}{5}=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6} \quad (1)$$

$$\frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}-1=\frac{y-x}{15}+\frac{x}{6}+\frac{1}{10} \quad (2)$$

Simplify (1), $24x-18y-42=9x-4y-25.$

Transpose and combine, $15x-14y=17 \quad (3)$

Simplify (2), $20y-20+30x-9y-60=4y-4x+10x+6.$

Transpose and combine, $24x+7y=86 \quad (4)$

Multiply (4) by 2, $48x+14y=172 \quad (5)$

(3) is $15x-14y=17$

Add (3) and (5), $63x=189$

$\therefore x=3.$

Substitute value of x in (3), $45-14y=17.$

$\therefore y=2.$

15. $$\frac{x-4}{5}=\frac{y+2}{10} \quad (1)$$

$$\frac{x}{6}+\frac{y-2}{4}=3 \quad (2)$$

Simplify (1), $2x-8=y+2.$

Transpose and combine, $2x-y=10 \quad (3)$

Simplify (2), $2x+3y-6=36.$

Transpose and combine, $2x+3y=42 \quad (4)$

$2x-y=10 \quad (3)$

Subtract, $4y=32$

$\therefore y=8.$

Substitute value of y in (4), $2x+24=42.$

$\therefore x=9.$

16. $$\frac{3x+12y}{11}=9 \qquad (1)$$
$$\frac{1-3x}{7}=\frac{11-3y}{5} \qquad (2)$$
Simplify (1) and (2),
$$3x+12y=99 \qquad (3)$$
$$15x-21y=-72 \qquad (4)$$
Divide (3) by 3 and (4) by 15,
$$x=\frac{99-12y}{3}.$$
$$x=\frac{-72+21y}{15}.$$
Equate values of x,
$$\frac{99-12y}{3}=\frac{-72+21y}{15}.$$
Simplify,
$$495-60y=-72+21y,$$
$$-81y=-567.$$
$$\therefore y=7.$$
Substitute value of y in (3),
$$3x+84=99.$$
$$\therefore x=5.$$

17. $$5x-\tfrac{1}{4}(5y+2)=32 \qquad (1)$$
$$3y+\tfrac{1}{3}(x+2)=9 \qquad (2)$$
Simplify (1) and (2),
$$20x-5y=130 \qquad (3)$$
$$x+9y=25 \qquad (4)$$
Multiply (4) by 20,
$$20x+180y=500 \qquad (5)$$
$$20x-5y=130 \qquad (3)$$
Subtract, $185y=370$
$$\therefore y=2.$$
Substitute value of y in (3),
$$20x-10=130.$$
$$\therefore x=7.$$

18. $$3x-0.25y=28 \qquad (1)$$
$$0.12x+0.7y=2.54 \qquad (2)$$
Multiply (1) by 0.04,
$$0.12x-0.01y=1.12 \qquad (3)$$
$$0.12x+0.7\ y=2.54 \qquad (2)$$
Subtract, $-0.71y=-1.42$
$$\therefore y=2.$$
Substitute value of y in (1),
$$3x-0.5=28.$$
$$\therefore x=9.5.$$

19. $$7(x-1)=3(y+8) \qquad (1)$$
$$\frac{4x+2}{9}=\frac{5y+9}{2} \qquad (2)$$
Simplify (1) and (2),
$$7x-7=3y+24,$$
$$7x-3y=31 \qquad (3)$$
$$8x+4=45y+81,$$
$$8x-45y=77 \qquad (4)$$
Multiply (3) by 8 and (4) by 7,
$$56x-24y=248$$
$$56x-315y=539$$
Subtract, $291y=-291$
$$\therefore y=-1.$$
Substitute value of y in (3),
$$7x+3=31.$$
$$\therefore x=4.$$

20. $7x + \frac{1}{5}(2y+4) = 16$ (1)

$3y - \frac{1}{4}(x+2) = 8$ (2)

Simplify (1),

$35x + 2y + 4 = 80.$

Transpose and combine,

$35x + 2y = 76$ (3)

Simplify (2),

$12y - x - 2 = 32.$

Transpose and combine,

$12y - x = 34$ (4)

Multiply (4) by 35,

$$\begin{aligned} -35x + 420y &= 1190 \\ 35x + 2y &= 76 \quad (3) \\ \hline 422y &= 1266 \end{aligned}$$

$\therefore y = 3.$

Substitute value of y in (3),

$35x + 6 = 76,$

$35x = 70.$

$\therefore x = 2.$

21. $\dfrac{5x-6y}{13} + 3x = 4y - 2$ (1)

$\dfrac{5x+6y}{6} - \dfrac{3x-2y}{4} = 2y - 2$ (2)

Simplify (1),

$5x - 6y + 39x = 52y - 26.$

Transpose and combine,

$44x - 58y = -26$ (3)

Simplify (2),

$10x + 12y - 9x + 6y = 24y - 24.$

Transpose and combine,

$x - 6y = -24$ (4)

Multiply (4) by 44,

$$\begin{aligned} 44x - 264y &= -1056 \\ 44x - 58y &= -26 \quad (3) \\ \hline 206y &= 1030 \end{aligned}$$

$\therefore y = 5.$

Substitute value of y in (4),

$x - 30 = -24.$

$\therefore x = 6.$

22.

$$\frac{5x-3}{2} - \frac{3x-19}{2} = 4 - \frac{3y-x}{3} \quad (1)$$

$$\frac{2x+y}{2} - \frac{9x-7}{8} = \frac{3(y+3)}{4} - \frac{4x+5y}{16} \quad (2)$$

Simplify (1), $15x - 9 - 9x + 57 = 24 - 6y + 2x.$

Simplify (2), $16x + 8y - 18x + 14 = 12y + 36 - 4x - 5y.$

Transpose and combine (1), $4x + 6y = -24$ (3)

Transpose and combine (2), $2x + y = 22$ (4)

Divide (1) by 2, $2x + 3y = -12$

$$\begin{aligned} \hline 2y &= -34 \end{aligned}$$

$\therefore y = -17.$

Substitute value of y in (4), $2x - 17 = 22,$

$2x = 39.$

$\therefore x = 19\frac{1}{2}.$

23.

$$3y+11=\frac{4x^2-y(x+3y)}{x-y+4}+31-4x \quad (1)$$

$$(x+7)(y-2)+3=2xy-(y-1)(x+1) \quad (2)$$

Simplify (1), $3xy-3y^2+12y+11x-11y+44$

$$=4x^2-xy-3y^2+31x-31y+124-4x^2+4xy-16x \quad (3)$$

Transpose and combine, $32y-4x=80$ (4)

Divide by 4, $8y-x=20$ (5)

Simplify (2), $xy+7y-2x-14+3=2xy-xy-y+x+1$ (6)

Transpose and combine, $8y-3x=12$

Subtract (5), $8y-\ \ x=20$

$$-2x=-8$$

$$\therefore x=4.$$

Substitute value of x in (5), $8y-4=20,$

$$8y=24.$$

$$\therefore y=3.$$

24.

$$\frac{6x+9}{4}+\frac{3x+5y}{4x-6}=3\tfrac{1}{4}+\frac{3x+4}{2} \quad (1)$$

$$\frac{8y+7}{10}+\frac{6x-3y}{2y-8}=4+\frac{4y-9}{5} \quad (2)$$

Multiply (1) by 4, $6x+9+\dfrac{6x+10y}{2x-3}=13+6x+8.$

Transpose and combine, $\dfrac{6x+10y}{2x-3}=12.$

Divide both sides by (2), $\dfrac{3x+5y}{2x-3}=6.$

Multiply by $2x-3$, $3x+5y=12x-18.$

Transpose and combine, $-9x+5y=-18$ (3)

Multiply (2) by 10 $8y+7+\dfrac{30x-15y}{y-4}=40+8y-18.$

Transpose and combine, $\dfrac{30x-15y}{y-4}=15.$

Divide both sides by 15, $\dfrac{2x-y}{y-4}=1.$

Multiply by $y-4$, $2x-y=y-4.$

Transpose and combine, $2x-2y=-4.$

Divide by 2, $x-y=-2$ (4)

Multiply (3) by 1 and (4) by 9, $-9x+5y=-18$

$$9x-9y=-18$$

Add, $-4y=-36$

$$\therefore y=9.$$

Substitute value of y in (4), $x-9=-2.$

$$\therefore x=7.$$

25. $$x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2} \quad (1)$$

$$y+\frac{y-3}{x-18}=30-\frac{73-3y}{3} \quad (2)$$

Multiply (1) by 2, $$2x-\frac{4y-2x}{23-x}=40-59+2x.$$

Transpose and combine, $$\frac{4y-2x}{23-x}=19.$$

Multiply by $23-x$, $$4y-2x=437-19x.$$

Transpose and combine, $$4y+17x=437 \quad (3)$$

Multiply both sides of (2) by 3, $$3y+\frac{3y-9}{x-18}=90-73+3y.$$

Transpose and combine, $$\frac{3y-9}{x-18}=17.$$

Multiply by $x-18$, $$3y-9=17x-306.$$

Transpose and combine, $$3y-17x=-297 \quad (4)$$

Add (3), $$4y+17x=437$$

$$7y \qquad = 140$$

$$\therefore y=20.$$

Substitute value of y in (3), $$80+17x=437,$$

$$17x=357.$$

$$\therefore x=21.$$

EXERCISE LXXII.

1. $$x+y=a \quad (1)$$

$$x-y=b \quad (2)$$

Add, $$2x \qquad =a+b$$

$$\therefore x=\frac{a+b}{2}.$$

Subtract (2) from (1),

$$2y=a-b.$$

$$\therefore y=\frac{a-b}{2}.$$

2. $$ax+by=c \quad (1)$$

$$px+qy=r \quad (2)$$

Multiply (1) by p and (2) by a,

$$apx+bpy=cp \quad (3)$$

$$apx+aqy=ar \quad (4)$$

Subt., $$y(bp-aq)=cp-ar.$$

$$\therefore y=\frac{cp-ar}{bp-aq}.$$

Multiply (1) by q and (2) by b,

$$aqx+bqy=cq$$

$$bpx+bqy=br$$

Subt., $$(aq-bp)x=cq-br$$

$$\therefore x=\frac{cq-br}{aq-bp}.$$

3.
$$mx + ny = a \qquad (1)$$
$$px + qy = b \qquad (2)$$

Multiply (1) by p and (2) by m,
$$mpx + npy = ap \qquad (3)$$
$$mpx + mqy = mb \qquad (4)$$
Sub., $(np - mq)y = ap - mb$
$$\therefore y = \frac{ap - mb}{np - mq}.$$

Multiply (1) by q and (2) by n,
$$mqx + nqy = aq$$
$$npx + nqy = nb$$
Sub., $(mq - np)x = aq - nb$
$$\therefore x = \frac{aq - nb}{mq - np}.$$

4.
$$ax + by = e \qquad (1)$$
$$ax + cy = d \qquad (2)$$
Subt., $(b - c)y = e - d$
$$\therefore y = \frac{e - d}{b - c}.$$

Multiply (1) by c and (2) by b,
$$acx + bcy = ce$$
$$abx + bcy = bd$$
Subt., $(ac - ab)x = ce - bd$
$$\therefore x = \frac{ce - bd}{a(c - b)}.$$

5.
$$mx - ny = r \qquad (1)$$
$$m'x + n'y = r' \qquad (2)$$

Multiply (1) by m', $mm'x - m'ny = m'r \qquad (3)$

Multiply (2) by m, $mm'x + m\,n'y = mr' \qquad (4)$

Subtract, $(m'n + m\,n')y = mr' - m'r$
$$\therefore y = \frac{mr' - m'r}{m'n + m\,n'}.$$

Multiply (1) by n', $mn'x - nn'y = n'r$

Multiply (2) by n, $m'nx + nn'y = nr'$

Add, $(mn' + m'n)x = n'r + nr'$
$$\therefore x = \frac{n'r + nr'}{mn' + m'n}.$$

6.
$$ax + by = c \qquad (1)$$
$$dx + fy = c^2 \qquad (2)$$

Multiply (1) by d, $adx + bdy = cd$

Multiply (2) by a, $adx + afy = ac^2$

Subtract, $bdy - afy = cd - ac^2$
$$\therefore y = \frac{c(d - ac)}{bd - af}.$$

Multiply (1) by f, $afx + bfy = cf$

Multiply (2) by b, $bdx + bfy = bc^2$

Subtract, $(af - bd)x = cf - bc^2$
$$\therefore x = \frac{c(f - bc)}{af - bd}.$$

7. $$\frac{x}{a}+\frac{y}{b}=c \quad (1)$$
$$\frac{x}{b}+\frac{y}{a}=-c \quad (2)$$
Simplify (1), $bx+ay=abc$ (3)
Simplify (2), $ax+bcy=0$ (4)
Multiply (3) by a and (4) by b,
$$abx+a^2y=a^2bc$$
$$abx+b^2cy=0$$
Subt., $$a^2y-b^2cy=a^2bc$$
$$(a^2-b^2c)y=a^2bc,$$
$$\therefore y=\frac{a^2bc}{a^2-b^2c}$$
Multiply (3) by bc and (4) by a,
$$b^2cx+abcy=ab^2c^2$$
$$a^2x+abcy=0$$
Subt., $$b^2cx-a^2x=ab^2c^2$$
$$(b^2c-a^2)x=ab^2c^2.$$
$$\therefore x=\frac{ab^2c^2}{b^2c-a^2}.$$

8. $$abx+cdy=2 \quad (1)$$
$$ax-cy=\frac{d-b}{bd} \quad (2)$$
Simplify (2),
$$abdx-bcdy=d-b \quad (3)$$
Multiply (1) by b,
$$ab^2x+bcdy=2b \quad (4)$$
(3) is $$abdx-bcdy=d-b \quad (5)$$
Add, $$(ab^2+abd)x=b+d$$
$$\therefore x=\frac{b+d}{ab(b+d)},$$
or, $x=\frac{1}{ab}$.
Multiply (1) by d,
$$abdx+cd^2y=2d$$
(3) is $$abdx-bcdy=d-b$$
Add, $$(cd^2+bcd)y=b+d$$
$$\therefore y=\frac{b+d}{cd(b+d)},$$
or, $y=\frac{1}{cd}$.

9. $$\frac{a}{b+y}=\frac{b}{3a+x} \quad (1)$$
$$ax+2by=d \quad (2)$$
Simplify (1),
$$3a^2+ax=b^2+by.$$
Transpose and combine,
$$ax-by=b^2-3a^2 \quad (3)$$
(2) is $$ax+2by=d$$
Subt., $$-3by=b^2-3a^2-d$$
$$\therefore y=\frac{3a^2-b^2+d}{3b}.$$
Multiply (3) by 2,
$$2ax-2by=2b^2-6a^2$$
(2) is $$ax+2by=d$$
Add, $$3ax=2b^2-6a^2+d$$
$$\therefore x=\frac{2b^2-6a^2+d}{3a}.$$

10. $$\frac{x}{a+b}-\frac{y}{a-b}=\frac{1}{a+b} \quad (1)$$
$$\frac{x}{a+b}+\frac{y}{a-b}=\frac{1}{a-b} \quad (2)$$
Add (1) and (2),
$$\frac{2x}{a+b}=\frac{1}{a+b}+\frac{1}{a-b}.$$
Simplify,
$$2x(a-b)=2a,$$
$$x(a-b)=a.$$
$$\therefore x=\frac{a}{a-b}.$$
Subtract (1) from (2),
$$\frac{2y}{a-b}=\frac{1}{a-b}-\frac{1}{a+b}$$
Simplify,
$$2y(a+b)=2b,$$
$$y(a+b)=b.$$
$$\therefore y=\frac{b}{a+b}.$$

11.

$$a(a-x)=b(x+y-a) \qquad (1)$$
$$a(y-b-x)=b(y-b) \qquad (2)$$

Simplify (1), $a^2-ax=bx+by-ab. \qquad (3)$

Simplify (2), $ay-ab-ax=by-b^2. \qquad (4)$

Transpose a^2 and bx in (3), $ax+bx=a^2+ab-by \qquad (5)$

Transpose $ay-ab$ in (4), $ax=ay-ab-by+b^2. \qquad (6)$

Divide (5) by $(a+b)$ and (6) by a, $x=\dfrac{a^2+ab-by}{a+b}$,

$$x=\frac{ay-ab-by+b^2}{a}.$$

Equate values of x, $\dfrac{a^2+ab-by}{a+b}=\dfrac{ay-ab-by+b^2}{a}.$

Simplify, $a^3+a^2b-aby=a^2y-a^2b-b^2y+b^3,$

$$a^2y+aby-b^2y=a^3+2a^2b-b^3.$$
$$\therefore y=a+b.$$

Substitute value of y in (5), $ax+bx=a^2+ab-ab-b^2.$

$$\therefore x=a-b.$$

12.

$$\frac{x-y+1}{x-y-1}=a \qquad (1)$$
$$\frac{x+y+1}{x+y-1}=b \qquad (2)$$

Simplify (1), $x-y+1=ax-ay-a.$

Simplify (2), $x+y+1=bx+by-b.$

Trans. and combine, $(a-1)x-(a-1)y=a+1 \qquad (3)$

$$(b-1)x+(b-1)y=b+1 \qquad (4)$$

Multiply (3) by $b-1$ and (4) by $(a-1)$,

$$(a-1)(b-1)x-(a-1)(b-1)y=(a+1)(b-1) \qquad (5)$$
$$(a-1)(b-1)x+(a-1)(b-1)y=(a-1)(b+1) \qquad (6)$$

Add, $2(a-1)(b-1)x=2(ab-1)$

$$\therefore x=\frac{ab-1}{(a-1)(b-1)}$$

Subtract (5) from (6), $2(a-1)(b-1)y=2(a-b).$

$$\therefore y=\frac{a-b}{(a-1)(b-1)}.$$

13. $$ax = by + \frac{a^2 + b^2}{2} \qquad (1)$$

$$(a - b)x = (a + b)y \qquad (2)$$

Simplify (1), $$2ax - 2by = a^2 + b^2 \qquad (3)$$

Simplify (2), $$ax - bx - ay - by = 0 \qquad (4)$$

In (3), $$x = \frac{a^2 + b^2 + 2by}{2a}.$$

In (4), $$x = \frac{ay + by}{a - b}.$$

Equate values of x, $$\frac{a^2 + b^2 + 2by}{2a} = \frac{ay + by}{a - b}.$$

Simplify,

$$a^3 + ab^2 + 2aby - a^2b - b^3 - 2b^2y = 2a^2y + 2aby.$$

Transpose and combine, $$2a^2y + 2b^2y = a^3 - a^2b + ab^2 - b^3.$$

$$\therefore y = \frac{a - b}{2}.$$

Substitute value of y in (1), $$ax = \frac{ab - b^2}{2} + \frac{a^2 + b^2}{2},$$

$$ax = \frac{a^2 + ab}{2}.$$

$$\therefore x = \frac{a + b}{2}.$$

14.

$$ax + by = c^2 \qquad (1)$$

$$\frac{a}{b + y} - \frac{b}{a + x} = 0 \qquad (2)$$

Simplify (2), $$ax - by = -a^2 + b^2 \qquad (3)$$

Add (1) and (3),
$$\begin{aligned} ax + by &= c^2 \\ ax - by &= -a^2 + b^2 \\ \hline 2ax &= c^2 - a^2 + b^2 \end{aligned}$$

$$\therefore x = \frac{c^2 - a^2 + b^2}{2a}.$$

Subtract (3) from (1),
$$\begin{aligned} ax + by &= c^2 \\ ax - by &= -a^2 + b^2 \\ \hline 2by &= c^2 + a^2 - b^2 \end{aligned}$$

$$\therefore y = \frac{c^2 + a^2 - b^2}{2b}.$$

15.

$$\frac{x}{a+b}+\frac{y}{a-b}=2a \qquad (1)$$

$$\frac{x-y}{2ab}=\frac{x+y}{a^2+b^2} \qquad (2)$$

Clear (1) of fractions, $ax-bx+ay+by=2a^3-2ab^2 \qquad (3)$

Clear (2) of fractions, $a^2x+b^2x-a^2y-b^2y=2abx+2aby \qquad (4)$

In (3), $y=\dfrac{2a^3-2ab^2-ax+bx}{a+b}.$

In (4), $y=\dfrac{a^2x-2abx+b^2x}{a^2+2ab+b^2}.$

Hence, $\dfrac{2a^3-2ab^2-ax+bx}{a+b}=\dfrac{a^2x-2abx+b^2x}{a^2+2ab+b^2},$

$$2a^4-2a^2b^2-a^2x+2a^3b-2ab^3+b^2x=a^2x-2abx+b^2x.$$

Transpose and combine, $2a^2x-2abx=2a^4-2a^2b^2+2a^3b-2ab^3.$

Divide by $2a$, $ax-bx=a^3-ab^2+a^2b-b^3$

$$\therefore x=\frac{a^3-ab^2+a^2b-b^3}{a-b}$$

or, $x=a^2+2ab+b^2.$

Substitute value of x in (3),

$$a^3+2a^2b+ab^2-a^2b-2ab^2-b^3+ay+by=2a^3-2ab^2.$$

Transpose and combine, $ay+by=a^3-a^2b-ab^2+b^3.$

$$\therefore y=\frac{a^3-a^2b-ab^2+b^3}{a+b},$$

or, $y=a^2-2ab+b^2.$

16.

$$bx-bc=ay-ac \qquad (1)$$

$$x-y=a-b \qquad (2)$$

Transpose (1), $bx-ay=(b-a)c$

Multiply (2) by a, $ax-ay=(a-b)a \qquad (3)$

Subtract, $(b-a)x=c(b-a)+a(b-a)$

$$\therefore x=c+a.$$

$$bx-ay=(b-a)c \qquad (1)$$

Multiply (2) by b, $bx-by=(a-b)b \qquad (4)$

Subtract, $(b-a)y=c(b-a)+b(b-a)$

$$\therefore y=c+b.$$

17.

$$\frac{x-a}{y-b}=c \qquad (1)$$

$$a(x-a)+b(y-b)+abc=0 \qquad (2)$$

Simplify (1), $x-cy=a-bc \qquad (3)$

(2) is $ax+by=a^2+b^2-abc \qquad (4)$

Multiply (3) by a, $ax-acy=a^2-abc$

Subtract, $by+acy=b^2$

$$\therefore y=\frac{b^2}{b+ac}.$$

Multiply (3) by b and (4) by c, $bx-bcy=ab-cb^2$

$acx+bcy=a^2c+b^2c-abc^2$

Add, $bx+acx=ab+a^2c-abc^2$

$$\therefore x=a-\frac{abc^2}{ac+b}$$

18.

$$(a+b)x-(a-b)y=4ab \qquad (1)$$

$$(a-b)x+(a+b)y=2a^2-2b^2 \qquad (2)$$

Multiply (1) by $(a-b)$,

$$(a^2-b^2)x-(a-b)^2y=4a^2b-4ab^2 \qquad (3)$$

Multiply (2) by $(a+b)$,

$$(a^2-b^2)x+(a+b)^2y=2a^3-2ab^2+2a^2b-2b^3 \qquad (4)$$

Subtract (3) from (4), $(2a^2+2b^2)y=2a^3-2a^2b+2ab^2-2b^3.$

$$\therefore y=a-b.$$

Multiply (1) by $(a+b)$ and (2) by $(a-b)$,

$$(a+b)^2x-(a^2-b^2)y=4a^2b+4ab^2$$

$$(a-b)^2x+(a^2-b^2)y=2a^3-2a^2b-2ab^2+2b^3$$

Add, $(2a^2+2b^2)x=2a^3+2a^2b+2ab^2+2b^3$

$$\therefore x=a+b.$$

19.

$$(x+a)(y+b)-(x-a)(y-b)=2(a-b)^2 \qquad (1)$$

$$x-y+2(a-b)=0 \qquad (2)$$

Simplify (1) and (2),

$$xy+bx+ay+ab-xy+ay+bx-ab=2(a-b)^2 \qquad (3)$$

$$x-y+2a-2b=0 \qquad (4)$$

Transpose and combine, $2ay+2bx=2a^2-4ab+2b^2 \qquad (5)$

$$x-y=2b-2a \qquad (6)$$

Divide (5) by 2, $ay+bx=a^2-2ab+b^2$

Multiply (6) by a, $-ay+ax=2ab-2a^2$

Add, $(b+a)x=b^2-a^2$

$$\therefore x=b-a.$$

Substitute value of x in (6), $b-a-y=2b-2a.$

$$\therefore y=a-b.$$

20.

$$(a+b)(x+y)-(a-b)(x-y)=a^2 \quad (1)$$
$$(a-b)(x+y)+(a+b)(x-y)=b^2 \quad (2)$$

Simplify (1), $\quad 2bx+2ay=a^2 \quad (3)$

Simplify (2), $\quad 2ax-2by=b^2. \quad (4)$

Multiply (3) by a, $\quad 2abx+2a^2y=a^3$

Multiply (4) by b, $\quad 2abx-2b^2y=b^3$

Subtract, $\quad (2a^2+2b^2)y=a^3-b^3$

$$\therefore y=\frac{a^3-b^3}{2(a^2+b^2)}.$$

Multiply (3) by b, $\quad 2b^2x+2aby=a^2b$

Multiply (4) by $-a$, $\quad -2a^2x+2aby=-ab^2$

Subtract, $\quad (2a^2+2b^2)x=a^2b+ab^2$

$$\therefore x=\frac{ab(a+b)}{2(a^2+b^2)}.$$

EXERCISE LXXIII.

1.

$$\frac{1}{x}+\frac{2}{y}=10 \quad (1)$$
$$\frac{4}{x}+\frac{3}{y}=20 \quad (2)$$

Multiply (1) by 4,

$$\frac{4}{x}+\frac{8}{y}=40 \quad (3)$$

(2) is $\quad \frac{4}{x}+\frac{3}{y}=20$

Subtract, $\quad \frac{5}{y}=20$

$$\therefore y=\tfrac{1}{4}.$$

Multiply (1) by 3,

$$\frac{3}{x}+\frac{6}{y}=30 \quad (4)$$

(2) by 2, $\quad \frac{8}{x}+\frac{6}{y}=40 \quad (5)$

Subtract, $\quad -\frac{5}{x}=-10$

or, $10x=5$.

$$\therefore x=\tfrac{1}{2}.$$

2.

$$\frac{1}{x}+\frac{2}{y}=a \quad (1)$$
$$\frac{3}{x}+\frac{4}{y}=b \quad (2)$$

Multiply (1) by 3,

$$\frac{3}{x}+\frac{6}{y}=3a \quad (3)$$

(2) is $\quad \frac{3}{x}+\frac{4}{y}=b$

Subtract, $\quad \frac{2}{y}=3a-b$

$$\therefore y=\frac{2}{3a-b}.$$

Multiply (1) by 2,

$$\frac{2}{x}+\frac{4}{y}=2a$$

(2) is $\quad \frac{3}{x}+\frac{4}{y}=b$

Subtract, $\quad \frac{1}{x}=b-2a.$

$$\therefore x=\frac{1}{b-2a}.$$

3.
$$\frac{2}{x}-\frac{5}{3y}=\frac{4}{27} \qquad (1)$$
$$\frac{1}{4x}+\frac{1}{y}=\frac{11}{72} \qquad (2)$$
(1) is $\frac{2}{x}-\frac{5}{3y}=\frac{4}{27}$

$8\times(2)$ is $\frac{2}{x}+\frac{8}{y}=\frac{11}{9} \qquad (3)$

Subtract, $\frac{29}{3y}=\frac{29}{27}$
$$\therefore y=9.$$
Substitute value of y in (1),
$$\frac{2}{x}-\frac{5}{27}=\frac{4}{27},$$
$$\frac{2}{x}=\frac{9}{27}.$$
$$\therefore x=6.$$

4.
$$\frac{1}{x}+\frac{2}{y}=4 \qquad (1)$$
$$\frac{3}{x}-\frac{2}{y}=4 \qquad (2)$$
Multiply (1) by 3,
$$\frac{3}{x}+\frac{6}{y}=12$$
(2) is $\frac{3}{x}-\frac{2}{y}=4$

Subtract, $\frac{8}{y}=8$
$$\therefore y=1.$$
(1) is $\frac{1}{x}+\frac{2}{y}=4$

(2) is $\frac{3}{x}-\frac{2}{y}=4$

Add, $\frac{4}{x}=8$
$$\therefore x=\tfrac{1}{2}.$$

5.
$$\frac{3}{x}-\frac{4}{y}=5 \qquad (1)$$
$$\frac{4}{x}-\frac{5}{y}=6 \qquad (2)$$
Multiply (1) by (4),
$$\frac{12}{x}-\frac{16}{y}=20$$
(2) by 3, $\frac{12}{x}-\frac{15}{y}=18$

Subtract, $-\frac{1}{y}=2$
$$\therefore y=-\tfrac{1}{2}.$$
Substitute value of y in (1),
$$\frac{3}{x}+8=5.$$
$$\therefore x=-1.$$

6.
$$\frac{a}{x}+\frac{b}{y}=\frac{ac}{b} \qquad (1)$$
$$\frac{b}{x}+\frac{a}{y}=\frac{bc}{a} \qquad (2)$$
Multiply (1) by b and (2) by a,
$$\frac{ab}{x}+\frac{b^2}{y}=ac$$
$$\frac{ab}{x}+\frac{a^2}{y}=bc$$
Subtract, $\frac{b^2-a^2}{y}=ac-bc$
$$\therefore y=-\frac{a+b}{c}.$$
Multiply (1) by a and (2) by b,
$$\frac{a^2}{x}+\frac{ab}{y}=\frac{a^2c}{b}$$
$$\frac{b^2}{x}+\frac{ab}{y}=\frac{b^2c}{a}$$
Subtract, $\frac{a^2-b^2}{x}=\frac{a^3c-b^3c}{ab}$
$$\therefore x=\frac{ab(a+b)}{c(a^2+ab+b^2)}.$$

7.

$$\frac{2}{ax}+\frac{3}{by}=5 \qquad (1)$$

$$\frac{5}{ax}-\frac{2}{by}=3 \qquad (2)$$

Multiply (1) by 5, $\frac{10}{ax}+\frac{15}{by}=25$

Multiply (2) by 2, $\frac{10}{ax}-\frac{4}{by}=6$

Subtract, $\frac{19}{by}=19$

$$\therefore y=\frac{1}{b}.$$

Multiply (1) by 2, $\frac{4}{ax}+\frac{6}{by}=10$

Multiply (2) by 3, $\frac{15}{ax}-\frac{6}{by}=9$

Subtract $\frac{19}{ax}=19.$

$$\therefore x=\frac{1}{a}.$$

8.

$$\frac{m}{nx}+\frac{n}{my}=m+n \qquad (1)$$

$$\frac{n}{x}+\frac{m}{y}=m^2+n^2 \qquad (2)$$

Multiply (1) by n, $\frac{mn}{nx}+\frac{n^2}{my}=n(m+n)$

Multiply (2) by $\frac{m}{n}$ $\frac{mn}{nx}+\frac{m^2}{ny}=\frac{m(m^2+n^2)}{n}$

Subtract, $\frac{n^3-m^3}{mny}=\frac{n^2(m+n)-m(m^2+n^2)}{n}$

$$y=\frac{n^3-m^3}{m^2n^2+mn^3-m^4-m^2n^2}.$$

$$\therefore y=\frac{1}{m}.$$

Substitute value of y in (2), $x=\frac{1}{n}.$

9. $\frac{a}{x}+\frac{b}{y}=m$ (1)

$\frac{b}{x}-\frac{a}{y}=n$ (2)

Multiply (1) by b, $\frac{ab}{x}+\frac{b^2}{y}=bm$

Multiply (2) by a, $\frac{ab}{x}-\frac{a^2}{y}=an$

Subtract, $\frac{a^2+b^2}{y}=bm-an$

$\therefore y=\frac{a^2+b^2}{bm-an}$

Multiply (1) by a, $\frac{a^2}{x}+\frac{ab}{y}=am$

Multiply (2) by b, $\frac{b^2}{x}-\frac{ab}{y}=bn$

Add, $\frac{a^2+b^2}{x}=am+bn$

$\therefore x=\frac{a^2+b^2}{am+bn}$

Exercise LXXIV.

1. $5x+3y-6z=4$ (1)
$3x-y+2z=8$ (2)
$x-2y+2z=2$ (3)

(1) is $5x+3y-6z=4$ (1)
$3\times(2)$ is $9x-3y+6z=24$
Add, $14x=28$

$\therefore x=2.$

(1) is $5x+3y-6z=4$ (1)
$3\times(3)$ is $3x-6y+6z=6$
Add, $8x-3y=10$ (4)

Substitute value of x in (4),

$16-3y=10,$
$-3y=-6.$
$\therefore y=2.$

Substitute values of x and y in (3),

$2-4+2z=2,$
$2z=4.$
$\therefore z=2.$

2. $4x-5y+2z=6$ (1)
$2x+3y-z=20$ (2)
$7x-4y+3z=35$ (3)

Multiply (1) by 3 and (3) by 2,

$12x-15y+6z=18$ (4)
$14x-8y+6z=70$

Subt., $-2x-7y=-52$ (5)

Multiply (2) by 3 and (3) by 1,

$6x+9y-3z=60$
$7x-4y+3z=35$

Add, $13x+5y=95$ (6)

Multiply (5) by 5 and (6) by 7,

$10x+35y=260$
$91x+35y=665$

Subt., $-81x=-405$

$\therefore x=5.$

Substitute value of x in (6),

$\therefore y=6.$
$z=8.$

3.
$$\begin{aligned} x + y + z &= 6 \quad (1)\\ 5x + 4y + 3z &= 22 \quad (2)\\ 15x + 10y + 6z &= 53 \quad (3)\end{aligned}$$

$$\begin{array}{ll} \text{(3) is} & 15x+10y+6z = 53 \\ 6\times(1)\text{ is} & \underline{6x+\ 6y+6z = 36} \\ \text{Subtract,} & 9x+4y \qquad = 17 \quad (4) \\ \text{(2) is} & 5x+4y+3z = 22 \\ 3\times(1)\text{ is} & \underline{3x+3y+3z = 18} \\ \text{Subtract,} & 2x+y \qquad = 4 \quad (5) \\ \text{(4) is} & 9x+4y = 17 \\ 4\times(5)\text{ is} & \underline{8x+4y = 16} \\ \text{Subtract,} & x \qquad = 1 \end{array}$$

Substitute value of x in (5),

$$2 + y = 4.$$
$$\therefore y = 2.$$

Substitute values of x and y in (1), $1 + 2 + z = 6.$

$$\therefore z = 3.$$

4.
$$\begin{aligned} 4x - 3y + z &= 9 \quad (1)\\ 9x + y - 5z &= 16 \quad (2)\\ x - 4y + 3z &= 2 \quad (3)\end{aligned}$$

$$\begin{array}{ll} \text{(1) is} & 4x-3y+\ z = 9 \\ 3\times(2)\text{ is} & \underline{27x+3y-15z = 48} \quad (4) \\ \text{Add,} & 31x \qquad -14z = 57 \quad (5) \end{array}$$

Multiply (2) by 4,

$$\begin{array}{ll} & 36x+4y-20z = 64 \quad (6) \\ \text{(3) is} & \underline{x-4y+\ 3z = \ 2} \\ \text{Add,} & 37x \qquad -17z = 66 \quad (7) \end{array}$$

Multiply (5) by 37,

$$\begin{array}{ll} & 1147x-527z = 2046 \\ 31\times(7)\text{ is} & \underline{1147x-518z = 2107} \\ \text{Subtract,} & -9z = -63 \end{array}$$
$$\therefore z = 7.$$

Substitute value of z in (7),

$$37x - 119 = 66,$$
$$37x = 185.$$
$$\therefore x = 5.$$

Substitute values of x and y in (1), $20 - 3y + 7 = 9,$

$$-3y = -18.$$
$$\therefore y = 6.$$

5.
$$\begin{aligned} 8x + 4y - 3z &= 6 \quad (1)\\ x + 3y - z &= 7 \quad (2)\\ 4x - 5y + 4z &= 8 \quad (3)\end{aligned}$$

$$\begin{array}{ll} \text{(1) is} & 8x+4y-3z = \ 6 \\ 3\times(2)\text{is} & \underline{3x+9y-3z = \ 21} \\ \text{Subt.,} & 5x-5y \qquad = -15 \quad (4) \end{array}$$

Multiply (2) by 4,

$$\begin{array}{ll} & 4x+12y-4z = 28 \\ \text{(3) is} & \underline{4x-\ 5y+4z = \ 8} \\ \text{Add,} & 8x+\ 7y \qquad = 36 \quad (5) \end{array}$$

Multiply (4) by 7 and (5) by 5,

$$\begin{array}{ll} & 35x - 35y = -105 \\ & \underline{40x + 35y = \ 180} \\ \text{Add,} & 75x \qquad = \ 75 \end{array}$$
$$\therefore x = 1.$$

Substitute value of x in (4),

$$5 - 5y = -15,$$
$$-5y = -20.$$
$$\therefore y = 4.$$

Substitute values of x and y in (2), $1+12-z = 7.$

$$\therefore z = 6.$$

6.
$$\begin{aligned} 12x + 5y - 4z &= 29 \quad (1)\\ 13x - 2y + 5y &= 58 \quad (2)\\ 17x - y - z &= 15 \quad (3)\end{aligned}$$

$$\begin{array}{ll} \text{(1) is} & 12x+5y-4z = 29 \\ 4\times(3)\text{ is} & \underline{68x-4y-4z = 60} \quad (4) \\ \text{Subt.,} & 56x-9y \qquad = 31 \quad (5) \\ \text{(2) is} & 13x-2y+5z = \ 58 \\ 5\times(3)\text{ is} & \underline{85x-5y-5z = \ 75} \quad (6) \\ \text{Add,} & 98x-7y \qquad = 133 \quad (7) \end{array}$$

Multiply (7) by 9 and (5) by 7,

$$\begin{array}{ll} & 882x - 63y = 1197 \quad (8) \\ & \underline{392x - 63y = \ 217} \quad (9) \\ \text{Subt.,} & 490x \qquad = \ 980 \end{array}$$
$$\therefore x = 2.$$

Substitute value of x in (7),

$$196 - 7y = 133.$$
$$\therefore y = 9.$$

Substitute values of x and y in (1), $24+45-4z = 29.$

$$\therefore z = 10.$$

7.

$$\begin{aligned} x-y-z&=5 \quad (1)\\ x+y-z&=25 \quad (2)\\ x+y+z&=35 \quad (3)\end{aligned}$$

$$\begin{array}{lrl} \text{(1) is} & x-y-z &= 5\\ \text{(3) is} & x+y+z &= 35\\ \hline \text{Add,} & 2x &= 40\end{array}$$

$$\therefore x=20.$$

Substitute value of x in (2) and (3),

$$\begin{array}{lrl} & y-z &= 5\\ & y+z &= 15\\ \hline \text{Add,} & 2y &= 20\end{array}$$

$$\therefore y=10.$$

$$\text{Subtract,} \quad -2z=-10.$$

$$\therefore z=5.$$

8.

$$\begin{aligned} x+y+z&=30 \quad (1)\\ 8x+4y+2z&=50 \quad (2)\\ 27x+9y+3z&=64 \quad (3)\end{aligned}$$

Multiply (1) by 2,

$$\begin{array}{lrll} & 2x+2y+2z &= 60 & (4)\\ \text{(2) is} & 8x+4y+2z &= 50 & \\ \hline \text{Sub.,} & -6x-2y &= 10 & (5)\end{array}$$

Multiply (1) by 3,

$$\begin{array}{lrll} & 3x+3y+3z &= 90 & (6)\\ \text{(3) is} & 27x+9y+3z &= 64 & \\ \hline \text{Sub.,} & -24x-6y &= 26 & (7)\end{array}$$

Multiply (5) by 3,

$$\begin{array}{lrll} & -18x-6y &= 30 & (8)\\ \text{(7) is} & -24x-6y &= 26 & \\ \hline \text{Subtract,} & 6x &= 4 & \end{array}$$

$$\therefore x=\tfrac{2}{3}.$$

Substitute value of x in (8),

$$-12-6y=30.$$
$$\therefore y=-7.$$

Substitute values of x and y in (1), $\quad \tfrac{2}{3}-7+z=30.$

$$\therefore z=36\tfrac{1}{3}.$$

9.

$$\begin{aligned} 15y&=24z-10x+41 \quad (1)\\ 15x&=12y-16z+10 \quad (2)\\ 18x-(7z-13)&=14y \quad (3)\end{aligned}$$

$$\begin{array}{lrll} \text{Multiply (1) by 2,} & 20x+30y-48z &= 82 & (4)\\ \text{Multiply (2) by 3,} & 45x-36y+48z &= 30 & (5)\\ \hline \text{Add,} & 65x-\ \ 6y &= 112 & (6)\end{array}$$

$$\begin{array}{lrll} \text{Multiply (2) by 7,} & 105x-\ \ 84y+112z &= 70 & (7)\\ \text{Multiply (3) by 16,} & 288x-224y-112z &= -208 & (8)\\ \hline \text{Add,} & 393x-308y &= -138 & (9)\end{array}$$

$$\begin{array}{lrl} \text{Multiply (6) by 154,} & 10{,}010x-924y &= 17{,}248\\ \text{Multiply (9) by 3,} & 1{,}179x-924y &= -414\\ \hline \text{Subtract,} & 8{,}831x &= 17{,}662\end{array}$$

$$\therefore x=2.$$

Substitute value of x in (6), $\quad 130-6y=112.$

$$\therefore y=3.$$

Substitute values of x and y in (1),

$$20+45-24z=41.$$
$$\therefore z=1.$$

10.
$$\begin{aligned} 3x - y + z &= 17 \quad (1)\\ 5x + 3y - 2z &= 10 \quad (2)\\ 7x + 4y - 5z &= 3 \quad (3) \end{aligned}$$

Multiply (1) by 2,

$$\begin{array}{lrl} & 6x - 2y + 2z = 34 & (4)\\ \text{(2) is} & \underline{5x + 3y - 2z = 10} & \\ \text{Add,} & 11x + y \quad = 44 & (5) \end{array}$$

Multiply (1) by 5,

$$\begin{array}{lrl} & 15x - 5y + 5z = 85 & (6)\\ \text{(3) is} & \underline{7x + 4y - 5z = 3} & \\ \text{Add,} & 22x - y \quad = 88 & (7)\\ \text{(5) is} & \underline{11x + y \quad = 44} & (5)\\ \text{Add,} & 33x \quad = 132 & \end{array}$$

$$\therefore x = 4.$$

Substitute value of x in (5),

$$44 + y = 44,$$
$$\therefore y = 0.$$

From (1), $z = 5.$

11.
$$\begin{aligned} x + y + z &= 5 \quad (1)\\ 3x - 5y + 7z &= 75 \quad (2)\\ 9x - 11z + 10 &= 0 \quad (3) \end{aligned}$$

Multiply (1) by 5,

$$\begin{array}{lrl} & 5x + 5y + 5z = 25 & (4)\\ \text{(2) is} & \underline{3x - 5y + 7z = 75} & \\ \text{Add,} & 8x \quad + 12z = 100 & (5) \end{array}$$

Multiply (5) by 9 and (3) by 8,

$$\begin{array}{lr} & 72x + 108z = 900\\ & \underline{72x - 88z = -80}\\ \text{Subtract,} & 196z = 980 \end{array}$$

$$\therefore z = 5.$$

Substitute value of z in (3),

$$9x - 55 = -10,$$
$$9x = 45.$$
$$\therefore x = 5.$$

Substitute values of x and z in (1),

$$5 + y + 5 = 5.$$
$$\therefore y = -5.$$

12.
$$\begin{aligned} x + 2y + 3z &= 6 \quad (1)\\ 2x + 4y + 2z &= 8 \quad (2)\\ 3x + 2y + 8z &= 101 \quad (3) \end{aligned}$$

Multiply (1) by 2,

$$\begin{array}{lrl} & 2x + 4y + 6z = 12 & (4)\\ \text{(2) is} & \underline{2x + 4y + 2z = 8} & (5)\\ \text{Subtract,} & 4z = 4 & \end{array}$$

$$\therefore z = 1.$$

$$\begin{array}{lrl} \text{(2) is} & 2x + 4y + 2z = 8 & \\ 2\times\text{(3) is} & \underline{6x + 4y + 16z = 202} & (7)\\ \text{Subt.,} & -4x \quad - 14z = -194 & (8) \end{array}$$

Substitute value of z in (8),

$$-4x - 14 = -194,$$
$$-4x = -180.$$
$$\therefore x = 45.$$

Substitute values of x and z in (1),

$$45 + 2y + 3 = 6,$$
$$2y = -42.$$
$$\therefore y = -21.$$

13.
$$\begin{aligned} x - 3y - 2z &= 1 \quad (1)\\ 2x - 3y + 5z &= -19 \quad (2)\\ 5x + 2y - z &= 12 \quad (3) \end{aligned}$$

Multiply (3) by 2,

$$\begin{array}{lrl} & 10x + 4y - 2z = 24 & (4)\\ \text{(1) is} & \underline{x - 3y - 2z = 1} & \\ \text{Subt.,} & 9x + 7y \quad = 23 & (5) \end{array}$$

Multiply (3) by 5,

$$\begin{array}{lrl} & 25x + 10y - 5z = 60 & (6)\\ \text{(2) is} & \underline{2x - 3y + 5z = -19} & \\ \text{Add,} & 27x + 7y \quad = 41 & (7)\\ \text{(5) is} & \underline{9x + 7y \quad = 23} & \\ \text{Sub.,} & 18x \quad = 18 & \end{array}$$

$$\therefore x = 1.$$

Substitute value of x in (5),

$$9 + 7y = 23.$$
$$\therefore y = 2.$$

Substitute values of x and y in (1),

$$1 - 6 - 2z = 1.$$
$$\therefore z = -3.$$

14.

$$\begin{aligned} 3x - 2y &= 5 & (1)\\ 4x - 3y + 2z &= 11 & (2)\\ x - 2y - 5z &= -7 & (3) \end{aligned}$$

Multiply (2) by 5 and (3) by 2,

$$\begin{aligned} 20x - 15y + 10z &= 55 & (4)\\ 2x - 4y - 10z &= -14 & (5) \end{aligned}$$

Add, $22x - 19y = 41$ (6)

Multiply (1) by 19 and (6) by 2,

$$\begin{aligned} 57x - 38y &= 95 & (7)\\ 44x - 38y &= 82 & (8) \end{aligned}$$

Subtract, $13x = 13$

$\therefore x = 1.$

Substitute value of x in (1),

$$\begin{aligned} 3 - 2y &= 5,\\ -2y &= 2.\\ \therefore y &= -1. \end{aligned}$$

Substitute values of x and y in (2),

$$\begin{aligned} 4 + 3 + 2z &= 11,\\ 2z &= 4.\\ \therefore z &= 2. \end{aligned}$$

15.

$$\begin{aligned} x + y \quad &= 1 & (1)\\ y + z &= 9 & (2)\\ x \quad + z &= 5 & (3) \end{aligned}$$

Add, $2x + 2y + 2z = 15$

$x + y + z = 7\frac{1}{2}$ (4)

Subtract (1) from (4),

$z = 6\frac{1}{2}.$

Subtract (2) from (4),

$x = -1\frac{1}{2}.$

Subtract (3) from (4),

$y = 2\frac{1}{2}.$

16.

$$\begin{aligned} 2x - 3y &= 3 & (1)\\ 3y - 4z &= 7 & (2)\\ -5x + 4z &= 2 & (3) \end{aligned}$$

(1) is $2x - 3y = 3$

(2) is $3y - 4z = 7$

Add, $2x - 4z = 10$

(3) is $-5x + 4z = 2$

Add, $-3x = 12$

$\therefore x = -4.$

Substitute value of x in (1),

$$\begin{aligned} -8 - 3y &= 3,\\ -3y &= 11.\\ \therefore y &= -3\tfrac{2}{3}. \end{aligned}$$

Substitute value of x in (3),

$$\begin{aligned} 20 + 4z &= 2,\\ 4z &= -18.\\ \therefore z &= -4\tfrac{1}{2}. \end{aligned}$$

17.

$$\begin{aligned} 3x - 4y + 6z &= 1 & (1)\\ 2x + 2y - z &= 1 & (2)\\ 7x - 6y + 7z &= 2 & (3) \end{aligned}$$

(1) is $3x - 4y + 6z = 1$

$6 \times (2)$ is $12x + 12y - 6z = 6$ (4)

Add, $15x + 8y = 7$ (5)

Multiply (2) by 7,

$14x + 14y - 7z = 7$ (6)

(3) is $7x - 6y + 7z = 2$

Add, $21x + 8y = 9$ (7)

Subtract (7) from (5),

$$\begin{aligned} -6x &= -2.\\ \therefore x &= \tfrac{1}{3}. \end{aligned}$$

Substitute value of x in (7),

$$\begin{aligned} 7 + 8y &= 9,\\ 8y &= 2.\\ \therefore y &= \tfrac{1}{4}. \end{aligned}$$

Substitute values of x and y in (2),

$$\begin{aligned} \tfrac{2}{3} + \tfrac{1}{2} - z &= 1,\\ -z &= 1 - \tfrac{2}{3} - \tfrac{1}{2}.\\ \therefore z &= \tfrac{1}{6}. \end{aligned}$$

18.

$$7x - 3y = 30 \quad (1)$$
$$9y - 5z = 34 \quad (2)$$
$$x + y + z = 33 \quad (3)$$

Multiply (3) by 7,

$$7x + 7y + 7z = 231$$

(1) is $7x - 3y \quad = 30$

Subtract, $10y + 7z = 201 \quad (5)$

Multiply (2) by 10 and (5) by 9,

$$90y - 50z = 340 \quad (6)$$
$$90y + 63z = 1809 \quad (7)$$

Subtract, $-113z = -1469$

$$\therefore z = 13.$$

Substitute value of z in (5),

$$10y + 91 = 201,$$
$$10y = 110.$$
$$\therefore y = 11.$$

Substitute values of y and z in (3),

$$x + 11 + 13 = 33.$$
$$\therefore x = 9.$$

19.

$$x + \frac{y}{2} + \frac{z}{3} = 6,$$

$$y + \frac{z}{2} + \frac{x}{3} = -1,$$

$$z + \frac{x}{2} + \frac{y}{3} = 17.$$

Simplify,

$$6x + 3y + 2z = 36 \quad (1)$$
$$2x + 6y + 3z = -6 \quad (2)$$
$$3x + 2y + 6z = 102 \quad (3)$$

$3\times(1)$ is $18x+9y+6z = 108$

(3) is $3x+2y+6z = 102$

Sub., $15x+7y \quad = 6 \quad (4)$

$2\times(2)$ is $4x+12y+6z = -12$

(3) is $3x+2y+6z = 102$

Sub., $x+10y \quad = -114 \quad (5)$

Subtract $7\times(5)$ from $10\times(4)$,

$$143x = 858$$
$$\therefore x = 6.$$

Substitute value of x in (5),

$$y = -12.$$

Substitute values of x and y in (1),

$$z = 18.$$

20.

$$\frac{1}{x} + \frac{2}{y} = 5 \quad (1)$$

$$\frac{3}{y} - \frac{4}{z} = -6 \quad (2)$$

$$\frac{3}{z} - \frac{4}{x} = 5 \quad (3)$$

Multiply (1) by 3 and (2) by 2,

$$\frac{3}{x} + \frac{6}{y} = 15 \quad (4)$$

$$-\frac{8}{z} + \frac{6}{y} = -12 \quad (5)$$

Subtract, $\frac{3}{x} + \frac{8}{z} = 27 \quad (6)$

Multiply (6) by 4 and (3) by 3,

$$\frac{12}{x} + \frac{32}{z} = 108$$

$$-\frac{12}{x} + \frac{9}{z} = 15$$

Add $\frac{41}{z} = 123$

$$\therefore z = \tfrac{1}{3}.$$

Substitute value of z in (3),

$$9 - \frac{4}{x} = 5.$$
$$\therefore x = 1.$$

Substitute value of x in (1),

$$1 + \frac{2}{y} = 5.$$
$$\therefore y = \tfrac{1}{2}.$$

21.

$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=a \quad (1)$$

$$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=b \quad (2)$$

$$\frac{1}{y}+\frac{1}{z}-\frac{1}{x}=c \quad (3)$$

Add, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=a+b+c \quad (4)$

(1) is $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=a$

Subtract, $\frac{2}{z}=b+c$

$$\therefore z=\frac{2}{b+c}$$

(4) is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=a+b+c$

(2) is $\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=b$

Subtract, $\frac{2}{y} \quad =a+c$

$$\therefore y=\frac{2}{a+c}$$

(4) is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=a+b+c$

(3) is $-\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c$

Subt., $\frac{2}{x} \quad =a+b$

$$\therefore x=\frac{2}{a+b}.$$

22.

$$bz+cy=a \quad (1)$$

$$az+cx=b \quad (2)$$

$$ay+bx=c \quad (3)$$

Multiply (1) by a, $abz+acy=a^2 \quad (4)$

Multiply (2) by b, $abz+bcx=b^2 \quad (5)$

Multiply (3) by c $acy+bcx=c^2 \quad (6)$

Add (4), (5), and (6), $2abz+2acy+2bcx=a^2+b^2+c^2 \quad (7)$

Subtract twice (4) from (7), $2bcx=b^2+c^2-a^2 \quad (8)$

Subtract twice (5) from (7), $2acy=a^2-b^2+c^2 \quad (9)$

Subtract twice (6) from (7), $2abz=a^2+b^2-c^2 \quad (10)$

In (8), $x=\frac{b^2+c^2-a^2}{2bc}.$

In (9), $y=\frac{a^2-b^2+c^2}{2ac}.$

In (10), $z=\frac{a^2+b^2-c^2}{2ab}.$

23.
$$\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\tfrac{3}{5} \qquad (1)$$
$$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=10\tfrac{1}{6} \qquad (2)$$
$$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\tfrac{1}{10} \qquad (3)$$

Multiply (1) by 60 and (2) by 30,
$$\frac{180}{x}-\frac{48}{y}+\frac{60}{z}=456 \qquad (4)$$
$$\frac{10}{x}+\frac{15}{y}+\frac{60}{z}=305 \qquad (5)$$
Sub., $$\frac{170}{x}-\frac{63}{y}=151 \qquad (6)$$

Multiply (2) by 60 and (3) by 30,
$$\frac{20}{x}+\frac{30}{y}+\frac{120}{z}=610 \qquad (7)$$
$$\frac{24}{x}-\frac{15}{y}+\frac{120}{z}=483 \qquad (8)$$
Sub., $$-\frac{4}{x}+\frac{45}{y}=127 \qquad (9)$$

Multiply (6) by 2 and (9) by 85,
$$\frac{340}{x}-\frac{126}{y}=302 \qquad (10)$$
$$-\frac{340}{x}+\frac{3825}{y}=10795 \qquad (11)$$
Add, $$\frac{3699}{y}=11097$$
$$\therefore y=\tfrac{1}{3}.$$

Substitute value of y in (9),
$$-\frac{4}{x}+135=127,$$
$$8x=4.$$
$$\therefore x=\tfrac{1}{2}.$$

Substitute values of x and y in (5),
$$20+45+\frac{60}{z}=305,$$
$$240z=60.$$
$$\therefore z=\tfrac{1}{4}.$$

24.
$$\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=2.9 \qquad (1)$$
$$\frac{5}{x}-\frac{6}{y}-\frac{7}{z}=-10.4 \qquad (2)$$
$$\frac{9}{y}+\frac{10}{z}-\frac{8}{x}=14.9 \qquad (3)$$
Add, $$-\frac{1}{x}+\frac{7}{z}=7.4 \qquad (4)$$

Multiply (1) by 2,
$$\frac{4}{x}-\frac{6}{y}+\frac{8}{z}=5.8 \qquad (5)$$
(2) is $$\frac{5}{x}-\frac{6}{y}-\frac{7}{z}=-10.4 \qquad (6)$$
Subt., $$-\frac{1}{x}+\frac{15}{z}=16.2 \qquad (7)$$
(4) is $$-\frac{1}{x}+\frac{7}{z}=7.4 \qquad (8)$$
Subtract, $$\frac{8}{z}=8.8$$
$$\therefore z=\tfrac{10}{11}.$$

Substitute value of z in (4),
$$-\frac{1}{x}+7.7=7.4.$$

Simplify, $$-1+7.7x=7.4x,$$
$$3x=10.$$
$$\therefore x=3\tfrac{1}{3}.$$

Substitute values of x and z in (1),
$$0.6-\frac{3}{y}+4.4=2.9.$$

Simplify,
$$0.6y-3+4.4y=2.9y.$$
$$\therefore y=1\tfrac{3}{7}.$$

25.

$$\frac{2}{x}+\frac{1}{y}-\frac{3}{z}=0 \qquad (1)$$

$$\frac{3}{z}-\frac{2}{y}=2 \qquad (2)$$

$$\frac{1}{x}+\frac{1}{z}=1\tfrac{1}{3} \qquad (3)$$

Multiply (1) by 2,

$$\frac{4}{x}+\frac{2}{y}-\frac{6}{z}=0 \qquad (4)$$

(2) is $$-\frac{2}{y}+\frac{3}{z}=2$$

Add, $$\frac{4}{x}\qquad -\frac{3}{z}=2 \qquad (5)$$

(5) is $$\frac{4}{x}-\frac{3}{z}=2$$

Mul. (3) by 3, $$\frac{3}{x}+\frac{3}{z}=4$$

Add, $$\frac{7}{x}\qquad =6$$

$$\therefore x=1\tfrac{1}{6}.$$

Substitute value of x in (5),

$$\frac{24}{7}-\frac{3}{z}=2.$$

$$\therefore z=2\tfrac{1}{10}.$$

Substitute values of x and y in (1),

$$\frac{12}{7}+\frac{1}{y}-\frac{10}{7}=0.$$

$$\therefore y=-3\tfrac{1}{2}.$$

26.

$$ax+by+cz=a \qquad (1)$$

$$ax-by-cz=b \qquad (2)$$

$$ax+cy+bz=c \qquad (3)$$

Add (1) and (2), $$2ax=a+b \qquad (4)$$

$$\therefore x=\frac{a+b}{2a}.$$

Multiply (2) by b and (3) by c,

$$abx \qquad -b^2y \qquad -bcz=b^2 \qquad (5)$$

$$acx \qquad +c^2y+bcz=c^2 \qquad (6)$$

Add, $$abx+acx-b^2y+c^2y \qquad =b^2+c^2 \qquad (7)$$

Substitute value of x in (7),

$$\frac{a^2b+a^2c+ab^2+abc}{2a}-(b^2-c^2)y=b^2+c^2.$$

$$\therefore y=\frac{ab+ac+bc-b^2-2c^2}{2(b^2-c^2)}.$$

Substitute values of x and y in (3),

$$\frac{a+b}{2}+\frac{abc+ac^2+bc^2-b^2c-2c^3}{2(b^2-c^2)}+bz=c,$$

$$bz=\frac{3b^2c-ab^2-abc-b^3}{2(b^2-c^2)}.$$

$$\therefore z=\frac{3bc-ab-ac-b^2}{2(b^2-c^2)}.$$

27.

$$\frac{2x-y}{3}=\frac{3y+2z}{4}=\frac{x-y-z}{5}=4 \qquad (1)$$

Simplify, $40x-20y=45y+30z=12x-12y-12z=240$

$$40x-20y=240 \qquad (1)$$
$$45y+30z=240 \qquad (2)$$
$$12x-12y-12z=240 \qquad (3)$$

Divide (1) by 20, $\quad 2x-y=12 \qquad (4)$
Divide (2) by 15, $\quad 3y+2z=16 \qquad (5)$
Divide (3) by 12, $\quad x-y-z=20 \qquad (6)$

Multiply (4) by 3, $\quad 6x-3y \quad =36 \qquad (7)$
(5) is $\quad 3y+2z=16$
Add, $\quad 6x \quad +2z=52 \qquad (8)$

(4) is $\quad 2x-y \quad = 12$
(6) is $\quad x-y-z= 20$
Subtract, $\quad x \quad +z=-8 \qquad (9)$

(8) ÷ 2 is $\quad 3x+z= 26$
(9) is $\quad x+z=-8$
Subtract, $\quad 2x \quad = 34$

$$\therefore x=17.$$

Substitute value of x in (4), $\quad y=22.$
Substitute value of y in (5), $\quad z=-25.$

28.

$$\frac{x-y}{a}=\frac{x-a-b}{a+b+c} \qquad (1)$$

$$\frac{y-z}{b}=\frac{x-a-b}{a+b+c} \qquad (2)$$

$$\frac{x+z}{c}=\frac{x-a-b}{a+b+c} \qquad (3)$$

$$x(a+b+c)-y(a+b+c)=ax-a^2-ab \qquad (4)$$
$$-z(a+b+c)+y(a+b+c)=bx-b^2-ab \qquad (5)$$
$$x(a+b+c)+z(a+b+c)=cx-ac-bc \qquad (6)$$

Add, $\quad 2x(a+b+c)=x(a+b+c)-a^2-b^2-2ab-ac-bc$

$$x(a+b+c)=-(a^2+b^2+2ab+ac+bc).$$
$$\therefore x=-(a+b).$$

From (4), $\quad -(a+b)(a+b+c)-y(a+b+c)=-2a^2-2ab,$
or, $\quad -y(a+b+c)=-a^2+b^2+ac+bc.$

$$\therefore y=\frac{(a+b)(a-b-c)}{a+b+c}.$$

From (6), $\quad -(a+b)(a+b+c)+z(a+b+c)=-2ac-2bc,$
$\quad z(a+b+c)=a^2+2ab+b^2-ac-bc.$

$$\therefore z=\frac{(a+b-c)(a+b)}{a+b+c}.$$

Exercise LXXV.

1. The sum of two numbers divided by 2 gives as a quotient 24, and the difference between them divided by 2 gives as a quotient 17. What are the numbers?

Let x = first number,
and y = second number.

Then $$\frac{x+y}{2} = 24 \quad (1)$$

and $$\frac{x-y}{2} = 17 \quad (2)$$

Add (1) and (2), $x = 41$
Subtract (2) from (1), $y = 7$.

2. The number 144 is divided into three numbers. When the first is divided by the second, the quotient is 3 and the remainder 2; and when the third is divided by the sum of the other two numbers, the quotient is 2 and the remainder 6. Find the numbers.

Let x = first number,
y = second number,
and z = third number.

Then $$x + y + z = 144 \quad (1)$$

$$\frac{x-2}{y} = 3 \quad (2)$$

and $$\frac{z-6}{x+y} = 2 \quad (3)$$

Simplify (2), $x - 3y = 2$ (4)
Simplify (3), $z - 2y - 2x = 6$ (5)
Multiply (1) by 2, $2x + 2y + 2z = 288$ (6)
Add (5) and (6), $3z = 294$
$\therefore z = 98$.
Substitute value of z in (1), $x + y + 98 = 144$,
$x + y = 46$ (7)
(4) is $x - 3y = 2$
Subtract, $4y = 44$
$\therefore y = 11$.
Substitute value of y in (7), $x + 11 = 46$.
$\therefore x = 35$.

3. Three times the greater of two numbers exceeds twice the less by 10; and twice the greater together with three times the less is 24. Find the numbers.

Let $x =$ greater number,
and $y =$ less number.

Then $3x - 2y = 10$ (1)
and $2x + 3y = 24$ (2)

Multiply (1) by 2, $6x - 4y = 20$
Multiply (2) by 3, $6x + 9y = 72$
Subtract, $-13y = -52$
$\therefore y = 4.$

Substitute value of y in (1), $3x - 8 = 10.$
$\therefore x = 6.$

4. If the smaller of two numbers is divided by the greater, the quotient is 0.21 and the remainder 0.0057; but if the greater be divided by the smaller, the quotient is 4 and the remainder 0.742. What are the numbers?

Let $x =$ larger number,
and $y =$ smaller number.

Then $\frac{y}{x} =$ smaller divided by larger.

$\frac{x}{y} =$ larger divided by smaller.

Hence $$\frac{y}{x} = 0.21 + \frac{0.0057}{x} \quad (1)$$

and $$\frac{x}{y} = 4 + \frac{0.742}{y} \quad (2)$$

Simplify (1), $y = 0.21x + 0.0057$ (3)
$y - 0.21x = 0.0057$

Simplify (2), $x = 4y + 0.742$ (4)
$x - 4y = 0.742$

Multiply (3) by 4, $4y - 0.84x = 0.0228$
(4) is $-4y + x = 0.742$
Add, $0.16x = 0.7648$
$\therefore x = 4.78.$

Substitute value of x in (4),
$-4y = -4.038$
$\therefore y = 1.0095.$

5. Seven years ago the age of a father was four times that of his son; seven years hence the age of the father will be double that of the son. What are their ages?

Let x = number of years in father's age.

Then $x + 7$ = number of years in father's age 7 years hence,

$x - 7$ = number of years in father's age 7 years ago.

Let y = number of years in son's age.

Then $y + 7$ = number of years in son's age 7 years hence,

$y - 7$ = number of years in son's age 7 years ago.

$$x - 7 = 4(y - 7) \qquad (1)$$

$$x + 7 = 2(y + 7) \qquad (2)$$

$$x - 4y = -21 \qquad (3)$$

$$x - 2y = 7 \qquad (4)$$

Subtract,

$$-2y = -28$$

$$\therefore y = 14.$$

Substitute value of y in (4),

$$x - 28 = 7.$$

$$\therefore x = 35.$$

6. The sum of the ages of a father and son is one-half what it will be in 25 years; the difference between their ages is one-third of what the sum will be in 20 years. What are their ages?

Let x = number of years in father's age,

and y = number of years in son's age.

Then $x + y$ = sum of ages,

$x + y + 50$ = sum of ages in twenty-five years.

$$\therefore x + y = \frac{x + y + 50}{2} \qquad (1)$$

$$x - y = \frac{x + y + 40}{3} \qquad (2)$$

Simplify (1), $\quad x + y = 50 \qquad (3)$

Simplify (2), $\quad 2x - 4y = 40 \qquad (4)$

(3) is $\quad x + y = 50$

(4) ÷ 2 is $\quad x - 2y = 20$

Subtract, $\quad 3y = 30$

$$\therefore y = 10.$$

Substitute value of y in (3),

$$x + 10 = 50.$$

$$\therefore x = 40.$$

7. If B give A \$25, they will have equal sums of money; but if A give B \$22, B's money will be double that of A. How much has each?

$$
\begin{aligned}
&\text{Let} && x = \text{number of dollars B has,}\\
&\text{and} && y = \text{number of dollars A has.}\\
&\text{Then} && x - 25 = \text{number of dollars B has after giving \$25 to A,}\\
& && y + 25 = \text{number of dollars A has after receiving \$25.}\\
& && x - 25 = y + 25 \qquad (1)\\
& && y - 22 = \text{number of dollars A has after giving \$25 to B.}\\
& && x + 22 = \text{number of dollars B has after receiving \$22.}\\
& && x + 22 = 2(y - 22) \qquad (2)
\end{aligned}
$$

Transpose and combine,

$$x - y = 50 \qquad (3)$$
$$x - 2y = -66 \qquad (4)$$

Subtract, $y = 116$

Substitute value of y in (3),

$$x - 116 = 50.$$
$$\therefore x = 166.$$

8. A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for \$67.50; to another person he sold 50 bushels of wheat and 30 bushels of barley for \$85. What was the price of the wheat and of the barley per bushel?

Let x = number of dollars received per bushel of wheat,
and y = number of dollars received per bushel of barley.

Then
$$30x + 40y = 67\tfrac{1}{2} \qquad (1)$$
$$50x + 30y = 85 \qquad (2)$$

Simplify (1), $60x + 80y = 135$ (3)

Multiply (2) by $\frac{6}{5}$, $60x + 36y = 102$ (4)

Subtract, $44y = 33$

$$\therefore y = \tfrac{3}{4}.$$

Substitute value of y in (3),

$$60x + 60 = 135,$$
$$60x = 75.$$
$$\therefore x = 1\tfrac{1}{4}.$$

9. If A give B \$5, he will then have \$6 less than B; but if he receive \$5 from B, three times his money will be \$20 more than four times B's. How much has each?

Let $x =$ number of dollars A has,
and $y =$ number of dollars B has.
Then $x - 5 =$ number of dollars A has after giving B \$5,
and $y + 5 =$ number of dollars B has after receiving \$5.

Hence, $x - 5 = y + 5 - 6$,
and $3(x + 5) = 4(y - 5) + 20$.

Transpose,
$$x - y = 4 \qquad (1)$$
$$3x - 4y = -15 \qquad (2)$$
Multiply (1) by 3,
$$3x - 3y = 12 \qquad (3)$$
(2) is
$$3x - 4y = -15$$
$$\therefore y = 27$$

Substitute value of y in (1),
$$x - 27 = 4.$$
$$\therefore x = 31.$$

10. The cost of 12 horses and 14 cows is \$1900; the cost of 5 horses and 3 cows is \$650. What is the cost of a horse and a cow respectively?

Let $x =$ number of dollars a horse costs,
and $y =$ number of dollars a cow costs.

Then
$$12x + 14y = 1900 \qquad (1)$$
and
$$5x + 3y = 650 \qquad (2)$$
Multiply (1) by 3,
$$36x + 42y = 5700 \qquad (3)$$
Multiply (2) by 14,
$$70x + 42y = 9100 \qquad (4)$$
Subtract,
$$34x = 3400$$
$$\therefore x = 100.$$

Substitute value of x in (2),
$$500 + 3y = 650,$$
$$3y = 150.$$
$$\therefore y = 50.$$

11. A certain fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Determine the fraction.

Let $\frac{x}{y}$ = required fraction.

By conditions, $$\frac{x+7}{y} = 2 \qquad (1)$$

and $$\frac{x}{y-1} = 1 \qquad (2)$$

Simplify (1), $x + 7 = 2y$ (3)

Simplify (2), $x = y - 1$ (4)

Transpose (3), $x - 2y = -7$ (5)

Transpose (4), $x - y = -1$ (6)

Subtract, $y = 6$

Substitute value of y in (5),

$$x - 12 = -7.$$

$$\therefore x = 5.$$

$\therefore$ fraction $= \frac{5}{6}$.

12. A certain fraction becomes equal to $\frac{1}{2}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Determine the fraction.

Let $\frac{x}{y}$ = required fraction.

By conditions, $$\frac{x}{y+7} = \frac{1}{2} \qquad (1)$$

and $$\frac{x+13}{y} = 2 \qquad (2)$$

Simplify (1), $2x - y = 7$ (3)

Simplify (2), $x - 2y = -13$ (4)

Multiply (3) by 2,

$$4x - 2y = 14$$

(4) is $x - 2y = -13$

Subtract, $3x = 27$

$$\therefore x = 9.$$

Substitute value of x in (3),

$$18 - y = 7.$$

$$\therefore y = 11.$$

$\therefore$ fraction $= \frac{9}{11}$.

13. **A certain fraction becomes equal to $\frac{7}{9}$ when the denominator is increased by 4, and equal to $\frac{20}{41}$ when the numerator is diminished by 15. Determine the fraction.**

Let $\frac{x}{y}$ = fraction.

Then $$\frac{x}{y+4}=\frac{7}{9} \quad (1)$$

$$\frac{x-15}{y}=\frac{20}{41} \quad (2)$$

Simplify (1), $9x = 7y + 28.$

Simplify (2), $41x - 615 = 20y.$

Transpose, $$9x - 7y = 28 \quad (3)$$
$$41x - 20y = 615 \quad (4)$$

Multiply (3) by 20, $180x - 140y = 560$ (5)

Multiply (4) by 7, $287x - 140y = 4305$ (6)

Subtract, $-107x = -3845$

$\therefore x = 35.$

Substitute value of x in (3), $315 - 7y = 28.$

$\therefore y = 41.$

$\therefore$ fraction $= \frac{35}{41}$.

14. **A certain fraction becomes equal to $\frac{2}{3}$ if 7 is added to the numerator, and equal to $\frac{3}{8}$ if 7 is subtracted from the denominator. Determine the fraction.**

Let $\frac{x}{y}$ = fraction.

Then $$\frac{x+7}{y}=\frac{2}{3} \quad (1)$$

and $$\frac{x}{y-7}=\frac{3}{8} \quad (2)$$

Simplify (1), $3x + 21 = 2y.$

Transpose, $3x - 2y = -21$ (3)

Simplify (2), $8x = 3y - 21.$

Transpose, $8x - 3y = -21$ (4)

Multiply (3) by 3, $9x - 6y = -63$ (5)

Multiply (4) by 2, $16x - 6y = -42$ (6)

Subtract, $-7x = -21$

$\therefore x = 3.$

Substitute value of x in (3), $9 - 2y = -21.$

$\therefore y = 15.$

$\therefore$ fraction $= \frac{3}{15}$.

15. **Find two fractions with numerators 2 and 5 respectively, such that their sum is $1\frac{1}{2}$; and if their denominators are interchanged their sum is 2.**

Let $x =$ denominator of first fraction,
and $y =$ denominator of second fraction.

Then $$\frac{2}{x} + \frac{5}{y} = \frac{3}{2} \quad (1)$$

and $$\frac{2}{y} + \frac{5}{x} = 2 \quad (2)$$

Multiply (1) by 2, $$\frac{4}{x} + \frac{10}{y} = 3 \quad (3)$$

Multiply (2) by 5, $$\frac{25}{x} + \frac{10}{y} = 10 \quad (4)$$

Subtract, $$\frac{21}{x} = 7$$

$$\therefore x = 3.$$

Substitute value of x in (2), $y = 6$.
$\therefore$ first fraction $= \frac{2}{3}$, second fraction $= \frac{5}{6}$.

16. **A fraction which is equal to $\frac{2}{3}$ is increased to $\frac{8}{11}$ when a certain number is added to both its numerator and denominator, and is diminished to $\frac{5}{9}$ when one more than the same number is subtracted from each. Determine the fraction.**

Let x equal numerator, y the denominator, and z the number to be added.

Then $$\frac{x}{y} = \frac{2}{3} \quad (1)$$

$$\frac{x+z}{y+z} = \frac{8}{11} \quad (2)$$

and $$\frac{x-(z+1)}{y-(z+1)} = \frac{5}{9} \quad (3)$$

Clear of fractions and transpose,

$$3x - 2y = 0 \quad (4)$$
$$11x - 8y + 3z = 0 \quad (5)$$
$$9x - 5y - 4z = 4 \quad (6)$$

Multiply (5) by 4, $44x - 32y + 12z = 0$
Multiply (6) by 3, $27x - 15y - 12z = 12$
Add, $71x - 47y = 12 \quad (7)$
Multiply (7) by 3, $213x - 141y = 36$
Multiply (4) by 71, $213x - 142y = 0$
Subtract, $y = 36$
Substitute value of y in (1), $x = 24$.
$\therefore$ fraction $= \frac{24}{36}$.

17. The sum of the two digits of a number is 10, and if 54 be added to the number the digits will be interchanged. What is the number?

Let $x =$ digit in tens' place,

and $y =$ digit in units' place.

Then $10x + y =$ number.

By conditions, $x + y = 10$ (1)

and $10x + y + 54 = 10y + x,$

$$9x - 9y = -54.$$

Divide by 9, $x - y = -6$ (2)

Add (1) and (2), $2x = 4.$

$$\therefore x = 2.$$

Subtract (2) from (1), $2y = 16.$

$$\therefore y = 8.$$

number $= 10x + y.$

$\therefore$ number $= 28.$

18. The sum of the two digits of a number is 6, and if the number be divided by the sum of the digits the quotient is 4. What is the number?

Let $x =$ digit in tens' place,

and $y =$ digit in units' place.

Then $10x + y =$ number,

and $x + y = 6$ (1)

But $\frac{10x + y}{6} = 4$ (2)

Clear of fractions, $10x + y = 24$

$$x + y = 6$$

Subtract, $9x = 18$

$$\therefore x = 2.$$

Substitute value of x in (1),

$$2 + y = 6.$$

$$\therefore y = 4.$$

$\therefore$ number $= 24.$

19. A certain number is expressed by two digits, of which the first is the greater. If the number is divided by the sum of its digits the quotient is 7; if the digits are interchanged, and the resulting number diminished by 12 is divided by the difference between the two digits, the quotient is 9. What is the number?

Let $x =$ digit in tens' place,
and $y =$ digit in units' place.
Then $10x + y =$ number.

By conditions, $$\frac{10x+y}{x+y} = 7 \quad (1)$$

$$\frac{10y+x-12}{x-y} = 9 \quad (2)$$

Simplify (1), $3x - 6y = 0$ (3)

Simplify (2), $-8x + 19y = 12$ (4)

Multiply (3) by $\frac{8}{3}$, $8x - 16y = 0$

Add, $3y = 12$.

$\therefore y = 4$.

Substitute in (3), $x = 8$.

$\therefore$ number $= 84$.

20. If a certain number is divided by the sum of its two digits, the quotient is 6 and the remainder 3; if the digits are interchanged, and the resulting number is divided by the sum of the digits, the quotient is 4 and the remainder 9. What is the number?

Let $x =$ digit in tens' place,
and $y =$ digit in units' place.
Then $10x + y =$ number.

By conditions, $$\frac{10x+y-3}{x+y} = 6 \quad (1)$$

$$\frac{10y+x-9}{x+y} = 4 \quad (2)$$

Clear (1) and (2) of fractions, transpose and combine,

$4x - 5y = 3$ (3)

$-3x + 6y = 9$ (4)

Divide (4) by 3, $-x + 2y = 3$ (5)

Add $4 \times (5)$ and (3), $-4x + 8y = 12$

$4x - 5y = 3$

$3y = 15$

$\therefore y = 5$.

Substitute value of y in (3), $x = 7$.

$\therefore$ number $= 75$.

21. If a certain number is divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits are interchanged, and the resulting number is divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.

Let $x =$ digit in tens' place,
and $y =$ digit in units' place.
Then $10x + y =$ number.

By conditions, $$\frac{10x+y}{x+y-2} = 5 + \frac{1}{x+y-2},$$

and $$\frac{10y+x}{x+y+2} = 5 + \frac{8}{x+y+2}.$$

Clear of fractions, $$10x + y = 5x + 5y - 10 + 1.$$
$$10y + x = 5x + 5y + 10 + 8.$$

Transpose and combine, $$5x - 4y = -9 \quad (1)$$
$$5y - 4x = 18 \quad (2)$$

Multiply (1) by 5, $$25x - 20y = -45$$
Multiply (2) by 4, $$-16x + 20y = 72$$

Add, $$9x = 27$$
$$\therefore x = 3.$$

Substitute value of x in (1), $y = 6$.

$\therefore$ number $= 36$.

22. The first of the two digits of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than five times the sum of the digits. What is the number?

Let $x =$ digit in tens' place,
and $y =$ digit in units' place.
Then $10x + y =$ number.

By conditions, $$2x = y + 3 \quad (1)$$
and $$10x + y + 6 = 5x + 5y \quad (2)$$

Transpose and combine, $$2x - y = 3 \quad (3)$$
$$5x - 4y = -6 \quad (4)$$

Multiply (3) by 4, $$8x - 4y = 12$$
(4) is $$5x - 4y = -6$$

Subtract, $$3x = 18$$
$$\therefore x = 6.$$

Substitute value of x in (3), $y = 9$.

$\therefore$ number $= 69$.

23. A number is expressed by three digits, of which the first and last are alike. By interchanging the digits in the units' and tens' places, the number is increased by 54; but if the digits in the tens' and hundreds' places are interchanged, 9 must be added to four times the resulting number to make it equal to the original number. What is the number?

Let $x =$ digit in hundreds' and units' place,
and $y =$ digit in tens' place.

Then $101x + 10y =$ number.

By conditions, $110x + y = 101x + 10y + 54$ (1)

$4(11x + 100y) + 9 = 101x + 10y$ (2)

Transpose and combine (1), $9x - 9y = 54$ (3)

Divide (3) by 9, $x - y = 6$ (4)

Transpose and combine (2),

$-57x + 390y = -9$

Multiply (4) by 57, $57x - 57y = 342$

Add, $333y = 333$

$\therefore y = 1.$

Substitute value of y in (4), $x = 7.$

$\therefore$ number $= 717.$

24. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

Let $x =$ digit in hundreds' place,
$y =$ digit in tens' place,
and $z =$ digit in units' place.

Then $100x + 10y + z =$ number.

By conditions, $x + y + z = 21$ (1)

and $x + y - z = 3$ (2)

$100x + 10y + z + 198 = 100z + 10y + x$ (3)

Subtract (2) from (1), $2z = 18.$

$\therefore z = 9.$

Divide (3) by 99, $x - z = -2$ (4)

Substitute value of z in (3), $x - 9 = -2.$

$\therefore x = 7.$

Substitute values of x and z in (2), $y = 5.$

$\therefore$ number $= 759.$

25. A number is expressed by three digits. The sum of the digits is 9; the number is equal to forty-two times the sum of the first and second digits; and the third digit is twice the sum of the other two. Find the number.

Let $x =$ digit in hundreds' place,
$y =$ digit in tens' place,
and $z =$ digit in units' place.

Then
$$x + y + z = 9 \quad (1)$$
$$100x + 10y + z = 42(x + y) \quad (2)$$
$$z = 2(x + y) \quad (3)$$

From (2), $58x - 32y + z = 0$
From (3), $-2x - 2y + z = 0$

Subtract, $60x - 30y = 0$

Divide by 30, $2x - y = 0$ (4)
Subtract (3) from (1), $3x + 3y = 9$

Divide by 3, $x + y = 3$ (5)
(4) is $2x - y = 0$

Add, $3x = 3$
$\therefore x = 1.$

Substitute value of x in (5), $y = 2.$
Substitute values of x and y in (1), $z = 6.$
$\therefore$ number $= 126.$

26. A certain number, expressed by three digits, is equal to forty-eight times the sum of its digits. If 198 be subtracted from the number, the digits in the units' and hundreds' places will be interchanged; and the sum of the extreme digits is equal to twice the middle digit. Find the number.

Let $x =$ digit in hundreds' place,
$y =$ digit in tens' place,
and $z =$ digit in units' place.

Then
$$100x + 10y + z = 48(x + y + z) \quad (1)$$
$$100x + 10y + z - 198 = 100z + 10y + x \quad (2)$$
and
$$x + z = 2y \quad (3)$$

From (1), $52x - 38y - 47z = 0$ (4)
From (2), $99x - 99z = 198.$
Divide by 99, $x - z = 2$ (5)
From (3), $x - 2y + z = 0$ (6)
Subtract $19 \times$ (6) from (4), $33x - 66z = 0.$
Divide by 33, $x - 2z = 0$ (7)
Subtract (7) from (5), $z = 2.$
Substitute value of z in (5), $x = 4.$
Substitute values of x and z in (6), $y = 3.$
$\therefore$ number $= 432.$

27. A waterman rows 30 miles and back in 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time he was rowing up and down respectively.

Let $x =$ number of hours he rowed down,
and $y =$ number of hours he rowed up.

By conditions, $x + y = 12$ (1)
and $5x = 3y$ (2)

Transpose (2), $5x - 3y = 0$
Multiply (1) by 3, $3x + 3y = 36$

Add, $8x = 36$
$\therefore x = 4\frac{1}{2}$.

Substitute value of x in (1), $4\frac{1}{2} + y = 12$.
$\therefore y = 7\frac{1}{2}$.

28. A crew, which can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to come up the river as to go down. At what rate does the stream flow?

Let $x =$ rate of pulling,
and $y =$ rate of stream.
$x + y =$ rate down stream,
$x - y =$ rate up stream.

Then $x + y = 12$ (1)
$x - y = 6$ (2)

Subtract, $2y = 6$
$\therefore y = 3 =$ rate stream flows.

Substitute value of y in (1),
$x + 3 = 12$,
$x = 12 - 3$.
$\therefore x = 9$.

29. A man sculls down a stream, which runs at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he pulled down the stream and the rate of his pulling.

Let $\qquad x =$ rate the man sculls,

and $\qquad y =$ number of miles he goes.

Then $\qquad x + 4 =$ rate going down the stream,

and $\qquad x - 4 =$ rate going up the stream.

$(x + 4)\frac{5}{3} =$ number of miles he goes.

$$\therefore (x + 4)\tfrac{5}{3} = y \qquad (1)$$

and

$$(x - 4)\tfrac{17}{4} = y - 3 \qquad (2)$$

$4 \times (1)$ is $\qquad 20x + 80 = 12y \qquad (3)$

$3 \times (2)$ is $\qquad 51x - 204 = 12y - 36 \qquad (4)$

Subtract, $\qquad -31x + 284 = 36$

$$\therefore x = 8, \text{ rate of pulling.}$$

Substitute value of x in (1), $\quad y = 20$.

30. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream; and also the rate of the stream.

Let $\qquad x =$ rate of rowing,

and $\qquad y =$ rate of stream.

Then

$$\frac{2}{x-y} = \frac{3}{x+y} \qquad (1)$$

$$\frac{20}{x+y} + \frac{20}{x-y} = 10 \qquad (2)$$

Simplify (1), $\qquad x = 5y \qquad (3)$

Substitute this value of x in (2),

$$\frac{20}{6y} + \frac{20}{4y} = 10.$$

$$\therefore y = \tfrac{5}{6}.$$

From (3), $\qquad x = 4\frac{1}{6}$.

Therefore,

$$\frac{20}{4\frac{1}{6} + \frac{5}{6}} = 4 \text{ (time of running down),}$$

and

$$\frac{20}{4\frac{1}{6} - \frac{5}{6}} = 6 \text{ (time of rowing up).}$$

31. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound, he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put into the mixture?

Let $x =$ number of pounds of tea at 42 cents,

and $y =$ number of pounds of tea at 54 cents.

Then
$$x + y = 30 \quad (1)$$
$$42x + 54y = 1800 - 420 \quad (2)$$

Multiply (1) by 42,
$$42x + 42y = 1260$$
$$42x + 54y = 1380$$
$$12y = 120$$
$$\therefore y = 10.$$
$$x + 10 = 30.$$
$$\therefore x = 20.$$

32. A grocer mixes tea that cost him 90 cents a pound with tea that cost him 28 cents a pound. The cost of the mixture is \$61.20. He sells the mixture at 50 cents a pound, and gains \$3.80. How many pounds of each did he put into the mixture?

Let $x =$ number of pounds of tea at 90 cents,

and $y =$ number of pounds of tea at 28 cents.

$28y =$ number of cents second kind cost.

Then $90x + 28y =$ number of cents whole cost,

$x + y =$ number of pounds in whole mixture,

and $50(x + y) =$ number of cents received.

Hence,
$$50x + 50y = 6500 \quad (1)$$
$$90x + 28y = 6120 \quad (2)$$

Multiply (1) by $\frac{14}{25}$,
$$28x + 28y = 3640 \quad (3)$$

Subtract,
$$62x = 2480$$
$$\therefore x = 40.$$

Substitute value of x in (1), $y = 90.$

33. A farmer has 28 bushels of barley worth 84 cents a bushel. With his barley he wishes to mix rye worth \$1.08 a bushel, and wheat worth \$1.44 a bushel, so that the mixture may be 100 bushels, and be worth \$1.20 a bushel. How many bushels of rye and of wheat must he take?

$$
\begin{array}{lrl}
\text{Let} & x = & \text{number of bushels of wheat,}\\
\text{and} & y = & \text{number of bushels of rye.}\\
\text{Then} & 2352 = & \text{cost in cents of barley,}\\
 & 144x = & \text{cost in cents of wheat,}\\
 & 108y = & \text{cost in cents of rye,}\\
\text{and} & 12{,}000 = & \text{cost in cents of mixture.}
\end{array}
$$

$$
\begin{array}{lrr}
 & x + y + 28 = 100, & \\
 & x + y = 72 & (1)\\
 & 144x + 108y + 2352 = 12000, & \\
 & 144x + 108y = 9648 & (2)\\
\text{Divide (2) by 36,} & 4x + 3y = 268 & (3)\\
\text{Multiply (1) by 3,} & 3x + 3y = 216 & (4)\\
\text{Subtract,} & x \qquad = 52 & \\
\text{Substitute value of } x \text{ in (1),} & y = 20. &
\end{array}
$$

34. A and B together earn \$40 in 6 days; A and C together earn \$54 in 9 days; B and C together earn \$80 in 15 days. What does each earn a day?

$$
\begin{array}{lrl}
\text{Let} & x = & \text{number of dollars A earns in one day,}\\
 & y = & \text{number of dollars B earns in one day,}\\
\text{and} & z = & \text{number of dollars C earns in one day.}
\end{array}
$$

$$
\begin{array}{lrr}
\text{Then} & x + y = \frac{40}{6} & (1)\\
 & x + z = 6 & (2)\\
\text{and} & y + z = \frac{80}{15} & (3)\\
\text{Simplify (1),} & 3x + 3y \qquad = 20 & (4)\\
\text{Simplify (3),} & 3y + 3z = 16 & (5)\\
\text{Subtract,} & 3x \qquad - 3z = 4 & (6)\\
\text{Multiply (2) by 3,} & 3x \qquad + 3z = 18 & (7)\\
\text{Add,} & 6x \qquad = 22 & \\
 & \therefore x = 3\tfrac{2}{3}. & \\
\text{Substitute value of } x \text{ in (1),} & y = 3. & \\
\text{Substitute value of } x \text{ in (2),} & z = 2\tfrac{1}{3}. &
\end{array}
$$

35. A cistern has three pipes, A, B, and C. A and B will fill it in 1 hour and 10 minutes; A and C in one hour and 24 minutes; B and C in 2 hours and 20 minutes. How long will it take each to fill it?

Let $x =$ number of minutes it takes A to fill it,

$y =$ number of minutes it takes B to fill it,

and $z =$ number of minutes it takes C to fill it.

$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, =$ parts A, B, C can fill in one minute.

Hence,
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{70} \quad (1)$$
$$\frac{1}{x}+\frac{1}{z}=\frac{1}{84} \quad (2)$$
$$\frac{1}{y}+\frac{1}{z}=\frac{1}{140} \quad (3)$$

Add, and divide by 2,
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{60}$$

Subtract (1), $z = 420.$

Subtract (2), $y = 210.$

Subtract (3), $x = 105.$

36. A warehouse will hold 24 boxes and 20 bales; 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?

Let $x =$ number of boxes it will hold,

and $y =$ number of bales it will hold.

Then $\frac{1}{x}, \frac{1}{y} =$ parts one box, one bale, can fill.

Hence,
$$\frac{24}{x}+\frac{20}{y}=1 \quad (1)$$

and
$$\frac{6}{x}+\frac{14}{y}=\frac{1}{2} \quad (2)$$

$4 \times (2) - (1)$
$$\frac{36}{y}=1.$$
$$\therefore y = 36.$$

Substitute value of y in (2), $x = 54.$

37. Two workmen together complete some work in 20 days; but if the first had worked twice as fast, and the second half as fast, they would have finished it in 15 days. How long would it take each alone to do the work?

Let $x =$ number of days it would take the first alone,

and $y =$ number of days it would take the second alone.

Then $\frac{1}{x}, \frac{1}{y} =$ parts they can do in one day,

$\frac{1}{x} + \frac{1}{y} =$ part both could do in one day,

and $\frac{2}{x} + \frac{1}{2y} =$ part they could do if first worked twice as fast, and second worked half as fast.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{20} \qquad (1)$$

and
$$\frac{2}{x} + \frac{1}{2y} = \frac{1}{15} \qquad (2)$$

(1) − ½ of (2) is
$$\frac{3}{4y} = \frac{1}{60}.$$
$$\therefore y = 45.$$

Substitute value of y in (1), $x = 36.$

38. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many of each alone will it hold?

Let $x =$ number of crowns bag holds,

$y =$ number of guineas bag holds.

Then $\frac{1}{x} =$ part of bag 1 crown occupies,

$\frac{1}{y} =$ part of bag 1 guinea occupies.

$$\therefore \frac{19}{x} + \frac{6}{y} = 1 \qquad (1)$$

and
$$\frac{4}{x} + \frac{5}{y} = \frac{17}{63} \qquad (2)$$

$5 \times (1) - 6 \times (2)$ is
$$\frac{71}{x} = \frac{213}{63}$$
$$\therefore x = 21.$$

Substitute value of x in (1), $\frac{19}{21} + \frac{6}{y} = 1.$

$$\therefore y = 63.$$

39. A piece of work can be completed by A, B, and C together in 10 days; by A and B together in 12 days; by B and C, if B work 15 days and C 30 days. How long will it take each alone to do the work?

Let x = number of days it takes A,

y = number of days it takes B,

and z = number of days it takes C.

Then $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, respectively, = part each can do in one day.

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10} \qquad (1)$$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{12} \qquad (2)$$

and $$\frac{15}{y}+\frac{30}{z}=1 \qquad (3)$$

Subtract (2) from (1), $$\frac{1}{z}=\frac{1}{60}.$$

$$\therefore z=60.$$

Substitute value of z in (3), $$\frac{15}{y}+\frac{1}{2}=1.$$

$$\therefore y=30.$$

Substitute value of y in (2), $$\frac{1}{x}+\frac{1}{30}=\frac{1}{12},$$

$$\frac{1}{x}=\frac{1}{20}.$$

$$\therefore x=20.$$

40. A cistern has three pipes, A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

Let x = number of minutes it takes A,

y = number of minutes it takes B,

and z = number of minutes it takes C.

Then $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, respectively, = part each fills in one minute,

and $\frac{1}{x}+\frac{1}{y}$ = part A and B fill in one minute.

But $\frac{1}{a}$ = part A and B fill in one minute.

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{1}{a} \quad (1)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{b} \quad (2)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{c} \quad (3)$$

Add, and divide by 2, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{bc+ac+ab}{2abc}$ (4)

Subtract (1) from (4), $\frac{1}{z}=\frac{ac+ab-bc}{2abc}$.

$$\therefore z=\frac{2abc}{ac+ab-bc}.$$

Subtract (2) from (4), $\frac{1}{y}=\frac{ab-ac+bc}{2abc}$.

$$\therefore y=\frac{2abc}{ab-ac+bc}.$$

Subtract (3) from (4), $\frac{1}{x}=\frac{ac-ab+bc}{2abc}$.

$$\therefore x=\frac{2abc}{ac-ab+bc}.$$

41. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 4 per cent; the income from his 5 per cent investment is \$50 more than from his 4 per cent. How much has he in each investment?

Let x = number of dollars invested at 5%,
and y = number of dollars invested at 4%.
Then $x+y$ = total number of dollars invested.

$$\therefore x+y=10,000 \quad (1)$$

As he receives 5% on x dollars, $\frac{5x}{100}$ = interest at 5%.

As he receives 4% on y dollars, $\frac{4y}{100}$ = interest at 4%.

But interest at 5% is \$50 more than that at 4%.

$$\therefore \frac{5x}{100}-\frac{4y}{100}=50 \quad (2)$$

Simplify (2), $5x-4y=5000$
Multiply (1) by 5, $5x+5y=50000$
Subtract, $-9y=-45000$

$$\therefore y=5000.$$

Substitute value of y in (1), $x=5000$.

42. A sum of money at simple interest amounted in 6 years to $26,000, and in 10 years to $30,000. Find the sum and the rate of interest.

Let $x =$ number of dollars at interest,

and $y =$ rate of interest.

Then $\frac{xy}{100} =$ interest on x dollars for one year,

$\frac{6xy}{100} =$ interest on x dollars for six years,

and $\frac{10xy}{100} =$ interest on x dollars for ten years.

$$\therefore \frac{6xy}{100} + x = 26{,}000 \qquad (1)$$

and

$$\frac{10xy}{100} + x = 30{,}000 \qquad (2)$$

Multiply (1) by 5, $\frac{30xy}{100} + 5x = 130{,}000$

Multiply (2) by 3, $\frac{30xy}{100} + 3x = 90{,}000$

Subtract, $2x = 40{,}000$

$\therefore x = 20{,}000.$

Substitute value of x in (1), $y = 5.$

43. A sum of money at simple interest amounted in 10 months to $26,250, and in 18 months to $27,250. Find the sum and the rate of interest.

Let $x =$ sum,

and $y =$ rate of interest.

Then $\frac{xy}{100} =$ interest for one year.

Since 10 months equals $\frac{5}{6}$ of a year,

$\frac{5}{6}$ of $\frac{xy}{100} =$ interest for 10 months,

and $\frac{3}{2}$ of $\frac{xy}{100} =$ interest for 18 months.

But $26{,}250 - x =$ interest for 10 months,

and $27{,}250 - x =$ interest for eighteen months.

$$\therefore \frac{5xy}{600} = 26{,}250 - x \qquad (1)$$

and $$\frac{3xy}{200} = 27{,}250 - x \qquad (2)$$

Multiply (1) by $\frac{9}{5}$, $$\frac{3xy}{200} = \frac{236{,}250 - 9x}{5}$$

Subtract, $$0 = \frac{100{,}000 - 4x}{5}$$

$$4x = 100{,}000.$$

$$\therefore x = 25{,}000.$$

Substitute value of x in (2), $$y = 6.$$

44. A sum of money at simple interest amounted in m years to a dollars, and in n years to b dollars. Find the sum and the rate of interest.

Let $x =$ sum,

and $y =$ rate of interest.

Then $\frac{mxy}{100} =$ interest on sum for m years,

and $\frac{nxy}{100} =$ interest on sum for n years.

$$\therefore \frac{mxy}{100} + x = a \qquad (1)$$

and $$\frac{nxy}{100} + x = b \qquad (2)$$

Multiply (1) and (2) by 100,

$$mxy + 100x = 100a \qquad (3)$$

$$nxy + 100x = 100b \qquad (4)$$

Multiply (3) by n, $$mnxy + 100nx = 100an$$

Multiply (4) by m, $$mnxy + 100mx = 100bm$$

Subtract, $$100nx - 100mx = 100an - 100bm$$

Divide by 100, $$nx - mx = an - bm,$$

$$\therefore x = \frac{an - bm}{n - m}.$$

Substitute value of x in (3),

$$\frac{many - m^2by}{n - m} + \frac{100an - 100bm}{n - m} = 100a.$$

Multiply by $n - m$,

$$many - m^2by + 100an - 100bm = 100an - 100am,$$

$$many - m^2by = 100bm - 100am,$$

$$any - mby = 100b - 100a.$$

$$\therefore y = \frac{100(b - a)}{an - bm}$$

45. A sum of money at simple interest amounted in a months to c dollars, and in b months to d dollars. Find the sum and the rate of interest.

Let $x = \text{sum},$

and $y = \text{rate of interest}.$

Then $\frac{xy}{100} = \text{interest of \$}x\text{ for one year},$

$$x + \frac{axy}{1200} = \text{amount of \$}x\text{ for } a \text{ months},$$

$$x + \frac{bxy}{1200} = \text{amount of \$}x\text{ for } b \text{ months}.$$

$$\frac{axy}{1200} + x = c \qquad (1)$$

$$\frac{bxy}{1200} + x = d \qquad (2)$$

Simplify (1), $axy + 1200x = 1200c \qquad (3)$

Simplify (2), $bxy + 1200x = 1200d \qquad (4)$

Multiply (3) by b and (4) by a,

$$abxy + 1200bx = 1200bc \qquad (5)$$

$$abxy + 1200ax = 1200ad \qquad (6)$$

(6) – (5) is $1200x(a-b) = 1200(ad-bc)$

$$\therefore x = \frac{ad-bc}{a-b}.$$

Substitute value of x in (3),

$$ay\left(\frac{ad-bc}{a-b}\right) + 1200\left(\frac{ad-bc}{a-b}\right) = 1200c.$$

Simplify, $ay(ad-bc) + 1200(ad-bc) = 1200(ac-bc).$

Transpose and unite, $ay(ad-bc) = 1200a(c-d).$

$$\therefore y = \frac{1200(c-d)}{ad-bc}.$$

46. A person has a certain capital invested at a certain rate per cent. Another person has \$1000 more capital, and his capital invested at one per cent better than the first, and receives an income \$80 greater. A third person has \$1500 more capital, and his capital invested at two per cent better than the first, and receives an income \$150 greater. Find the capital of each, and the rate at which it is invested.

Let $x =$ capital,
and $y =$ rate.

Then $\frac{(x+1000)(y+1)}{100} =$ interest on capital \$1000 greater, at 1% greater,

and $\frac{(x+1500)(y+2)}{100} =$ interest on capital \$1500 greater, at 2% greater.

$\frac{xy}{100} + 80 =$ interest on first capital, increased by \$80,

$\frac{xy}{100} + 150 =$ interest on first capital, increased by \$150.

$$\frac{(x+1000)(y+1)}{100} = \frac{xy}{100} + 80 \qquad (1)$$

$$\frac{(x+1500)(y+2)}{100} = \frac{xy}{100} + 150 \qquad (2)$$

Simplify, $xy + 1000y + x + 1000 = xy + 8000,$
$xy + 1500y + 2x + 3000 = xy + 15000$

Combining, $1000y + x = 7000 \qquad (3)$
$1500y + 2x = 12000 \qquad (4)$

Multiply (3) by 2, $2000y + 2x = 14000 \qquad (5)$
(4) is $1500y + 2x = 12000$

Subtract, $500y = 2000$
$\therefore y = 4.$

Substitute value of y in (5), $x = 3000.$

$\therefore$ the capitals are \$3000, \$4000, \$4500; and the rates 4%, 5%, 6%.

47. A person has \$12,750 to invest. He can buy three per cent bonds at 81, and five per cents at 120. Find the amount of money he must invest in each in order to have the same income from each investment.

Let $x =$ number of dollars in three per cent bonds,
and $y =$ number of dollars in five per cent bonds.

Then $\frac{300x}{81} =$ interest of money invested in three per cents.

But $\frac{500y}{120} =$ interest of money invested in five per cents.

$$\therefore \frac{300x}{81} = \frac{500y}{120} \qquad (1)$$

$$x + y = 12750 \qquad (2)$$

Reduce (1), $8x - 9y = 0$
Multiply (2) by 8, $8x + 8y = 102000$
Subtract, $17y = 102000$
$\therefore y = 6000.$

Substitute value of y in (2), $x = 6750.$

48. **A and B each invested \$1500 in bonds; A in three per cents and B in four per cents. The bonds were bought at such prices that B received \$5 interest more than A. Both classes of bonds rose ten points, and they sold out, A receiving \$50 more than B. What price was paid for each class of bonds?**

Let $x =$ amount paid for \$1 three per cents,

and $y =$ amount paid for \$1 four per cents.

$$\frac{1500}{x} = \text{face value of three per cents},$$

$$\frac{1500}{y} = \text{face value of four per cents}.$$

$$\frac{1500}{x} \times \frac{3}{100} = \text{income from three per cents},$$

$$\frac{1500}{y} \times \frac{4}{100} = \text{income from four per cents}.$$

Then $$\left(\frac{1500}{y} \times \frac{4}{100}\right) - \left(\frac{1500}{x} \times \frac{3}{100}\right) = 5,$$

$$\left(\frac{1500}{x} \times \frac{10}{100}\right) - \left(\frac{1500}{y} \times \frac{10}{100}\right) = 50.$$

Simplify, $$\frac{60}{y} - \frac{45}{x} = 5 \qquad (1)$$

$$\frac{3}{x} - \frac{3}{y} = 1 \qquad (2)$$

Multiply (2) by 20, $$-\frac{60}{y} + \frac{60}{x} = 20 \qquad (3)$$

Add (1) and (3), $$\frac{3}{x} = 5.$$

$$\therefore x = 0.60.$$

Substitute value of x in (2), $$5 - \frac{3}{y} = 1.$$

$$\therefore y = 0.75.$$

That is, the three per cents were bought at 60 and the four per cents at 75.

49. A person invests $10,000 in three per cent bonds, $16,500 in three and one-half per cents, and has an income from both investments of $1056.25. If his investments had been $2750 more in the three per cents, and less in the three and one-half per cents, his income would have been $62\frac{1}{2}$ cents greater. What price was paid for each class of bonds?

Let $x =$ amount paid on $1 three per cent bonds,

and $y =$ amount paid on $1 three and one-half per cent bonds.

Then $\dfrac{10000}{x} \times \dfrac{3}{100} =$ number of dollars income from first investment,

and $\dfrac{16500}{y} \times \dfrac{3\frac{1}{2}}{100} =$ number of dollars income from second investment.

$$\therefore \frac{30000}{100x} + \frac{115500}{200y} = 1056.25 \quad (1)$$

$\dfrac{12750}{x} \times \dfrac{3}{100} =$ number of dollars income of 3 per cents if the stated addition in the amount invested had been made,

$\dfrac{13750}{y} \times \dfrac{3\frac{1}{2}}{100} =$ number of dollars income of $3\frac{1}{2}$ per cents if the stated deduction in the amount invested had been made.

Then $\dfrac{38250}{100x} + \dfrac{96250}{200y} =$ number of dollars income on both.

$$\therefore \frac{38250}{100x} + \frac{96250}{200y} = \$1056.87\tfrac{1}{2} \quad (2)$$

Multiply (1) by 5, $$\frac{150000}{100x} + \frac{577500}{200y} = 5281.25 \quad (3)$$

Multiply (2) by 6, $$\frac{229500}{100x} + \frac{577500}{200y} = 6341.25 \quad (4)$$

Subtract, $$\frac{79500}{100x} = 1060,$$

$$106000x = 79500.$$

$$\therefore x = 0.75.$$

That is, the 3 per cent bonds were bought at 75.

Substitute value of x in (1),

$$\frac{30000}{75} + \frac{115500}{200y} = 1056.25,$$

$$\frac{115500}{200y} = 656.25.$$

$$\therefore y = 0.88.$$

That is, the $3\frac{1}{2}$ per cent bonds were bought at 88.

50. The sum of $2500 was divided into two unequal parts and invested, the smaller part at two per cent more than the larger. The *rate* of interest on the larger sum was afterwards increased by 1, and that of the smaller sum diminished by 1; and thus the *interest* of the whole was increased by one-fourth of its value. If the interest of the larger sum had been so increased, and no change been made in the interest of the smaller sum, the interest of the whole would have been increased one-third of its value. Find the sums invested, and the rate per cent of each.

Let $x =$ number of dollars in larger part,

and $y =$ number of dollars in smaller part.

Then $x + y = 2500$ (1)

Let $z =$ rate per cent on larger part,

and $z + 2 =$ rate per cent on smaller part.

Then $xz + y(z + 2) =$ interest on whole amount.

Changing rate per cent,

$z + 1 =$ rate per cent on larger part,

and $z + 1 =$ rate per cent on smaller part.

Then $x(z+1)+y(z+1) =$ interest on whole after change.

Then $x(z+1)+y(z+1) = \frac{5}{4}[xz+y(z+2)]$ (2)

Changing rate per cent again,

$z + 1 =$ rate of larger part,

$z + 2 =$ rate of smaller part.

Then $x(z+1)+y(z+2) = \frac{4}{3}[xz+y(z+2)]$ (3)

Simplify (2), $4x - 6y - xz - yz = 0.$

Simplify (3), $3x - 2y - xz - yz = 0.$ (4)

Subtract, $x - 4y = 0$ (5)

Subtract (5) from (1), $5y = 2500.$

$\therefore y = 500.$

Substitute value of y in (4), $x = 2000.$

Substitute values of x and y in (3),

$$6000 - 1000 - 2000z - 500z = 0,$$

$$-2500z = -5000.$$

$$\therefore z = 2.$$

51. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards; if the length were increased and the breadth were diminished each by 5 yards, the area would be diminished by 185 square yards. What is its area?

Let x = number of yards in length,
and y = number of yards in width.
Then xy = number of yards in area.

$$(x+2)(y+2) = xy + 220 \quad (1)$$
$$(x+5)(y-5) = xy - 185 \quad (2)$$

Simplify (1), $xy + 2x + 2y + 4 = xy + 220,$
$$2x + 2y = 216,$$
$$x + y = 108 \quad (3)$$

Simplify (2), $xy - 5x + 5y - 25 = xy - 185,$
$$5x - 5y = 160$$
$$x - y = 32 \quad (4)$$

Add (4) and (3), $2x = 140$
$$\therefore x = 70.$$

Subtract (4) from (3), $2y = 76.$
$$\therefore y = 38.$$

$\therefore xy = 2660$ square yards.

52. If a given rectangular floor had been 3 feet longer and 2 feet broader it would have contained 64 square feet more; but if it had been 2 feet longer and 3 feet broader it would have contained 68 square feet more. Find the length and breadth of the floor.

Let x = number of feet in length,
and y = number of feet in breadth.
Then xy = number of feet in surface.

$$(x+3)(y+2) = xy + 64 \quad (1)$$
$$(x+2)(y+3) = xy + 68 \quad (2)$$

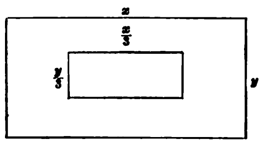
Simplify (1), $xy + 3y + 2x + 6 = xy + 64.$
$$3y + 2x = 58 \quad (3)$$

Simplify (2), $xy + 2y + 3x + 6 = xy + 68.$
$$2y + 3x = 62 \quad (4)$$

Multiply (3) by 2, $6y + 4x = 116 \quad (5)$
Multiply (4) by 3, $6y + 9x = 186 \quad (6)$
Subtract, $-5x = -70$
$$\therefore x = 14.$$

Substitute value of x in (3), $3y + 28 = 58,$
$$3y = 30.$$
$$\therefore y = 10.$$

53. In a certain rectangular garden there is a strawberry-bed whose sides are one-third of the lengths of the corresponding sides of the garden. The perimeter of the garden exceeds that of the bed by 200 yards; and if the greater side of the garden be increased by 3, and the other by 5 yards, the garden will be enlarged by 645 square yards. Find the length and breadth of the garden.



Let $x =$ number of yds. in length of garden,

and $y =$ number of yds. in width of garden.

Then $2x + 2y =$ perimeter of garden,

and $xy =$ area of garden.

Also, $\frac{x}{3} =$ number of yds. in length of bed,

and $\frac{y}{3} =$ number of yds. in width of bed.

Then $\frac{2x}{3} + \frac{2y}{3} =$ perimeter of bed.

Add 3 to one side of garden, $x + 3$.

Add 5 to other side of garden, $y + 5$.

Then $(x+3)(y+5) =$ area.

$$2x + 2y - \left(\frac{2x}{3} + \frac{2y}{3}\right) = 200.$$

Simplify, $x + y = 150$ (1)

$$(x + 3)(y + 5) = xy + 645.$$

Simplify, $5x + 3y = 630$ (2)

Multiply (1) by 5, $5x + 5y = 750$

Subtract, $2y = 120$

$\therefore y = 60.$

Substitute value of y in (1), $x + 60 = 150.$

$\therefore x = 90.$

54. In a mile race A gives B a start of 100 yards, and beats him by 15 seconds. In the second trial A gives B a start of 45 seconds, and is beaten by 22 yards. Find the rate of each in miles per hour.

Let $x =$ number of yards A runs in one second,

and $y =$ number of yards B runs in one second.

Since there are 1760 yards in one mile,

$\frac{1760}{x}$ and $\frac{1738}{x} =$ number of seconds A ran in first and second trials respectively,

$\frac{1660}{y}$ and $\frac{1760}{y} =$ number of seconds B ran in first and second trials respectively.

Then
$$\frac{1660}{y} - \frac{1760}{x} = 15 \qquad (1)$$

and
$$\frac{1738}{x} - \frac{1760}{y} = -45 \qquad (2)$$

Multiply (1) by 88,
$$\frac{146080}{y} - \frac{154880}{x} = 1320$$

Multiply (2) by 83,
$$-\frac{146080}{y} + \frac{144254}{x} = -3735$$

Add,
$$-\frac{10626}{x} = -2415$$

$$\therefore x = 4\tfrac{966}{2415}.$$

Therefore, A runs $4\frac{966}{2415}$ yards, or $\frac{1}{400}$ of a mile, in one second, and in one hour (= 3600 seconds), 9 miles.

Substitute value of x in (1), $y = 4$.

Therefore, B runs 4 yards in one second, or $8\frac{2}{11}$ miles in one hour.

55. In a mile race A gives B a start of 44 yards, and beats him by 51 seconds. In the second trial A gives B a start of 1 minute and 15 seconds, and is beaten by 88 yards. Find the rate of each in miles per hour.

Let $x =$ number of yards A ran in one second,

and $y =$ number of yards B ran in one second.

$\frac{1760}{x}, \frac{1672}{x} =$ number of seconds A ran in first and second trials, respectively.

$\frac{1716}{y}, \frac{1760}{y} =$ number of seconds B ran in first and second trials, respectively.

Then $$\frac{1716}{y} - \frac{1760}{x} = 51 \qquad (1)$$

and $$\frac{1672}{x} - \frac{1760}{y} = -75 \qquad (2)$$

Multiply (1) by 19, $$-\frac{33440}{x} + \frac{32604}{y} = 969$$

Multiply (2) by 20, $$\frac{33440}{x} - \frac{35200}{y} = -1500$$

Add, $$-\frac{2596}{y} = -531$$

$$\therefore y = 4\tfrac{8}{9}.$$

Therefore, B runs $4\frac{8}{9}$ yards per second, or 10 miles per hour.

Substitute value of y in (1), $x = 5\frac{13}{15}$.

Therefore, A runs $5\frac{13}{15}$ yards per second, or 12 miles per hour.

56. The time which an express-train takes to go 120 miles is $\frac{9}{11}$ of the time taken by an accommodation-train. The slower train loses as much time in stopping at different stations as it would take to travel 20 miles without stopping; the express-train loses only half as much time by stopping as the accommodation-train, and travels 15 miles an hour faster. Find the rate of each train in miles per hour.

Let $x =$ the rate of the accommodation-train,

and $y =$ the rate of the express-train.

$\frac{120}{x} =$ number of hours accommodation-train goes 120 miles without stopping,

$\frac{120}{y} =$ number of hours it takes express-train to go 120 miles without stopping,

$\frac{20}{x} =$ number of hours accommodation-train loses in stopping,

$\frac{10}{x} =$ number of hours express-train loses in stopping,

$\frac{120}{x} + \frac{20}{x} =$ number of hours accommodation-train goes 120 miles including stops,

$\frac{120}{y} + \frac{10}{x} =$ number of hours express-train goes 120 miles including stops.

$$\therefore \frac{120}{y} + \frac{10}{x} = \frac{9}{14}\left(\frac{120}{x} + \frac{20}{x}\right) \qquad (1)$$

$$y - x = 15 \qquad (2)$$

Simplify (1), $120x - 80y = 0$

Multiply (2) by 80, $-80x + 80y = 1200$

Add, $40x = 1200$

$\therefore x = 30.$

Substitute value of x in (2), $y = 45.$

57. A train moves from P towards Q, and an hour later a second train starts from Q and moves towards P at a rate of 10 miles an hour more than the first train; the trains meet half-way between P and Q. If the train from P had started an hour after the train from Q, its rate must have been increased by 28 miles in order that the trains should meet at the half-way point. Find the distance from P to Q.

Let x = number of hours first goes half the distance.

Then $x - 1$ = number of hours second goes half the distance.

Let y = rate of first.

Then $y + 10$ = rate of second.

Hence, xy = one-half of the whole distance.

But $(x-1)(y+10)$ = one-half the whole distance.

$\therefore (x-1)(y+10) = xy.$

Simplify, $10x - y = 10 \qquad (1)$

In second statement,

if $x - 2$ = number of hours first goes half distance,

and $y + 28$ = rate of first,

then $(x-2)(y+28)$ = one-half of the whole distance.

But, from (1), xy = one-half of the whole distance.

$\therefore (x-2)(y+28) = xy \qquad (2)$

Simplify (2), $28x - 2y = 56$

$2 \times (1)$ is $20x - 2y = 20$

Subtract, $8x = 36$

$\therefore x = 4\frac{1}{2}.$

Substitute value of x in (1), $y = 35.$

Therefore, one-half the distance, xy, is $157\frac{1}{2}$ miles, and the whole distance is 315 miles.

58. A passenger-train, after travelling an hour, meets with an accident which detains it one-half an hour; after which it proceeds at four-fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Determine the usual rate of the train.

Let $x =$ number of miles train usually goes per hour,

and $y =$ number of miles train travels.

$\frac{y-x}{x} =$ number of hours usually required,

and $\frac{y-x}{\frac{4x}{5}} =$ number of hours actually required after accident.

Since the detention is $\frac{1}{2}$ hour, and the train is $1\frac{1}{4}$ hours late, the loss in running-time is $\frac{3}{4}$ of an hour.

$$\therefore \frac{y-x}{\frac{4x}{5}} - \frac{y-x}{x} = \frac{3}{4} \qquad (1)$$

If the accident had occurred 30 miles farther on, the loss in running-time would have been $\frac{1}{2}$ an hour.

$$\therefore \frac{y-x-30}{\frac{4x}{5}} - \frac{y-x-30}{x} = \frac{1}{2} \qquad (2)$$

Simplify (1), $y - 4x = 0$

Simplify (2), $y - 3x = 30$

Subtract, $x = 30$

59. A passenger-train, after travelling an hour, is detained 15 minutes; after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles farther on, the train would have been only 21 minutes late. Determine the usual rate of the train.

Let $x =$ usual rate of train per hour,

and $y =$ number of miles train has to run.

$y - x =$ number of miles train has to run after detention,

$\frac{y-x}{x} =$ number of hours usually required to run $y-x$ miles,

and $\frac{y-x}{\frac{3x}{4}} =$ number of hours actually required to run the $y-x$ miles.

Since the detention was 15 minutes, and the train is 24 minutes late, the loss in running-time is 9 minutes $= \frac{3}{20}$ of an hour.

$$\therefore \frac{y-x}{\frac{3x}{4}} - \frac{y-x}{x} = \frac{3}{20} \qquad (1)$$

If the detention had occurred 5 miles farther on, the loss in running-time would have been 6 minutes $= \frac{1}{10}$ of an hour.

$$\therefore \frac{y-x-5}{\frac{3x}{4}} - \frac{y-x-5}{x} = \frac{1}{10} \qquad (2)$$

Simplify (1), $\quad 20y - 29x = 0$

Simplify (2), $\quad 20y - 26x = 100$

Subtract, $\quad 3x = 100$

$$\therefore x = 33\tfrac{1}{3}.$$

60. A man bought 10 oxen, 120 sheep, and 46 lambs. The cost of 3 sheep was equal to that of 5 lambs; an ox, a sheep, and a lamb together cost a number of dollars less by 57 than the whole number of animals bought; and the whole sum spent was \$2341.50. Find the price of an ox, a sheep, and a lamb, respectively.

Let $\quad x =$ number of dollars paid for an ox,

$y =$ number of dollars paid for a sheep,

and $\quad z =$ number of dollars paid for a lamb.

$10x + 120y + 46z =$ number of dollars paid for all.

$$\therefore 10x + 120y + 46z = 2341.50 \qquad (1)$$

$$x + y + z = 119 \qquad (2)$$

$$3y = 5z \qquad (3)$$

(1) is $\quad 10x + 120y + 46z = 2341.50$

Multiply (2) by 10, $\quad 10x + 10y + 10z = 1190$

Subtract, $\quad 110y + 36z = 1151.50 \qquad (4)$

Multiply (4) by 3, $\quad 330y + 108z = 3454.50.$

Multiply (3) by 110, $\quad 330y - 550z = 0.$

Subtract, $\quad 658z = 3454.50.$

$$\therefore z = 5.25.$$

Substitute value of z in (3), $\quad y = 8.75.$

Substitute values of y and z in (2), $\quad x = 105.$

61. A farmer sold 100 head of stock, consisting of horses, oxen, and sheep, so that the whole realized $11.75 a head; while a horse, an ox, and a sheep were sold for $110, $62.50, and $7.50, respectively. Had he sold one-fourth of the number of oxen that he did, and 25 more sheep, he would have received the same sum. Find the number of horses, oxen, and sheep, respectively, which were sold.

Let $x =$ number of horses,

$y =$ number of oxen,

and $z =$ number of sheep.

Then
$$x + y + z = 100 \qquad (1)$$
$$110x + 62\tfrac{1}{2}y + 7\tfrac{1}{2}z = 1175 \qquad (2)$$
and
$$110x + \left(\frac{1}{4} \times \frac{125y}{2}\right) + \frac{15z}{2} + \frac{375}{2} = 1175 \qquad (3)$$

Multiply (3) by 8, $880x + 125y + 60z = 7900$

Multiply (2) by 8, $880x + 500y + 60z = 9400$

Subtract, $-375y = -1500$

$$\therefore y = 4.$$

Substitute value of y in (1), $x + z = 96 \qquad (4)$

Substitute value of y in (2),
$$110x + 250 + 7\tfrac{1}{2}z = 1175.$$
$$110x + 7\tfrac{1}{2}z = 925.$$

Multiply by 2, $220x + 15z = 1850$

Multiply (4) by 15, $15x + 15z = 1440$

Subtract, $205x = 410$

$$\therefore x = 2.$$

Substitute values of x and y in (1), $z = 94.$

62. A, B, and C together subscribed \$100. If A's subscription had been one-tenth less, and B's one-tenth more, C's must have been increased by \$2 to make up the sum; but if A's had been one-eighth more, and B's one-eighth less, C's subscription would have been \$17.50. What did each subscribe?

Let x = number of dollars A subscribed,

y = number of dollars B subscribed,

and z = number of dollars C subscribed.

$\frac{9x}{10} = \frac{9}{10}$ of A's subscription,

$\frac{11y}{10} = \frac{11}{10}$ of B's subscription,

$z + 2 =$ \$2 more than C's subscription,

$\frac{9x}{8} = \frac{9}{8}$ of A's subscription,

$\frac{7y}{8} = \frac{7}{8}$ of B's subscription,

$100 =$ number of dollars all subscribed,

$\frac{9x}{10} + \frac{11y}{10} + z + 2 =$ number of dollars all subscribed,

$\frac{9x}{8} + \frac{7y}{8} + 17.5 =$ number of dollars all subscribed.

$$\therefore x + y + z = 100 \quad (1)$$

$$\frac{9x}{10} + \frac{11y}{10} + z + 2 = 100 \quad (2)$$

$$\frac{9x}{8} + \frac{7y}{8} + 17.5 = 100 \quad (3)$$

Multiply (1) by 10, $10x + 10y + 10z = 1000$

Simplify (2), $9x + 11y + 10z = 980$

Subtract, $x - y = 20$ (4)

Simplify (3), $9x + 7y = 660$

Multiply (4) by 7, $7x - 7y = 140$

Add, $16x = 800$

$\therefore x = 50.$

Substitute value of x in (4), $y = 30.$

Substitute values of x and y in (1), $z = 20.$

63. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. In the end each of them has \$6. How much had each at first?

Let $x =$ number of dollars A had at first,

$y =$ number of dollars B had at first,

and $z =$ number of dollars C had at first.

$x - y - z =$ number of dollars A had at 1st distribution,

$2y =$ number of dollars B had at 1st distribution,

$2z =$ number of dollars C had at 1st distribution,

$2y - \{(x-y-z)+2z\} =$ number of dollars B had at 2d distribution,

or $3y - x - z =$ number of dollars B had at 2d distribution,

$2x - 2y - 2z =$ number of dollars A had at 2d distribution,

$4z =$ number of dollars C had at 2d distribution,

$4z - \{(2x-2y-2z)+(3y-x-z)\}$, or $7z - x - y$

$=$ number of dollars C had at 3d distribution,

$4x - 4y - 4z =$ number of dollars A had at 3d distribution,

$6y - 2x - 2z =$ number of dollars B had at 3d distribution.

$$\therefore\ 7z - x - y = 6 \qquad (1)$$

$$4x - 4y - 4z = 6 \qquad (2)$$

$$6y - 2x - 2z = 6 \qquad (3)$$

Multiply (1) by 4, $28z - 4x - 4y = 24$

(2) is $-4z + 4x - 4y = 6$

Add, $24z \qquad - 8y = 30 \qquad (4)$

Multiply (1) by 2, $14z - 2x - 2y = 12$

(3) is $-2z - 2x + 6y = 6$

Subtract, $16z \qquad - 8y = 6 \qquad (5)$

(4) is $24z \qquad - 8y = 30$

Subtract, $8z \qquad = 24$

$$\therefore\ z = 3.$$

Substitute value of z in (5), $y = 5\frac{1}{4}$.

Substitute values of y and z in (1), $x = 9\frac{3}{4}$.

64. A pays to B and C as much as each of them has; B pays to A and C one-half as much as each of them then has; and C pays to A and B one-third of what each of them then has. In the end A finds that he has \$1.50, B \$$4.16\frac{2}{3}$, C \$$0.58\frac{1}{3}$. How much had each at first?

Let x = number of dollars A had at first,

y = number of dollars B had at first,

and z = number of dollars C had at first.

$x-y-z$ = number of dollars A has left after giving to B and C,

$2y$ = number of dollars B has after A pays him,

$2z$ = number of dollars C has after A pays him,

$\dfrac{3x-3y-3z}{2}$ = number of dollars A has after B pays him,

$\dfrac{5y-x-z}{2}$ = number of dollars B has left after paying A and C,

$3z$ = number of dollars C has after receiving B's money,

$\dfrac{4x-4y-4z}{2}$ = number of dollars A has after C pays him,

$\dfrac{10y-2x-2z}{3}$ = number of dollars B has after C pays him,

$\dfrac{11z-x-y}{3}$ = number of dollars C has after paying A and B.

$$\therefore \frac{4x-4y-4z}{2}=1.50 \qquad (1)$$

$$\frac{10y-2x-2z}{3}=4.16\tfrac{2}{3} \qquad (2)$$

$$\frac{11z-x-y}{3}=0.58\tfrac{1}{3} \qquad (3)$$

Simplify (1), $x-y-z=0.75$ (4)

Simplify (2), $10y-2x-2z=12.50$ (5)

Simplify (3), $11z-x-y=1.75$ (6)

(4) is $x-y-z=0.75$

Add (4) and (6), $10z-2y=2.50$ (7)

(5) is $10y-2x-2z=12.50$

Multiply (4) by 2, $-2y+2x-2z=1.50$

Add, $8y-4z=14.00$ (8)

Multiply (7) by 4, $-8y+40z=10.00$

(8) is $8y-4z=14.00$

Add, $36z=24.00$

$\therefore z=0.66\frac{2}{3}$.

Substitute value of z in (8), $y=2.08\frac{1}{3}$.

Substitute values of y and z in (4), $x=3.50$.

EXERCISE LXXVI.

1. $(a^3)^2$
$= a^{3\times 2}$
$= a^6.$

2. $(x^5)^3$
$= x^{5\times 3}$
$= x^{15}.$

3. $(x^2y^3)^2$
$= x^{2\times 2}y^{3\times 2}$
$= x^4y^6.$

4. $\left(\dfrac{a^3b^2}{2}\right)^4$
$= \dfrac{a^{3\times 4}b^{2\times 4}}{2^4}$
$= \dfrac{a^{12}b^8}{16}.$

5. $\left(\dfrac{3x^2y}{2a^3b^2}\right)^5$
$= \dfrac{3^5x^{2\times 5}y^5}{2^5a^{3\times 5}b^{2\times 5}}$
$= \dfrac{243x^{10}y^5}{32a^{15}b^{10}}.$

6. $(x+2)^3$
$= (x)^3 + 3(x)^2(2) + 3(x)(2)^2 + (2)^3$
$= x^3 + 6x^2 + 12x + 8.$

7. $(x-2)^4$
$= x^4 - 4(x)^3(2) + 6(x)^2(2)^2 - 4(x)(2)^3 + 2^4$
$= x^4 - 8x^3 + 24x^2 - 32x + 16.$

8. $(x+3)^5$
$= (x)^5 + 5(x)^4(3) + 10(x)^3(3)^2 + 10(x)^2(3)^3 + 5(x)(3)^4 + (3)^5$
$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243.$

9. $(1+2x)^5$
$= 1^5 + 5(2x) + 10(2x)^2 + 10(2x)^3 + 5(2x)^4 + (2x)^5$
$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$

10. $(2m-1)^3$
$= (2m)^3 - 3(2m)^2 + 3(2m) - (1)^3$
$= 8m^3 - 12m^2 + 6m - 1.$

11. $(2a^2bc^3)^4$
$= 2^4a^{2\times 4}b^4c^{3\times 4}$
$= 16a^8b^4c^{12}.$

12. $(-5ax^3y^2)^3$
$= -5^3a^3x^{3\times 3}y^{2\times 3}$
$= -125a^3x^9y^6.$

13. $(-7m^3nx^2y^4)^2$
$= -7^2m^{3\times 2}n^2x^{2\times 2}y^{4\times 2}$
$= 49m^6n^2x^4y^8.$

14. $\left(-\dfrac{2x^3y}{3abc}\right)^5$
$= -\dfrac{2^5x^{3\times 5}y^5}{3^5a^5b^5c^5}$
$= -\dfrac{32x^{15}y^5}{243a^5b^5c^5}.$

15. $(3x+1)^4$
$= (3x)^4 + 4(3x)^3 + 6(3x)^2 + 4(3x) + 1$
$= 81x^4 + 108x^3 + 54x^2 + 12x + 1.$

16. $(2x-a)^4$
$= (2x)^4 - 4(2x)^3(a) + 6(2x)^2(a)^2 - 4(2x)(a)^3 + (a)^4$
$= 16x^4 - 32ax^3 + 24a^2x^2 - 8a^3x + a^4.$

17. $(3x+2a)^5$
$= (3x)^5 + 5(3x)^4(2a) + 10(3x)^3(2a)^2$
$\qquad + 10(3x)^2(2a)^3 + 5(3x)(2a)^4 + (2a)^5$
$= 243x^5 + 810ax^4 + 1080a^2x^3 + 720a^3x^2 + 240a^4x + 32a^5.$

18. $(2x-y)^4$
$= (2x)^4 - 4(2x)^3(y) + 6(2x)^2(y)^2 - 4(2x)(y)^3 + (y)^4$
$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4.$

19. $(x^2y - 2xy^2)^6$
$= (x^2y)^6 - 6(x^2y)^5(2xy^2) + 15(x^2y)^4(2xy^2)^2 - 20(x^2y)^3(2xy^2)^3$
$\quad + 15(x^2y)^2(2xy^2)^4 - 6(x^2y)(2xy^2)^5 + (2xy^2)^6$
$= x^{12}y^6 - 12x^{11}y^7 + 60x^{10}y^8 - 160x^9y^9 + 240x^8y^{10}$
$\quad - 192x^7y^{11} + 64x^6y^{12}.$

20. $(ab-3)^7$
$= (ab)^7 - 7(ab)^6(3) + 21(ab)^5(3)^2 - 35(ab)^4(3)^3$
$\quad + 35(ab)^3(3)^4 - 21(ab)^2(3)^5 + 7(ab)(3)^6 - (3)^7$
$= a^7b^7 - 21a^6b^6 + 189a^5b^5 - 945a^4b^4$
$\quad + 2835a^3b^3 - 5103a^2b^2 + 5103ab - 2187.$

21. $(-3a^2b^2c)^5$
$= -3^5a^{2\times5}b^{2\times5}c^5$
$= -243a^{10}b^{10}c^5.$

22. $(-3xy^3)^6$
$= -3^6x^6y^{3\times6}$
$= 729x^6y^{18}.$

23. $(-5a^2bx^3)^5$
$= -5^5a^{2\times5}b^5x^{3\times5}$
$= -3125a^{10}b^5x^{15}.$

24. $\left(-\dfrac{3ab^2}{4c^3}\right)^4$
$= \dfrac{81a^4b^8}{256c^{12}}.$

25. $\left(-\dfrac{x^2y^3z^4}{2}\right)^7$
$= -\dfrac{x^{2\times7}y^{3\times7}z^{4\times7}}{2^7}$
$= -\dfrac{x^{14}y^{21}z^{28}}{128}.$

26. $(1-a-a^2)^2$
$= \{1-(a+a^2)\}^2$
$= 1^2 - 2(a+a^2) + (a+a^2)^2$
$= 1 - 2a - 2a^2 + a^2 + 2a^3 + a^4$
$= 1 - 2a - a^2 + 2a^3 + a^4.$

27. $(2-3x+4x^2)^3$
$= [2-(3x-4x^2)]^3$
$= (2)^3 - 3(2)^2(3x-4x^2) + 3(2)(3x-4x^2)^2 - (3x-4x^2)^3$
$= 8 - 36x + 48x^2 + 54x^2 - 144x^3$
$\quad + 96x^4 - 27x^3 + 108x^4 - 144x^5 + 64x^6$
$= 8 - 36x + 102x^2 - 171x^3 + 204x^4 - 144x^5 + 64x^6.$

28. $(1-2x+x^2)^3$
$= \{(1-2x)+x^2\}^3$
$= (1-2x)^3 + 3(1-2x)^2x^2 + 3(1-2x)(x^2)^2 + (x^2)^3$
$= 1 - 6x + 12x^2 - 8x^3 + 3x^2 - 12x^3 + 12x^4 + 3x^4 - 6x^5 + x^6$
$= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6.$

29. $(1-x+x^2)^3$
$= \{1-(x-x^2)\}^3$
$= 1 - 3(x-x^2) + 3(x-x^2)^2 - (x-x^2)^3$
$= 1 - 3x + 3x^2 + 3x^2 - 6x^3 + 3x^4 - x^3 + 3x^4 - 3x^5 + x^6$
$= 1 - 3x + 6x^2 - 7x^3 + 6x^4 - 3x^5 + x^6.$

30. $(1 + x + x^2)^4$

$$= \{1 + (x + x^2)\}^4$$
$$= 1^4 + 4(1)^3(x + x^2) + 6(1)^2(x + x^2)^2 + 4(1)(x + x^2)^3 + (x + x^2)^4$$
$$= 1 + 4x + 4x^2 + 6x^2 + 12x^3 + 6x^4 + 4x^3 + 12x^4 + 12x^5$$
$$+ 4x^6 + x^4 + 4x^5 + 6x^6 + 4x^7 + x^8$$
$$= 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8.$$

Exercise LXXVII.

1. $\sqrt{a^4} = \pm a^2.$

$\sqrt[4]{x^8} = \pm x^2,$

$\sqrt{4a^6b^2} = \pm 2a^3b,$

$\sqrt[3]{64} = 4,$

$\sqrt[5]{a^5x^{10}y^{15}} = ax^2y^3,$

$\sqrt[4]{16a^{12}b^4c^8} = \pm 2a^3bc^2,$

$\sqrt[5]{-32a^{15}} = -2a^3.$

2. $\sqrt[3]{-1728c^6d^{12}x^3y^9} = -12c^2d^4xy^3,$

$\sqrt[3]{3375b^{21}z^{15}} = 15b^7z^5,$

$\sqrt[4]{3111696} = \sqrt[4]{2^4 \times 3^4 \times 7^4},$

$\sqrt[4]{2^4 \times 3^4 \times 7^4 c^{16} z^4} = \pm 42c^4z.$

3. $\sqrt{53361b^4c^8y^{12}z^{16}}$

3^2	53361
7	5929
7	847
11^2	121

$\sqrt{53361} = \sqrt{3^2 \times 7^2 \times 11^2}.$

$\therefore \sqrt{53361b^4c^8y^{12}z^{16}} = \pm 231b^2c^4y^6z^8.$

$$\sqrt[3]{-\frac{216b^3c^{15}}{343z^{24}}} = -\frac{6bc^5}{7z^8},$$

$$\sqrt[6]{\frac{64x^{18}}{729z^{30}}} = \pm\frac{2x^3}{3z^5}.$$

4. $\sqrt{25a^2b^4c^2} + \sqrt[3]{8a^3b^6c^3} - \sqrt[4]{81a^4b^8c^4} - \sqrt[5]{32a^5b^{10}c^5}$

$$= \sqrt{5^2a^2b^4c^2} + \sqrt[3]{2^3a^3b^6c^3} - \sqrt[4]{3^4a^4b^8c^4} - \sqrt[5]{2^5a^5b^{10}c^5}$$
$$= 5ab^2c + 2ab^2c - 3ab^2c - 2ab^2c$$
$$= 2ab^2c.$$

5. $\sqrt[3]{27x^3y^6} \times \sqrt[5]{243y^5z^5} \times \sqrt{16x^4z^2}$

$$= 3xy^2 \times 3yz \times 4x^2z$$
$$= 36x^3y^3z^2.$$

6. $4\sqrt{2x} - \sqrt{abxy} + 5\sqrt{a^2b^3xy}$

$$= 4\sqrt{2 \times 2} - \sqrt{1 \times 3 \times 2 \times 6} + 5\sqrt{1^2 \times 3^3 \times 2 \times 6}$$
$$= 4\sqrt{4} - \sqrt{36} + 5\sqrt{324}$$
$$= 4 \times 2 - 6 + 5 \times 18$$
$$= 92.$$

7. $2a\sqrt{8ax} + b\sqrt[3]{12by} + 4abx\sqrt{bxy}$

$$= 2 \times 1\sqrt{8 \times 1 \times 2} + 3\sqrt[3]{12 \times 3 \times 6} + 4 \times 1 \times 3 \times 2\sqrt{3 \times 2 \times 6}$$
$$= 2\sqrt{16} + 3\sqrt[3]{216} + 24\sqrt{36}$$
$$= 2 \times 4 + 3 \times 6 + 24 \times 6$$
$$= 170.$$

8. $\sqrt{a^2 + 2ab + b^2} \times \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}$
$= (a + b) \times (a + b)$
$= (1 + 3) \times (1 + 3)$
$= 16.$

9. $\sqrt[3]{b^3 - 3b^2a + 3ba^2 - a^3} \div \sqrt{b^2 + a^2 - 2ab}$
$= \sqrt[3]{(b - a)^3} \div \sqrt{(b - a)^2}$
$= (b - a) \div (b - a)$
$= 1.$

Exercise LXXVIII.

1.

$$
\begin{array}{r|l}
 & a^4 + 4a^3 + 2a^2 - 4a + 1 \;\underline{|\, a^2 + 2a - 1} \\
 & a^4 \\
2a^2 + 2a & 4a^3 + 2a^2 \\
 & \underline{4a^3 + 4a^2} \\
2a^2 + 4a - 1 & -2a^2 - 4a + 1 \\
 & \underline{-2a^2 - 4a + 1}
\end{array}
$$

2.

$$
\begin{array}{r|l}
 & x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 \;\underline{|\, x^2 - xy + y^2} \\
 & x^4 \\
2x^2 - xy & -2x^3y + 3x^2y^2 \\
 & \underline{-2x^3y + x^2y^2} \\
2x^2 - 2xy + y^2 & 2x^2y^2 - 2xy^3 + y^4 \\
 & \underline{2x^2y^2 - 2xy^3 + y^4}
\end{array}
$$

3.

$$
\begin{array}{r|l}
 & 4a^6 - 12a^5x + 5a^4x^2 + 6a^3x^3 + a^2x^4 \;\underline{|\, 2a^3 - 3a^2x - ax^2} \\
 & 4a^6 \\
4a^3 - 3a^2x & -12a^5x + 5a^4x^2 \\
 & \underline{-12a^5x + 9a^4x^2} \\
4a^3 - 6a^2x - ax^2 & -4a^4x^2 + 6a^3x^3 + a^2x^4 \\
 & \underline{-4a^4x^2 + 6a^3x^3 + a^2x^4}
\end{array}
$$

4.

$$
\begin{array}{r|l}
 & 9x^6 - 24x^4y^2 - 12x^3y^3 + 16x^2y^4 + 16xy^5 + 4y^6 \;\underline{|\, 3x^3 - 4xy^2 - 2y^3} \\
 & 9x^6 \\
6x^3 - 4xy^2 & -24x^4y^2 - 12x^3y^3 + 16x^2y^4 \\
 & \underline{-24x^4y^2 + 16x^3y^4 + 16x^2y^4} \\
6x^3 - 8xy^2 - 2y^3 & -12x^3y^3 \qquad +16xy^5 + 4y^6 \\
 & \underline{-12x^3y^3 \qquad +16xy^5 + 4y^6}
\end{array}
$$

5.

$$\begin{array}{r|l}
 & 4a^8 + 16a^6c^2 - 32a^2c^6 + 16c^8 \;\big|\; \underline{2a^4 + 4a^2c^2 - 4c^4} \\
 & 4a^8 \\
4a^4 + 4a^2c^2 & 16a^6c^2 - 32a^2c^6 \\
 & 16a^6c^2 + 16a^4c^4 \\
4a^4 + 8a^2c^2 - 4c^4 & -16a^4c^4 - 32a^2c^6 + 16c^8 \\
 & -16a^4c^4 - 32a^2c^6 + 16c^8
\end{array}$$

6.

$$\begin{array}{r|l}
 & 4x^4 - 20x^3 + 37x^2 - 30x + 9 \;\big|\; \underline{2x^2 - 5x + 3} \\
 & 4x^4 \\
4x^2 - 5x & -20x^3 + 37x^2 \\
 & -20x^3 + 25x^2 \\
4x^2 - 10x + 3 & 12x^2 - 30x + 9 \\
 & 12x^2 - 30x + 9
\end{array}$$

7.

$$\begin{array}{r|l}
 & 16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4 \;\big|\; \underline{4x^2 - 2ab + 2b^2} \\
 & 16x^4 \\
8x^2 - 2ab & -16abx^2 + 16b^2x^2 + 4a^2b^2 \\
 & -16abx^2 \qquad\qquad + 4a^2b^2 \\
8x^2 - 4ab + 2b^2 & 16b^2x^2 \qquad\qquad - 8ab^3 + 4b^4 \\
 & 16b^2x^2 \qquad\qquad - 8ab^3 + 4b^4
\end{array}$$

8.

$$\begin{array}{r|l}
 & 16 - 24x + 25x^2 - 20x^3 + 10x^4 - 4x^5 + x^6 \;\big|\; \underline{4 - 3x + 2x^2 - x^3} \\
 & 16 \\
8 - 3x & -24x + 25x^2 \\
 & -24x + 9x^2 \\
8 - 6x + 2x^2 & 16x^2 - 20x^3 + 10x^4 \\
 & 16x^2 - 12x^3 + 4x^4 \\
8 - 6x + 4x^2 - x^3 & -8x^3 + 6x^4 - 4x^5 + x^6 \\
 & -8x^3 + 6x^4 - 4x^5 + x^6
\end{array}$$

9.

$$\begin{array}{r|l}
 & x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6 \;\big|\; \underline{x^3 - 2x^2y + 2xy^2 - y^3} \\
 & x^6 \\
2x^3 - 2x^2y & -4x^5y + 8x^4y^2 \\
 & -4x^5y + 4x^4y^2 \\
2x^3 - 4x^2y + 2xy^2 & 4x^4y^2 - 10x^3y^3 + 8x^2y^4 \\
 & 4x^4y^2 - 8x^3y^3 + 4x^2y^4 \\
2x^3 - 4x^2y + 4xy^2 - y^3 & -2x^3y^3 + 4x^2y^4 - 4xy^5 + y^6 \\
 & -2x^3y^3 + 4x^2y^4 - 4xy^5 + y^6
\end{array}$$

10.

$$4a^6-4a^5-11a^4+14a^3+5a^2-12a+4 \,\big|\, \underline{2a^3-a^2-3a+2}$$
$$\underline{4a^6}$$
$$4a^3-a^2 \,\big|\, -4a^5-11a^4$$
$$\underline{-4a^5+a^4}$$
$$4a^3-2a^2-3a \,\big|\, -12a^4+14a^3+5a^2$$
$$\underline{-12a^4+6a^3+9a^2}$$
$$4a^3-2a^2-6a+2 \,\big|\, 8a^3-4a^2-12a+4$$
$$\underline{8a^3-4a^2-12a+4}$$

11.

$$\big|\, \underline{3a-b+5c+d}$$
$$9a^2-6ab+b^2+30ac-10bc+25c^2+6ad-2bd+10cd+d^2$$
$$\underline{9a^2}$$
$$6a-b \,\big|\, -6ab+b^2$$
$$\underline{-6ab+b^2}$$
$$6a-2b+5c \,\big|\, 30ac-10bc+25c^2$$
$$\underline{30ac-10bc+25c^2}$$
$$6a-2b+10c+d \,\big|\, 6ad-2bd+10cd+d^2$$
$$\underline{6ad-2bd+10cd+d^2}$$

12.

$$\big|\, \underline{5x^3-3x^2y-4xy^2+y^3}$$
$$25x^6-30x^5y-31x^4y^2+34x^3y^3+10x^2y^4-8xy^5+y^6$$
$$\underline{25x^6}$$
$$10x^3-3x^2y \,\big|\, -30x^5y-31x^4y^2$$
$$\underline{-30x^5y+9x^4y^2}$$
$$10x^3-6x^2y-4xy^2 \,\big|\, -40x^4y^2+34x^3y^3+10x^2y^4$$
$$\underline{-40x^4y^2+24x^3y^3+16x^2y^4}$$
$$10x^3-6x^2y-8xy^2+y^3 \,\big|\, 10x^3y^3-6x^2y^4-8xy^5+y^6$$
$$\underline{10x^3y^3-6x^2y^4-8xy^5+y^6}$$

13.

$$\big|\, \underline{m^4-2m^3+3m^2-4m+5}$$
$$m^8-4m^7+10m^6-20m^5+35m^4-44m^3+46m^2-40m+25$$
$$\underline{m^8}$$
$$2m^4-2m^3 \,\big|\, -4m^7+10m^6$$
$$\underline{-4m^7+4m^6}$$
$$2m^4-4m^3+3m^2 \,\big|\, 6m^6-20m^5+35m^4$$
$$\underline{6m^6-12m^5+9m^4}$$
$$2m^4-4m^3+6m^2-4m \,\big|\, -8m^5+26m^4-44m^3+46m^2$$
$$\underline{-8m^5+16m^4-24m^3+16m^2}$$
$$2m^4-4m^3+6m^2-8m+5 \,\big|\, 10m^4-20m^3+30m^2-40m+25$$
$$\underline{10m^4-20m^3+30m^2-40m+25}$$

14.

$$
\begin{array}{r|l}
 & x^4 - x^3y - \tfrac{7}{4}x^2y^2 + xy^3 + y^4 \;\big|\; \underline{x^2 - \tfrac{1}{2}xy - y^2} \\
 & x^4 \\
2x^2 - \tfrac{1}{2}xy & -x^3y - \tfrac{7}{4}x^2y^2 \\
 & \underline{-x^3y + \tfrac{1}{4}x^2y^2} \\
2x^2 - xy - y^2 & -2x^2y^2 + xy^3 + y^4 \\
 & \underline{-2x^2y^2 + xy^3 + y^4}
\end{array}
$$

15.

$$
\begin{array}{r|l}
 & x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2} \;\Big|\; \underline{x^2 - 2xy + y^2 - \frac{y^3}{x}} \\
 & x^4 \\
2x^2 - 2xy & -4x^3y + 6x^2y^2 \\
 & \underline{-4x^3y + 4x^2y^2} \\
2x^2 - 4xy + y^2 & 2x^2y^2 - 6xy^3 + 5y^4 \\
 & \underline{2x^2y^2 - 4xy^3 + y^4} \\
2x^2 - 4xy + 2y^2 - \frac{y^3}{x} & -2xy^3 + 4y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2} \\
 & \underline{-2xy^3 + 4y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2}}
\end{array}
$$

16.

$$
\begin{array}{r|l}
 & \frac{a^4}{9} - \frac{a^3x}{2} + \frac{43a^2x^2}{48} - \frac{3ax^3}{4} + \frac{x^4}{4} \;\Big|\; \underline{\frac{a^2}{3} - \frac{3ax}{4} + \frac{x^2}{2}} \\
 & \frac{a^4}{9} \\
\frac{2a^2}{3} - \frac{3ax}{4} & -\frac{a^3x}{2} + \frac{43a^2x^2}{48} \\
 & \underline{-\frac{a^3x}{2} + \frac{27a^2x^2}{48}} \\
2\left(\frac{a^2}{3} - \frac{3ax}{4}\right) + \frac{x^2}{2} & \frac{a^2x^2}{3} - \frac{3ax^3}{4} + \frac{x^4}{4} \\
 & \underline{\frac{a^2x^2}{3} - \frac{3ax^3}{4} + \frac{x^4}{4}}
\end{array}
$$

17.

$$
\begin{array}{r|l}
 & 1+\frac{4}{x}+\frac{10}{x^2}+\frac{20}{x^3}+\frac{25}{x^4}+\frac{24}{x^5}+\frac{16}{x^6} \;\Big|\; 1+\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3} \\
 & 1 \\
\hline
2+\frac{2}{x} & \frac{4}{x}+\frac{10}{x^2} \\
 & \frac{4}{x}+\frac{4}{x^2} \\
\hline
2+\frac{4}{x}+\frac{3}{x^2} & \frac{6}{x^2}+\frac{20}{x^3}+\frac{25}{x^4} \\
 & \frac{6}{x^2}+\frac{12}{x^3}+\frac{9}{x^4} \\
\hline
2+\frac{4}{x}+\frac{6}{x^2}+\frac{4}{x^3} & \frac{8}{x^3}+\frac{16}{x^4}+\frac{24}{x^5}+\frac{16}{x^6} \\
 & \frac{8}{x^3}+\frac{16}{x^4}+\frac{24}{x^5}+\frac{16}{x^6} \\
\hline
\end{array}
$$

18.

$$
\begin{array}{r|l}
 & \frac{a^2}{b^2}-\frac{2a}{b}+3-\frac{2b}{a}+\frac{b^2}{a^2} \;\Big|\; \frac{a}{b}-1+\frac{b}{a} \\
 & \frac{a^2}{b^2} \\
\hline
\frac{2a}{b}-1 & -\frac{2a}{b}+3 \\
 & -\frac{2a}{b}+1 \\
\hline
\frac{2a}{b}-2+\frac{b}{a} & 2-\frac{2b}{a}+\frac{b^2}{a^2} \\
 & 2-\frac{2b}{a}+\frac{b^2}{a^2} \\
\hline
\end{array}
$$

19.

$$
\begin{array}{r|l}
 & x^4+x^3-\frac{5x^2}{12}-\frac{x}{3}+\frac{1}{9} \;\Big|\; x^2+\frac{x}{2}-\frac{1}{3} \\
 & x^4 \\
\hline
2x^2+\frac{x}{2} & x^3-\frac{5x^2}{12} \\
 & x^3+\frac{x^2}{4} \\
\hline
2x^2+x-\frac{1}{3} & -\frac{2x^2}{3}-\frac{x}{3}+\frac{1}{9} \\
 & -\frac{2x^2}{3}-\frac{x}{3}+\frac{1}{9} \\
\hline
\end{array}
$$

EXERCISE LXXIX.

1.

(1) $1\dot{2}0\dot{4}0\dot{9}$ (347
9
64) 304
256
687) 4809
4809

(2) $4\dot{8}1\dot{6}.3\dot{6}$ (69.4
36
129) 1216
1161
1384) 5536
5536

(3) $1\dot{8}6\dot{7}.1\dot{0}4\dot{1}$ (43.21
16
83) 267
249
862) 1810
1724
8641) 8641
8641

(4) $1\dot{4}3\dot{5}.6\dot{5}2\dot{1}$ (37.89
9
67) 535
469
748) 6665
5984
7569) 68121
68121

(5) $6\dot{4}.1\dot{2}8\dot{0}6\dot{4}$ (8.008
64
16008) 128064
128064

2.

(1) $\dot{1}6\dot{8}0\dot{3}.9\dot{3}6\dot{9}$ (129.63
1
22) 68
44
249) 2403
2241
2586) 16293
15516
25923) 77769
77769

(2) $\dot{4}.5\dot{4}4\dot{9}9\dot{7}6\dot{1}$ (2.1319
4
41) 54
41
423) 1349
1269
4261) 8097
4261
42629) 383661
383661

(3) $0.2\dot{4}3\dot{7}3\dot{9}6\dot{9}$ (0.4937
16
89) 837
801
983) 3639
2949
9867) 69069
69069

(4) $0.5\dot{6}8\dot{7}5\dot{7}3\dot{0}5\dot{6}$ (0.75416
49
145) 787
725
1504) 6257
6016
15081) 24130
15081
150826) 904956
904956

3.

(1) $0.9\dot{0}0\dot{0}0\dot{0}0\dot{0}0\dot{0}$ (0.94868
81
184) 900
736
1888) 16400
15104
18966) 129600
113796
189728) 1580400
1517824

(2) $\dot{6}.2\dot{1}$ (2.4919
4
44) 221
176
489) 4500
4401
4981) 9900
4981
49829) 491900
448461

(3) $0.4\dot{3}$ (0.6557
36
125) 700
625
1305) 7500
6525
13107) 97500
91749

(4) $0.0\dot{0}8\dot{5}2\dot{0}$ (0.0923
81
182) 420
364
1843) 5600
5529

(5) $1\dot{7}.0\dot{0}$ (4.1231
16
81) 100
81
822) 1900
1644
8243) 25600
24729
82461) 87100
82461

(6) $\dot{1}2\dot{9}.0\dot{0}0\dot{0}0\dot{0}0\dot{0}$ (11.3578
1
21) 29
21
223) 800
669
2265) 13100
11325
22707) 177500
158949
227148) 1855100
1817184

(7) $\dot{3}4\dot{7}.2\dot{5}9\dot{0}$ (18.6348
1
28) 247
224
366) 2325
2196
3723) 12990
11169
37264) 182100
149056
372688) 3304400
2981504

4.

(1) 14295.3870(119.5633
1
21)42
21
229)2195
2061
2385)13438
11925
23906)151370
143436
239123)793400
717369
2391263)7603100
7173789

(2) 2.50000(1.5811
1
25)150
125
308)2500
2464
3161)3600
3161
31621)43900
31621

(3) 2000(44.7213
16
84)400
336
887)6400
6209
8942)19100
17884
89441)121600
89441

(4) 0.30000000(0.5477
25
104)500
416
1087)8400
7609
10947)79100
76629

(5) 0.03000000(0.1732
1
27)200
189
343)1100
1029
3462)7100
6924

(6) 111(10.5356
1
205)1100
1025
2103)7500
6309
21065)119100
105325
210706)1377500
1264236

5.

(1) 0.00111(0.0333
9
63)210
189
663)2100
1989

(2) 0.00400000(0.0632
36
123)400
369
1262)3100
2524

(3) 0.0050(0.07071
49
1407)10000
9849
14141)15100
14141

(4) $\dot{2}.0\dot{0}0\dot{0}0\dot{0}0\dot{0}$ (1.4142
1
24) 100
96
281) 400
281
2824) 11900
11296
28282) 60400
56564

(5) $\dot{5}.0\dot{0}$ (2.2360
4
42) 100
84
443) 1600
1329
4466) 27100
26796

(6) $\dot{3}.2\dot{5}$ (1.8027
1
28) 225
224
3602) 10000
7204
36047) 279600
252329

(7) $\dot{8}.6\dot{0}0\dot{0}0\dot{0}$ (2.9325
4
49) 460
441
583) 1900
1749
5862) 15100
11724

6.

(1) $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

(2) $\sqrt{\frac{16}{49}} = \frac{4}{7}$.

(3) $\sqrt{\frac{100}{144}} = \frac{10}{12} = \frac{5}{6}$.

(4) $\sqrt{\frac{169}{225}} = \frac{13}{15}$.

(5) $\sqrt{\frac{289}{324}} = \frac{17}{18}$.

(6) $\sqrt{\frac{400}{625}} = \frac{20}{25} = \frac{4}{5}$.

7.

(1) $\frac{1}{2} = 0.5$.
$0.5\dot{0}$ (0.7071
49
1407) 10000
9849
14141) 15100
14141

(2) $\frac{2}{3} = 0.666666$.
$0.6\dot{6}6\dot{6}6\dot{6}$ (0.8164
64
161) 266
161
1626) 10566
9756

(3) $\frac{3}{4} = 0.75$.
$0.7\dot{5}0\dot{0}0\dot{0}$ (0.8660
64
166) 1100
996
1726) 10400
10356
1732) 4400

(4) $\frac{1}{32} = 0.03125$.
$0.0\dot{3}1\dot{2}5\dot{0}$ (0.1767
1
27) 212
189
346) 2350
2076
3527) 27400
24689

(5) $\frac{7}{128} = 0.0546875.$

$$\begin{array}{rl}
 & 0.0\dot{5}4\dot{6}8\dot{7}5\dot{0}\,(0.2338 \\
 & 4 \\
43) & 146 \\
 & 129 \\
463) & 1787 \\
 & 1389 \\
4668) & 39850 \\
 & 37344
\end{array}$$

(6) $\frac{6}{125} = 0.048.$

$$\begin{array}{rl}
 & 0.0\dot{4}8\dot{0}\,(0.2190 \\
 & 4 \\
41) & 80 \\
 & 41 \\
429) & 3900 \\
 & 3861
\end{array}$$

(7) $\frac{6}{7} = 0.857142.$

$$\begin{array}{rl}
 & 0.8\dot{5}7\dot{1}4\dot{2}\,(0.9258 \\
 & 81 \\
182) & 471 \\
 & 364 \\
1845) & 10742 \\
 & 9225 \\
18508) & 151700 \\
 & 148064
\end{array}$$

(8) $\frac{1}{12} = 0.08333333.$

$$\begin{array}{rl}
 & 0.0\dot{8}3\dot{3}3\dot{3}3\dot{3}\,(0.2886 \\
 & 4 \\
48) & 433 \\
 & 384 \\
568) & 4933 \\
 & 4544 \\
5766) & 38933 \\
 & 34596
\end{array}$$

Exercise LXXX.

1.

$$\begin{array}{rr|l}
 & x^3 + 6x^2y + 12xy^2 + 8y^3 & \underline{x + 2y} \\
 & x^3 & \\
 3x^2 & 6x^2y + 12xy^2 + 8y^3 & \\
(3x+2y)2y = \quad 6xy + 4y^2 & 6x^2y + 12xy^2 + 8y^3 & \\
\hline
3x^2 + 6xy + 4y^2 & &
\end{array}$$

2.

$$\begin{array}{rr|l}
 & a^3 - 9a^2 + 27a - 27 & \underline{a - 3} \\
 & a^3 & \\
3a^2 & -9a^2 + 27a - 27 & \\
-3(3a-3) = \quad -9a + 9 & -9a^2 + 27a - 27 & \\
\hline
3a^2 - 9a + 9 & &
\end{array}$$

3.

$$\begin{array}{rr|l}
 & x^3 + 12x^2 + 48x + 64 & \underline{x + 4} \\
 & x^3 & \\
3x^2 & 12x^2 + 48x + 64 & \\
(3x+4)4 = \quad 12x + 16 & 12x^2 + 48x + 64 & \\
\hline
3x^2 + 12x + 16 & &
\end{array}$$

4.

$$\begin{array}{rr|l}
 & & x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6 \;\underline{|\, x^2 - ax - a^2} \\
 & & x^6 \\
 & 3x^4 & -3ax^5 + 5a^3x^3 - 3a^5x \\
(3x^2 - ax)(-ax) = & \underline{-3ax^3 + a^2x^2} & \\
 & 3x^4 - 3ax^3 + a^2x^2 & \underline{-3ax^5 + 3a^2x^4 - a^3x^3} \\
3(x^2 - ax)^2 = & 3x^4 - 6ax^3 + 3a^2x^2 & -3a^2x^4 + 6a^3x^3 - 3a^5x - a^6 \\
(3x^2 - 3ax - a^2)(-a^2) = & \underline{-3a^2x^2 + 3a^3x + a^4} & \\
 & 3x^4 - 6ax^3 \qquad + 3a^3x + a^4 & \underline{-3a^2x^4 + 6a^3x^3 - 3a^5x - a^6}
\end{array}$$

5.

$$\begin{array}{rr|l}
 & & x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1 \;\underline{|\, x^2 + x + 1} \\
 & & x^6 \\
 & 3x^4 & 3x^5 + 6x^4 + 7x^3 \\
(3x^2 + x)x = & \underline{+3x^3 + x^2} & \\
 & 3x^4 + 3x^3 + x^2 & \underline{3x^5 + 3x^4 + x^3} \\
3(x^2 + x)^2 = & 3x^4 + 6x^3 + 3x^2 & 3x^4 + 6x^3 + 6x^2 + 3x + 1 \\
(3x^2 + 3x + 1)(1) = & \underline{3x^2 + 3x + 1} & \underline{3x^4 + 6x^3 + 6x^2 + 3x + 1} \\
 & 3x^4 + 6x^3 + 6x^2 + 3x + 1 &
\end{array}$$

6.

$$\begin{array}{rr|l}
 & & \underline{|\, 1 - 3x + 4x^2} \\
 & & 1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6 \\
 & & 1 \\
 & 3 & -9x + 39x^2 - 99x^3 \\
 & \underline{-9x + 9x^2} & \\
 & 3 - 9x + 9x^2 & \underline{-9x + 27x^2 - 27x^3} \\
 & 3 - 18x + 27x^2 & 12x^2 - 72x^3 + 156x^4 - 144x^5 + 64x^6 \\
 & \underline{12x^2 - 36x^3 + 16x^4} & \\
 & 3 - 18x + 39x^2 - 36x^3 + 16x^4 & \underline{12x^2 - 72x^3 + 156x^4 - 144x^5 + 64x^6}
\end{array}$$

7.

$$\begin{array}{rr|l}
 & & \underline{|\, a^2 - 2a - 1} \\
 & & a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1 \\
 & & a^6 \\
 & 3a^4 & -6a^5 + 9a^4 + 4a^3 \\
(3a^2 - 2a)(-2a) = & \underline{-6a^3 + 4a^2} & \\
 & 3a^4 - 6a^3 + 4a^2 & \underline{-6a^5 + 12a^4 - 8a^3} \\
3(a^2 - 2a)^2 = & 3a^4 - 12a^3 + 12a^2 & -3a^4 + 12a^3 - 9a^2 - 6a - 1 \\
(3a^2 - 6a - 1)(-1) = & \underline{-\; 3a^2 + 6a + 1} & \underline{-3a^4 + 12a^3 - 9a^2 - 6a - 1} \\
 & 3a^4 - 12a^3 + 9a^2 + 6a + 1 &
\end{array}$$

8.

$$\underline{|4x^2+4x-1}$$

$$64x^6+192x^5+144x^4-32x^3-36x^2+12x-1$$

$$\underline{64x^6}$$

$$\begin{array}{rl|l}
 & 48x^4 & 192x^5+144x^4-32x^3 \\
(12x^2+4x)4x= & 48x^3+16x^2 & \\
\hline
 & 48x^4+48x^3+16x^2 & 192x^5+192x^4+64x^3 \\
\hline
3(4x^2+4x)^2= & 48x^4+96x^3+48x^2 & -48x^4-96x^3-36x^2+12x-1 \\
(12x^2+12x-1)(-1)= & -12x^2-12x+1 & \\
\hline
 & 48x^4+96x^3+36x^2-12x+1 & -48x^4-96x^3-36x^2+12x-1 \\
\hline
\end{array}$$

9.

$$\underline{|1-x+x^2-x^3}$$

$$1-3x+6x^2-10x^3+12x^4-12x^5+10x^6-6x^7+3x^8-x^9$$

$$\underline{1}$$

$$\begin{array}{r|l}
3 & -3x+6x^2-10x^3 \\
-3x+x^2 & \\
\hline
3-3x+x^2 & -3x+3x^2-x^3 \\
\hline
3-6x+3x^2 & 3x^2-9x^3+12x^4-12x^5+10x^6 \\
3x^2-3x^3+x^4 & 3x^2-6x^3+6x^4-3x^5+x^6 \\
\hline
3-6x+6x^2-3x^3+x^4 & -3x^3+6x^4-9x^5+9x^6-6x^7+3x^8-x^9 \\
3-6x+9x^2-6x^3+3x^4 & \\
-3x^3+3x^4-3x^5+x^6 & \\
\hline
3-6x+9x^2-9x^3+6x^4-3x^5+x^6 & -3x^3+6x^4-9x^5+9x^6-6x^7+3x^8-x^9 \\
\hline
\end{array}$$

10.

$$\underline{|a^2+3ab-9b^2}$$

$$a^6+9a^5b-135a^3b^3+729ab^5-729b^6$$

$$\underline{a^6}$$

$$\begin{array}{r|l}
3a^4 & 9a^5b-135a^3b^3+729ab^5 \\
9a^3b+9a^2b^2 & \\
\hline
3a^4+9a^3b+9a^2b^2 & 9a^5b+27a^4b^2+27a^3b^3 \\
\hline
3a^4+18a^3b+27a^2b^2 & -27a^4b^2-162a^3b^3+729ab^5-729b^6 \\
-27a^2b^2-81ab^3+81b^4 & \\
\hline
3a^4+18a^3b\qquad -81ab^3+81b^4 & -27a^4b^2-162a^3b^3+729ab^5-729b^6 \\
\hline
\end{array}$$

11.

$$\underline{|c^2-4bc+4b^2}$$

$$c^6-12bc^5+60b^2c^4-160b^3c^3+240b^4c^2-192b^5c+64b^6$$

$$\underline{c^6}$$

$$\begin{array}{r|l}
3c^4 & -12bc^5+60b^2c^4-160b^3c^3 \\
-12bc^3+16b^2c^2 & -12bc^5+48b^2c^4-64b^3c^3 \\
\hline
3c^4-12bc^3+16b^2c^2 & 12b^2c^4-96b^3c^3+240b^4c^2-192b^5c+64b^6 \\
3c^4-24bc^3+48b^2c^2 & \\
12b^2c^2-48b^3c+16b^4 & \\
\hline
3c^4-24bc^3+60b^2c^2-48b^3c+16b^4 & 12b^2c^4-96b^3c^3+240b^4c^2-192b^5c+64b^6 \\
\hline
\end{array}$$

12.

$$\begin{array}{r|l}
 & \underline{|2a^2+4ab-3b^2} \\
 & 8a^6+48a^5b+60a^4b^2-80a^3b^3-90a^2b^4+108ab^5-27b^6 \\
 & 8a^6 \\
12a^4 & 48a^5b+60a^4b^2-80a^3b^3 \\
24a^3b+16a^2b^2 & \\
\hline
12a^4+24a^3b+16a^2b^2 & 48a^5b+96a^4b^2+64a^3b^3 \\
\hline
12a^4+48a^3b+48a^2b^2 & -36a^4b^2-144a^3b^3-90a^2b^4+108ab^5-27b^6 \\
-18a^2b^2-36ab^3+9b^4 & \\
\hline
12a^4+48a^3b+30a^2b^2-36ab^3+9b^4 & -36a^4b^2-144a^3b^3-90a^2b^4+108ab^5-27b^6 \\
\hline
\end{array}$$

Exercise LXXXI.

1.

$$\begin{array}{rr|l}
 & 27\dot{4}62\dot{5} & \underline{|65} \\
6^3 = & 216 & \\
3(60)^2 = & 10800 & 58625 \\
3(60\times 5) = & 900 & \\
5^2 = & 25 & \\
\hline
 & 11725 & 58625 \\
\end{array}$$

3.

$$\begin{array}{rr|l}
 & 26\dot{2}14\dot{4} & \underline{|64} \\
6^3 = & 216 & \\
3(60)^2 = & 10800 & 46144 \\
3(60\times 4) = & 720 & \\
4^2 = & 16 & \\
\hline
 & 11536 & 46144 \\
\end{array}$$

2.

$$\begin{array}{rr|l}
 & 11\dot{0}59\dot{2} & \underline{|48} \\
4^3 = & 64 & \\
3(40)^2 = & 4800 & 46592 \\
3(40\times 8) = & 960 & \\
8^2 = & 64 & \\
\hline
 & 5824 & 46592 \\
\end{array}$$

4.

$$\begin{array}{rr|l}
 & 88\dot{4}.73\dot{6} & \underline{|9.6} \\
9^3 = & 729 & \\
3(90)^2 = & 24300 & 155736 \\
3(90\times 6) = & 1620 & \\
6^2 = & 36 & \\
\hline
 & 25956 & 155736 \\
\end{array}$$

5.

$$\begin{array}{rr|l}
 & 10\dot{9}21\dot{5}35\dot{2} & \underline{|478} \\
4^3 = & 64 & \\
3(40)^2 = & 4800 & 45215 \\
3(40\times 7) = & 840 & \\
7^2 = & 49 & \\
\hline
 & 5689 & 39823 \\
\hline
 & & 5392352 \\
3(470)^2 = & 662700 & \\
3(470\times 8) = & 11280 & \\
8^2 = & 64 & \\
\hline
 & 674044 & 5392352 \\
\end{array}$$

6.

$$\begin{array}{rr|l}
 & & \dot{1}48\dot{1}54\dot{4}\,\underline{|114} \\
1^3 = & & 1 \\
3(10)^2 = & 300 & 481 \\
3(10 \times 1) = & 30 & \\
1^2 = & 1 & \\
 & 331 & 331 \\
 & & 150544 \\
3(110)^2 = & 36300 & \\
3(110 \times 4) = & 1320 & \\
4^2 = & 16 & \\
 & 37636 & 150544
\end{array}$$

7.

$$\begin{array}{rr|l}
 & & \dot{1}60\dot{1}.61\dot{3}\,\underline{|11.7} \\
1^3 = & & 1 \\
3(10)^2 = & 300 & 601 \\
3(10 \times 1) = & 30 & \\
1^2 = & 1 & \\
 & 331 & 331 \\
 & & 270613 \\
3(110)^2 = & 36300 & \\
3(110 \times 7) = & 2310 & \\
(7)^2 = & 49 & \\
 & 38659 & 270613
\end{array}$$

8.

$$\begin{array}{rr|l}
 & & \dot{1}25\dot{9}71\dot{2}\,\underline{|108} \\
1^3 = & & 1 \\
3(10)^2 = & 300 & 259712 \\
3(100)^2 = & 30000 & \\
3(100 \times 8) = & 2400 & \\
8^2 = & 64 & \\
 & 32464 & 259712
\end{array}$$

9.

$$\begin{array}{rr|l}
 & & \dot{2}.80\dot{3}22\dot{1}\,\underline{|1.41} \\
1^3 = & & 1 \\
3(10)^2 = & 300 & 1803 \\
3(10 \times 4) = & 120 & \\
4^2 = & 16 & \\
 & 436 & 1744 \\
 & & 59221 \\
3(140)^2 = & 58800 & \\
3(140 \times 1) = & 420 & \\
1^2 = & 1 & \\
 & 59221 & 59221
\end{array}$$

10.

$$\begin{array}{rr|l}
 & & \dot{7}07\dot{7}88\dot{8}\,\underline{|192} \\
1^3 = & & 1 \\
3(10)^2 = & 300 & 6077 \\
3(10 \times 9) = & 270 & \\
9^2 = & 81 & \\
 & 651 & 5859 \\
 & & 218888 \\
3(190)^2 = & 108300 & \\
3(190 \times 2) = & 1140 & \\
2^2 = & 4 & \\
 & 109444 & 218888
\end{array}$$

11.

$$\begin{array}{rr|l}
 & & 1\dot{2}.81\dot{2}90\dot{4}\,\underline{|2.34} \\
2^3 = & & 8 \\
3(20)^2 = & 1200 & 4812 \\
3(20 \times 3) = & 180 & \\
3^2 = & 9 & \\
 & 1389 & 4167 \\
 & & 645904 \\
3(20 \times 3) = & 180 & \\
2(3)^2 = & 18 & \\
3(230)^2 = & 158700 & \\
3(230 \times 4) = & 2760 & \\
4^2 = & 16 & \\
 & 161476 & 645904
\end{array}$$

12.

$$
\begin{array}{rr|l}
 & & 5\dot{6}.62\dot{3}10\dot{4}\ \underline{|3.84} \\
3^3 = & & \underline{27} \\
3(30)^2 = & 2700 & 29623 \\
3(30\times 8) = & 720 & \\
8^2 = & \underline{64} & \\
 & 3484 & \underline{27872} \\
 & & 1751104 \\
3(380)^2 = & 433200 & \\
3(380\times 4) = & 4560 & \\
4^2 = & \underline{16} & \\
 & 437776 & \underline{1751104}
\end{array}
$$

13.

$$
\begin{array}{rr|l}
 & & 3\dot{3}07\dot{6}.16\dot{1}\ \underline{|32.1} \\
3^3 = & & \underline{27} \\
3(30)^2 = & 2700 & 6076 \\
3(30\times 2) = & 180 & \\
(2)^2 = & \underline{4} & \\
 & 2884 & \underline{5768} \\
 & & 308161 \\
3(30\times 2) = & 180 & \\
2(2)^2 = & \underline{8} & \\
3(320)^2 = & 307200 & \\
3(320\times 1) = & 960 & \\
1^2 = & \underline{1} & \\
 & 308161 & \underline{308161}
\end{array}
$$

14.

$$
\begin{array}{rr|l}
 & & 10\dot{2}50\dot{3}.23\dot{2}\ \underline{|46.8} \\
4^3 = & & \underline{64} \\
3(40)^2 = & 4800 & 38503 \\
3(40\times 6) = & 720 & \\
6^2 = & \underline{36} & \\
 & 5556 & \underline{33336} \\
 & & 5167232 \\
3(460)^2 = & 634800 & \\
3(460\times 8) = & 11040 & \\
8^2 = & \underline{64} & \\
 & 645904 & \underline{5167232}
\end{array}
$$

15.

		$820.025856 \,\lfloor 9.36$
$9^3 =$		729
$3(90)^2 =$	24300	91025
$3(90 \times 3) =$	810	
$3^2 =$	9	
	25119	75357
		15668856
	810	
	18	
$3(930)^2 =$	2594700	
$3(930 \times 6) =$	16740	
$6^2 =$	36	
	2611476	15668856

16.

		$8653.002877 \,\lfloor 20.53$
$2^3 =$		8
$3(200)^2 =$	120000	653002
$5(200 \times 5) =$	3000	
$(5)^2 =$	25	
	123025	615125
		37877877
$3(2050)^2 =$	12607500	
$3(2050 \times 3) =$	18450	
$(3)^2 =$	9	
	12625959	37877877

17.

		$1.371330631 \,\lfloor 1.111$
$1^3 =$		1
$3(10)^2 =$	300	371
$3(10 \times 1) =$	30	
$1^3 =$	1	
	331	331
		40330
$3(110)^2 =$	36300	
$3(110 \times 1) =$	330	
$(1)^2 =$	1	
	36631	36631
		3699631
$3(1110)^2 =$	3696300	
$3(1110 \times 1) =$	3330	
$1^2 =$	1	
	3699631	3699631

18.

		20910.518875 | 27.55
$2^3 =$		8
$3(20)^2 =$	1200	12910
$3(20 \times 7) =$	420	
$7^2 =$	49	
	1669	11683
		1227518
	420	
	98	
$3(270)^2 =$	218700	
$3(270 \times 5) =$	4050	
$5^2 =$	25	
	222775	1113875
		113643875
	4050	
	50	
$3(2750)^2 =$	22687500	
$3(2750 \times 5) =$	41250	
$5^2 =$	25	
	22728775	113643875

19.

		91.398648466125 | 4.5045
$4^3 =$		64
$3(40)^2 =$	4800	27398
$3(40 \times 5) =$	600	
$(5)^2 =$	25	
	5425	27125
	600	273648466
	50	
	60750000	
$3(4500 \times 4) =$	54000	
$4^2 =$	16	243216064
	60804016	30432402125
	54000	
	32	
	6085804800	
$3(45040 \times 5) =$	675600	
$5^2 =$	25	
	6086480425	30432402125

20.

		5.340104393239	1.7479
$1^3 =$		1	
$3(10)^2 =$	300	4340	
$3(10 \times 7) =$	210		
$(7)^2 =$	49		
	559	3913	
	210	427104	
	98		
$3(170)^2 =$	86700		
$3(170 \times 4) =$	2040		
$(4)^2 =$	16		
	88756	355024	
	2040	72080393	
	32		
$3(1740)^2 =$	9082800		
$3(1740 \times 7) =$	36540		
$7^2 =$	49		
	9119389	63835723	
	36540	8244670239	
	98		
$3(17470)^2 =$	915602700		
$3(17470 \times 9) =$	471690		
$9^2 =$	81		
	916074471	8244670239	

21.

(1)

		2.500	1.3572
$1^3 =$		1	
$3(10)^2 =$	300	1500	
$3(10 \times 3) =$	90		
$3^2 =$	9	1197	
	399	303000	
	90		
	18		
$3(130)^2 =$	50700		
$3(130 \times 5) =$	1950		
$5^2 =$	25	263375	
	52675	39625000	
	1950		
	50		
$3(1350)^2 =$	5467500		
$3(1350 \times 7) =$	28350		
$7^2 =$	49	38471293	
	5495899	1153707000	

(2)

		$0.20\dot{0}00\dot{0}00\dot{0}00\dot{0}$	0.5848
$5^3 =$		125	
$3(50)^2 =$	7500	75000	
$3(50 \times 8) =$	1200		
$8^2 =$	64		
	8764	70112	
	1200	4888000	
	128		
$3(580)^2 =$	1009200		
$3(580 \times 4) =$	6960		
$4^2 =$	16		
	1016176	4064704	
	6960	823296000	
	32		
$3(5840)^2 =$	102316800		
$3(5840 \times 8) =$	140160		
$8^2 =$	64		
	102457024	819656192	

(3)

		$0.01\dot{0}00\dot{0}00\dot{0}00\dot{0}$	0.2154
$2^3 =$		8	
$3(20)^2 =$	1200	2000	
$3(20 \times 1) =$	60		
$1^2 =$	1		
	1261	1261	
	60	739000	
	2		
$3(210)^2 =$	132300		
$3(210 \times 5) =$	3150		
$5^2 =$	25		
	135475	677375	
	3150	61625000	
	50		
$3(2150)^2 =$	13867500		
$3(2150 \times 4) =$	25800		
$4^2 =$	16		
	13893316	55573264	

(4)

		4.000000000000	1.5874
$1^3 =$		1	
$3(10)^2 =$	300	3000	
$3(10 \times 5) =$	150		
$5^2 =$	25		
	475	2375	
	150	625000	
	50		
$3(150)^2 =$	67500		
$3(150 \times 8) =$	3600		
$8^2 =$	64		
	71164	569312	
	3600	55688000	
	128		
$3(1580)^2 =$	7489200		
$3(1580 \times 7) =$	33180		
$7^2 =$	49		
	7522429	52657003	
	33180	3030997000	
	98		
$3(15870)^2 =$	755570700		
$3(15870 \times 4) =$	190440		
$4^2 =$	16		
	755761156	3023044624	

(5)

		0.400000000000	0.7368
$7^3 =$		343	
$3(70)^2 =$	14700	57000	
$3(70 \times 3) =$	630		
$3^2 =$	9		
	15339	46017	
	630	10983000	
	18		
$3(730)^2 =$	1598700		
$3(730 \times 6) =$	13140		
$6^2 =$	36		
	1611876	9671256	
	13140	1611744000	
	72		
$3(7360)^2 =$	162508800		
$3(7360 \times 8) =$	196640		
$8^2 =$	64		
	162705504	1301644032	

Exercise LXXXII.

1.

$$\begin{array}{rl}
 & 81a^4-540a^3b+1350a^2b^2-1500ab^3+625b^4 \;\big|\; \underline{9a^2-30ab+25b^2} \\
 & 81a^4 \\
18a^2-30ab & \big|\; -540a^3b+1350a^2b^2 \\
 & \big|\; \underline{-540a^3b+\;900a^2b^2} \\
18a^2-60ab+25b^2 & \big|\; 450a^2b^2-1500ab^3+625b^4 \\
 & \big|\; \underline{450a^2b^2-1500ab^3+625b^4}
\end{array}$$

$$\begin{array}{rl}
 & 9a^2-30ab+25b^2 \;\big|\; \underline{3a-5b} \\
 & 9a^2 \\
6a-5b & \big|\; -30ab+25b^2 \\
 & \big|\; \underline{-30ab+25b^2}
\end{array}$$

2.

$$\begin{array}{rl}
 & x^8-4x^7+10x^6-16x^5+19x^4-16x^3+10x^2-4x+1 \;\big|\; \underline{x^4-2x^3+3x^2-2x+1} \\
 & x^8 \\
2x^4-2x^3 & \big|\; -4x^7+10x^6 \\
 & \big|\; \underline{-4x^7+\;4x^6} \\
2x^4-4x^3+3x^2 & \big|\; 6x^6-16x^5+19x^4 \\
 & \big|\; \underline{6x^6-12x^5+\;9x^4} \\
2x^4-4x^3+6x^2-2x & \big|\; -4x^5+10x^4-16x^3+10x^2 \\
 & \big|\; \underline{-4x^5+\;8x^4-12x^3+\;4x^2} \\
2x^4-4x^3+6x^2-4x+1 & \big|\; 2x^4-\;4x^3+\;6x^2-4x+1 \\
 & \big|\; \underline{2x^4-\;4x^3+\;6x^2-4x+1}
\end{array}$$

$$\begin{array}{rl}
 & x^4-2x^3+3x^2-2x+1 \;\big|\; \underline{x^2-x+1} \\
 & x^4 \\
2x^2-x & \big|\; -2x^3+3x^2 \\
 & \big|\; \underline{-2x^3+\;x^2} \\
2x^2-2x+1 & \big|\; 2x^2-2x+1 \\
 & \big|\; \underline{2x^2-2x+1}
\end{array}$$

3.

$$\begin{array}{rl}
 & 64-192x+240x^2-160x^3+60x^4-12x^5+x^6 \;\big|\; \underline{8-12x+6x^2-x^3} \\
 & 64 \\
16-12x & \big|\; -192x+240x^2 \\
 & \big|\; \underline{-192x+144x^2} \\
16-24x+6x^2 & \big|\; 96x^2-160x^3+60x^4 \\
 & \big|\; \underline{96x^2-144x^3+36x^4} \\
16-24x+12x^2-x^3 & \big|\; -\;16x^3+24x^4-12x^5+x^6 \\
 & \big|\; \underline{-\;16x^3+24x^4-12x^5+x^6}
\end{array}$$

$$
\begin{array}{rr|l}
 & 8 - 12x + 6x^2 - x^3 & \underline{2 - x} \\
2^3 = & 8 & \\
3(2)^2 = 12 & & -12x + 6x^2 - x^3 \\
(6 - x)(-x) = & -6x + x^2 & \\
 & 12 - 6x + x^2 & -12x + 6x^2 - x^3
\end{array}
$$

4.

$$
\begin{array}{rl}
 & 729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1 \mid \underline{9x^2 - 6x + 1} \\
(9x^2)^3 = & 729x^6 \\
243x^4 & \\
-162x^3 & -1458x^5 + 1215x^4 - 540x^3 \\
+36x^2 & \\
243x^4 - 162x^3 + 36x^2 & -1458x^5 + 972x^4 - 216x^3 \\
 & 243x^4 - 324x^3 + 135x^2 - 18x + 1 \\
243x^4 - 324x^3 + 108x^2 & \\
27x^2 - 18x & \\
+1 & \\
243x^4 - 324x^3 + 135x^2 - 18x + 1 & 243x^4 - 324x^3 + 135x^2 - 18x + 1.
\end{array}
$$

$$
\begin{array}{rl}
 & 9x^2 - 6x + 1 \mid \underline{3x - 1} \\
 & 9x^2 \\
6x - 1 & -6x + 1 \\
 & -6x + 1
\end{array}
$$

5.

$$
\begin{array}{rl}
 & 1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8 \mid \underline{1 - 4y + 6y^2 - 4y^3 + y^4} \\
 & 1 \\
2 - 4y & -8y + 28y^2 \\
 & -8y + 16y^2 \\
2 - 8y + 6y^2 & 12y^2 - 56y^3 + 70y^4 \\
 & 12x^2 - 48y^3 + 36y^4 \\
2 - 8y + 12y^2 - 4y^3 & -8y^3 + 34y^4 - 56y^5 + 28y^6 \\
 & -8y^3 + 32y^4 - 48y^5 + 16y^6 \\
2 - 8y + 12y^2 - 8y^3 + y^4 & 2y^4 - 8y^5 + 12y^6 - 8y^7 + y^8 \\
 & 2y^4 - 8y^5 + 12y^6 - 8y^7 + y^8
\end{array}
$$

$$
\begin{array}{rl}
 & 1 - 4y + 6y^2 - 4y^3 + y^4 \mid \underline{1 - 2y + y^2} \\
 & 1 \\
2 - 2y & -4y + 6y^2 \\
 & -4y + 4y^2 \\
2 - 4y + y^2 & 2y^2 - 4y^3 + y^4 \\
 & 2y^2 - 4y^3 + y^4
\end{array}
$$

$$
\begin{array}{rl}
 & 1 - 2y + y^2 \mid \underline{1 - y} \\
 & 1 \\
2 - y & -2y + y^2 \\
 & -2y + y^2
\end{array}
$$

Exercise LXXXIII.

1. $$x^2 - 3 = 46,$$
$$x^2 = 49.$$
$$\therefore x = \pm 7.$$

2. $2(x^2-1)-3(x^2+1)+14 = 0.$
Simplify,
$$2x^2-2-3x^2-3+14 = 0,$$
$$x^2 = 9.$$
$$\therefore x = \pm 3.$$

3. $$\frac{x^2-5}{3} + \frac{2x^2+1}{6} = \frac{1}{2}.$$
Simplify,
$$2x^2 - 10 + 2x^2 + 1 = 3,$$
$$4x^2 = 12,$$
$$x^2 = 3.$$
$$\therefore x = \pm\sqrt{3}.$$

4. $$\frac{3}{1+x} + \frac{3}{1-x} = 8.$$
Simplify,
$$3 - 3x + 3 + 3x = 8 - 8x^2,$$
$$8x^2 = 2,$$
$$x^2 = \tfrac{1}{4}.$$
$$\therefore x = \pm\tfrac{1}{2}.$$

5. $$\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}.$$
Simplify, $9 - 2 = 28x^2,$
$$-28x^2 = -7,$$
$$x^2 = \tfrac{1}{4}.$$
$$\therefore x = \pm\tfrac{1}{2}.$$

6. $$5x^2 - 9 = 2x^2 + 24,$$
$$3x^2 = 33,$$
$$x^2 = 11.$$
$$\therefore x = \pm\sqrt{11}.$$

7. $$(x+2)^2 = 4x + 5.$$
Simplify,
$$x^2 + 4x + 4 = 4x + 5.$$
Transpose and combine,
$$x^2 = 1.$$
$$\therefore x = \pm 1.$$

8. $$\frac{x^2}{5} - \frac{x^2-10}{15} = 7 - \frac{50+x^2}{25}.$$
Simplify,
$$15x^2 - 5x^2 + 50 = 525 - 150 - 3x^2.$$
Transpose and combine,
$$13x^2 = 325,$$
$$x^2 = 25.$$
$$\therefore x = \pm 5.$$

9. $$\frac{3x^2-27}{x^2+3} + \frac{90+4x^2}{x^2+9} = 7.$$
Simplify, $(3x^2-27)(x^2+9) + (90+4x^2)(x^2+3) = 7(x^2+3)(x^2+9).$
$$3x^4 - 243 + 4x^4 + 102x^2 + 270 = 7x^4 + 84x^2 + 189,$$
$$18x^2 = 162,$$
$$x^2 = 9.$$
$$\therefore x = \pm 3.$$

10. $$8x + \frac{7}{x} = \frac{65x}{7}.$$
Simplify, $56x^2 + 49 = 65x^2.$
Transpose and combine,
$$-9x^2 = -49,$$
$$x^2 = \tfrac{49}{9}.$$
$$\therefore x = \pm\tfrac{7}{3}$$
$$= \pm 2\tfrac{1}{3}.$$

11. $$\frac{4x^2+5}{10} - \frac{2x^2-5}{15} = \frac{7x^2-25}{20}.$$
Simplify,
$$24x^2 + 30 - 8x^2 + 20 = 21x^2 - 75,$$
$$24x^2 - 8x^2 - 21x^2 = -75 - 30 - 20,$$
$$-5x^2 = -125,$$
$$x^2 = 25.$$
$$\therefore x = \pm 5.$$

12.

$$\frac{10x^2+17}{18}-\frac{12x^2+2}{11x^2-8}=\frac{5x^2-4}{9}.$$

Simplify, $110x^4+107x^2-136-216x^2-36=110x^4-168x^2+64.$

Transpose and combine, $59x^2=236,$

$$x^2=4.$$

$$\therefore x=\pm 2.$$

13.

$$\frac{14x^2+16}{21}-\frac{2x^2+8}{8x^2-11}=\frac{2x^2}{3}.$$

Simplify, $112x^4-26x^2-176-42x^2-168=112x^4-154x^2.$

Transpose and combine, $86x^2=344,$

$$x^2=4.$$

$$\therefore x=\pm 2.$$

14.
$$x^2+bx+a=bx(1-bx),$$
$$x^2+bx+a=bx-b^2x^2,$$
$$x^2+b^2x^2=-a,$$
$$x^2=-\frac{a}{1+b^2}.$$
$$\therefore x=\pm\sqrt{-\frac{a}{1+b^2}}.$$

15.
$$mx^2+n=q,$$
$$mx^2=q-n.$$
$$\therefore x=\pm\sqrt{\frac{q-n}{m}}.$$

16.
$$x^2-ax+b=ax(x-1),$$
$$x^2-ax+b=ax^2-ax,$$
$$x^2-ax^2=-b,$$
$$x^2(1-a)=-b,$$
$$x^2=\frac{b}{a-1}.$$
$$\therefore x=\pm\sqrt{\frac{b}{a-1}}.$$

Exercise LXXXIV.

1. $x^2+4x=12,$

Complete the square,

$$x^2+4x+4=16.$$

Extract the root,

$$x+2=\pm 4.$$

$$\therefore x=2 \text{ or } -6.$$

2. $x^2-6x=16.$

Complete the square,

$$x^2-6x+9=25.$$

Extract the root,

$$x-3=\pm 5.$$

$$\therefore x=8 \text{ or } -2.$$

3.
$$x^2 - 12x + 6 = \tfrac{1}{4}.$$
$$x^2 - 12x = -\tfrac{23}{4}.$$
Complete the square,
$$x^2 - 12x + 36 = \tfrac{121}{4}.$$
Extract the root,
$$x - 6 = \pm \tfrac{11}{2}.$$
$$\therefore x = 11\tfrac{1}{2} \text{ or } \tfrac{1}{2}.$$

4.
$$x^2 - 7x = 8.$$
Multiply by 4,
$$4x^2 - 28x = 32.$$
Complete the square,
$$4x^2 - (\ ) + 49 = 81.$$
Extract the root,
$$2x - 7 = \pm 9,$$
$$2x = 7 \pm 9.$$
$$\therefore x = 8 \text{ or } -1.$$

5.
$$3x^2 - 4x = 7.$$
Multiply by 3,
$$9x^2 - 12x = 21.$$
Complete the square,
$$9x^2 - 12x + 4 = 25.$$
Extract the root,
$$3x - 2 = \pm 5,$$
$$3x = 7 \text{ or } -3.$$
$$\therefore x = 2\tfrac{1}{3} \text{ or } -1.$$

6.
$$12x^2 + x - 1 = 0.$$
$$12x^2 + x = 1.$$
Multiply by 3,
$$36x^2 + 3x = 3.$$
Complete the square,
$$36x^2 + (\ ) + \frac{1}{16} = \frac{49}{16}.$$
Extract the root,
$$6x + \tfrac{1}{4} = \pm \tfrac{7}{4},$$
$$6x = \tfrac{6}{4} \text{ or } -2.$$
$$\therefore x = \tfrac{1}{4} \text{ or } -\tfrac{1}{3}.$$

7.
$$x^2 - x = 6.$$
Complete the square,
$$x^2 - x + \frac{1}{4} = \frac{25}{4}.$$
Extract the root,
$$x - \frac{1}{2} = \pm \frac{5}{2},$$
$$x = \frac{1 \pm 5}{2}.$$
$$\therefore x = 3 \text{ or } -2.$$

8.
$$5x^2 - 3x = 2.$$
Multiply by 5,
$$25x^2 - 15x = 10,$$
Complete the square,
$$25x^2 - (\ ) + \frac{9}{4} = \frac{49}{4}.$$
Extract the root,
$$5x - \frac{3}{2} = \pm \frac{7}{2},$$
$$5x = \frac{3 \pm 7}{2}.$$
$$\therefore x = 1 \text{ or } -\tfrac{2}{5}.$$

9.
$$2x^2 - 27x = 14.$$
Multiply by 8,
$$16x^2 - 216x = 112.$$
Complete the square,
$$16x^2 - (\ ) + 729 = 841.$$
Extract the root,
$$4x - 27 = \pm 29,$$
$$4x = 56 \text{ or } -2.$$
$$\therefore x = 14 \text{ or } -\tfrac{1}{2}.$$

10. $$x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$$

$$x^2 - \frac{2x}{3} = -\frac{1}{12}.$$

Complete the square,

$$x^2 - \frac{2x}{3} + \frac{1}{9} = \frac{1}{36}.$$

Extract the root,

$$x - \frac{1}{3} = \pm\frac{1}{6},$$

$$x = \frac{1}{3} \pm \frac{1}{6}.$$

$$\therefore\ x = \frac{1}{2} \text{ or } \frac{1}{6}.$$

11. $$\frac{x^2}{2} - \frac{x}{3} = 2(x+2).$$

Simplify, $3x^2 - 2x = 12x + 24,$

$$3x^2 - 14x = 24.$$

Multiply by 3,

$$9x^2 - 42x = 72.$$

Complete the square,

$$9x^2 - (\ ) + 49 = 121.$$

Extract the root,

$$3x - 7 = \pm 11,$$

$$3x = 18 \text{ or } -4.$$

$$\therefore\ x = 6 \text{ or } -1\tfrac{1}{3}.$$

12. $$\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$$

Simplify,

$$9x^2 - 26x = -16.$$

Complete the square,

$$9x^2 - 26x + \frac{169}{9} = \frac{25}{9}.$$

Extract the root,

$$3x - \frac{13}{3} = \pm\frac{5}{3},$$

$$3x = 6 \text{ or } \frac{8}{3}.$$

$$\therefore\ x = 2 \text{ or } \frac{8}{9}.$$

13. $$\frac{x+1}{x+4} = \frac{2x-1}{x+6}.$$

Simplify,

$$(x+1)(x+6) = (2x-1)(x+4).$$

$$x^2 + 7x + 6 = 2x^2 + 7x - 4,$$

$$x^2 - 2x^2 + 7x - 7x = -4 - 6,$$

$$-x^2 = -10,$$

$$x^2 = 10.$$

$$\therefore\ x = \pm\sqrt{10}.$$

14. $$\frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}.$$

Simplify, $$18x^2 + 72x - 9x^2 - 36x - 27 = -x^2 - 5x - 4$$

Transpose and combine, $$10x^2 + 41x = 23.$$

Multiply by 10, $$100x^2 + 410x = 230.$$

Complete the square, $$100x^2 + (\ ) + \frac{1681}{4} = \frac{2601}{4}.$$

Extract the root, $$10x + \frac{41}{2} = \pm\frac{51}{2},$$

$$10x = -\frac{41}{2} \pm \frac{51}{2}.$$

$$\therefore\ x = \tfrac{1}{2} \text{ or } -4\tfrac{3}{5}.$$

15. $$\frac{2}{x-1}=\frac{3}{x-2}+\frac{2}{x-4}.$$

Simplify, $2x^2-12x+16=3x^2-15x+12+2x^2-6x+4.$

Transpose and combine, $-3x^2+9x=0.$

Divide by -3, $x^2-3x=0,$

$$\text{or } x(x-3)=0.$$

$$\therefore x=3 \text{ or } 0.$$

16.

$$5x(x-3)-2(x^2-6)=(x+3)(x+4).$$

Simplify, $5x^2-15x-2x^2+12=x^2+7x+12.$

Transpose and combine, $2x^2-22x=0,$

$$x^2-11x=0,$$

$$\text{or } x(x-11)=0.$$

$$\therefore x=11 \text{ or } 0.$$

17.

$$\frac{3x}{2(x+1)}-\frac{5}{8}=\frac{3x^2}{x^2-1}-\frac{23}{4(x-1)}.$$

Simplify, $12x^2-12x-5x^2+5=24x^2-46x-46.$

Transpose and combine, $17x^2-34x=51,$

Divide by 17, $x^2-2x=3.$

Complete the square, $x^2-(\ )+1=4.$

Extract the root, $x-1=\pm 2.$

$$\therefore x=3 \text{ or } -1.$$

18.

$$(x-2)(x-4)-2(x-1)(x-3)=0.$$

Simplify, $x^2-6x+8-2x^2+8x-6=0.$

Transpose and combine, $x^2-2x=2.$

Complete the square, $x^2-2x+1=3.$

Extract the root, $x-1=\pm\sqrt{3}.$

$$\therefore x=1\pm\sqrt{3}.$$

19.

$$\frac{1}{7}(x-4)-\frac{2}{5}(x-2)=\frac{1}{x}(2x+3).$$

Simplify, $5x^2-20x-14x^2+28x=70x+105.$

Transpose and combine, $9x^2+62x=-105.$

Complete the square, $324x^2+(\ )+(62)^2=64.$

Extract the root, $18x+62=\pm 8,$

$$18x=-54 \text{ or } -70.$$

$$\therefore x=-3 \text{ or } -3\tfrac{8}{9}.$$

20.

$$\frac{2}{5}(3x^2 - x - 5) - \frac{1}{3}(x^2 - 1) = 2(x - 2)^2.$$

Simplify, $18x^2 - 6x - 30 - 5x^2 + 5 = 30x^2 - 120x + 120,$

$17x^2 - 114x = -145.$

7

Multiply by 17, $289x^2 - 1938x = -2465.$

Complete the square, $289x^2 - (\ ) + 3249 = 784.$

Extract the root, $17x - 57 = \pm 28.$

$\therefore x = 5 \text{ or } 1\frac{12}{17}.$

6

21.

$$\frac{2x}{15} + \frac{3x - 50}{3(10 + x)} = \frac{12x + 70}{190}.$$

Simplify, $760x + 78x^2 + 510x - 9500 = 38x^2 + 510x + 2100,$

$40x^2 + 760x = 11600.$

7

Divide by 10, $4x^2 + 76x = 1160.$

Complete the square, $4x^2 + (\ ) + 361 = 1521.$

Extract the root, $2x + 19 = \pm 39.$

$\therefore x = 10 \text{ or } -29.$

22.

$$\frac{x}{x^2 - 1} = \frac{15 - 7x}{8(1 - x)},$$

$$\text{or } \frac{x}{(x + 1)(x - 1)} = \frac{7x - 15}{8(x - 1)}.$$

Simplify, $8x = 7x^2 - 8x - 15,$

$7x^2 - 16x = 15.$

Multiply by 7, $49x^2 - 112x = 105.$

Complete the square, $49x^2 - (\ ) + 64 = 169.$

Extract the root, $7x - 8 = \pm 13.$

$\therefore x = 3 \text{ or } -\frac{5}{7}.$

23.

$$\frac{2x - 1}{x - 1} + \frac{1}{6} = \frac{2x - 3}{x - 2}.$$

Simplify, $12x^2 - 30x + 12 + x^2 - 3x + 2 = 12x^2 - 30x + 18.$

Transpose and combine, $x^2 - 3x = 4.$

Complete the square, $4x^2 - (\ ) + 9 = 25.$

Extract the root, $2x - 3 = \pm 5.$

$\therefore x = 4 \text{ or } -1.$

24.

$$\frac{x+2}{x-1}-\frac{4-x}{2x}=\frac{7}{3}.$$

Simplify, $6x^2+12x+3x^2-15x+12=14x^2-14x.$

Transpose and combine, $5x^2-11x=12.$

Multiply by 5, $25x^2-55x=60.$

Complete the square, $25x^2-(\ )+\frac{121}{4}=\frac{361}{4}.$

Extract the root, $5x-\frac{11}{2}=\pm\frac{19}{2},$

$$5x=15 \text{ or } -4.$$

$$\therefore x=3 \text{ or } -\tfrac{4}{5}.$$

25.

$$x-\frac{14x-9}{8x-3}=\frac{x^2-3}{x+1}.$$

Simplify, $8x^3+5x^2-3x-14x^2-5x+9=8x^3-24x-3x^2+9.$

Transpose and combine, $-6x^2+16x=0.$

Divide by -2, $3x^2-8x=0,$

$$x(3x-8)=0.$$

$$\therefore x=0 \text{ or } 2\tfrac{2}{3}.$$

26.

$$1-\frac{x+5}{2x+1}=\frac{x-6}{x-2}.$$

Simplify, $2x^2-3x-2-x^2-3x+10=2x^2-11x-6.$

Transpose and combine, $x^2-5x=14.$

Complete the square, $x^2-(\ )+\frac{25}{4}=\frac{81}{4}.$

Extract the root, $x-\frac{5}{2}=\pm\frac{9}{2}.$

$$\therefore x=7 \text{ or } -2.$$

27.

$$\frac{x}{7-x}+\frac{7-x}{x}=2\tfrac{9}{10}.$$

Simplify, $10x^2+490-140x+10x^2=203x-29x^2.$

Transpose and combine, $49x^2-343x=-490.$

Divide by 49, $x^2-7x=-10.$

Complete the square, $x^2-(\ )+\frac{49}{4}=\frac{9}{4}.$

Extract the root, $x-\frac{7}{2}=\pm\frac{3}{2}.$

$$\therefore x=5 \text{ or } 2.$$

28.

$$\frac{2x+3}{2(2x-1)}-\frac{7-x}{2(x+1)}=\frac{7-3x}{4-3x}.$$

Simplify, $-14x^2-12x^3+22x+24-12x^3+106x^2-162x+56$
$= 44x^2-40x-24x^3-28.$

Transpose and combine, $48x^2-180x=-108.$

Divide by 12, $4x^2-15x=-9.$

Multiply by 16 and complete the square,

$$64x^2-(\ )+225=81.$$

Extract the root, $8x-15=\pm 9,$

$$8x=24 \text{ or } 6.$$

$$\therefore x=3 \text{ or } \tfrac{3}{4}.$$

29.

$$\frac{12x^3-11x^2+10x-78}{8x^2-7x+6}=1\tfrac{1}{2}\,x-\frac{1}{2}.$$

Simplify, $24x^3-22x^2+20x-156=24x^3-21x^2+18x-8x^2+7x-6.$

Transpose and combine, $7x^2-5x=150.$

Multiply by 28, $196x^2-140x=4200.$

Complete the square, $196x^2-(\ )+25=4225.$

Extract the root, $14x-5=\pm 65.$

$$\therefore x=5 \text{ or } -4\tfrac{2}{7}.$$

30.

$$\frac{6}{x-1}-\frac{18}{x+5}=\frac{7}{x+1}-\frac{8}{x-5},$$

$$\text{or } \frac{6}{x-1}-\frac{7}{x+1}=\frac{18}{x+5}-\frac{8}{x-5}.$$

Combine, $\dfrac{13-x}{x^2-1}=\dfrac{10x-130}{x^2-25},$

$$\text{or } \frac{13-x}{x^2-1}=-\frac{10(13-x)}{x^2-25}.$$

$$\therefore x=13.$$

Hence, if $13-x=0$, the equation is satisfied.

Otherwise we may divide by $13-x$.

$$\frac{1}{x^2-1}=\frac{-10}{x^2-25}.$$

Simplify, $x^2-25=10-10x^2,$

$$11x^2=35,$$

$$x^2=\frac{35}{11}.$$

$$\therefore x=\pm\sqrt{\frac{35}{11}}.$$

Exercise LXXXV.

1. $x^2 + 2ax = a^2$.

Complete the square,

$$x^2 + 2ax + a^2 = 2a^2.$$

Extract the root,

$$x + a = \pm a\sqrt{2}.$$

$$\therefore x = -a \pm a\sqrt{2}.$$

2. $x^2 = 4ax + 7a^2$.

Transpose,

$$x^2 - 4ax = 7a^2.$$

Complete the square,

$$x^2 - 4ax + 4a^2 = 11a^2.$$

Extract the root,

$$x - 2a = \pm a\sqrt{11}.$$

$$\therefore x = 2a \pm a\sqrt{11}.$$

3. $x^2 = \dfrac{7m^2}{4} - 3mx.$

$$4x^2 = 7m^2 - 12mx,$$

$$4x^2 + 12mx = 7m^2,$$

$$4x^2 + (\ ) + (3m)^2 = 16m^2,$$

$$2x + 3m = \pm 4m,$$

$$2x = \pm 4m - 3m.$$

$$\therefore x = \frac{m}{2} \text{ or } -\frac{7m}{2}.$$

4. $x^2 - \dfrac{5nx}{2} - \dfrac{3n^2}{2} = 0.$

$$2x^2 - 5nx - 3n^2 = 0,$$

$$4x^2 - 10nx = 6n^2,$$

$$4x^2 - (\ ) + \frac{25n^2}{4} = \frac{49n^2}{4},$$

$$2x - \frac{5n}{2} = \pm\frac{7n}{2},$$

$$2x = 6n \text{ or } -n.$$

$$\therefore x = 3n \text{ or } -\frac{n}{2}.$$

5. $\dfrac{a^2}{(x+a)^2} = \dfrac{b^2}{(x-a)^2}.$

$$a^2(x-a)^2 = b^2(x+a)^2,$$

$$a(x-a) = \pm b(x+a),$$

$$ax - a^2 = bx + ab,$$

$$x(a-b) = a^2 + ab.$$

$$\therefore x = \frac{a(a+b)}{a-b};$$

$$\text{or } ax - a^2 = -bx - ab,$$

$$x(a+b) = a^2 - ab.$$

$$\therefore x = \frac{a(a-b)}{a+b}.$$

6. $cx = ax^2 + bx^2 - \dfrac{ac}{a+b}.$

$$acx + bcx = a^2x^2 + 2abx^2 + b^2x^2 - ac.$$

$$a^2x^2 + 2abx^2 + b^2x^2 - acx - bcx = ac,$$

$$x^2(a^2 + 2ab + b^2) - x(ac + bc) = ac,$$

$$x^2(a^2 + 2ab + b^2) - (\ ) + \frac{c^2}{4} = \frac{4ac + c^2}{4},$$

$$x(a+b) - \frac{c}{2} = \pm\frac{\sqrt{4ac + c^2}}{2},$$

$$x(a+b) = \frac{c \pm \sqrt{4ac + c^2}}{2}.$$

$$\therefore x = \frac{c \pm \sqrt{4ac + c^2}}{2(a+b)}.$$

7.
$$\frac{a^2x^2}{b^2}+\frac{b^2}{c^2}=\frac{2ax}{c}.$$
$$a^2c^2x^2+b^4=2ab^2cx,$$
$$a^2c^2x^2-2ab^2cx=-b^4,$$
$$a^2c^2x^2-(\ )+b^4=0,$$
$$acx-b^2=0,$$
$$acx=b^2,$$
$$\therefore x=\frac{b^2}{ac}.$$

8
$$(a^2+1)x=ax^2+a.$$
$$a^2x+x=ax^2+a,$$
$$ax^2-(a^2+1)x=-a,$$
$$4a^2x^2-(\ )+(a^2+1)^2=a^4-2a^2+1,$$
$$2ax-(a^2+1)=\pm(a^2-1),$$
$$2ax=(a^2+1)\pm(a^2-1).$$
$$\therefore x=a \text{ or } \frac{1}{a}.$$

9.
$$\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}.$$
$$a(x-b)(x-c)+b(x-a)(x-c)=2c(x-a)(x-b),$$
$$ax^2-abx-acx+abc+bx^2-abx-bcx+abc=2cx^2-2acx-2bcx+2abc,$$
$$x^2(a+b-2c)+x(ac+bc-2ab)=0.$$
$$\therefore x=0 \text{ or } \frac{2ab-ac-bc}{a+b-2c}.$$

10.
$$\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}.$$
$$abx=abx+b^2x+bx^2+a^2x+abx+ax^2+a^2b+ab^2+abx,$$
$$x^2(a+b)+x(a+b)^2=-ab(a+b).$$
Divide by $(a+b)$, $x^2+(a+b)x=-ab,$
$$4x^2+(\ )+(a+b)^2=a^2-2ab+b^2,$$
$$2x+(a+b)=\pm(a-b),$$
$$2x=-2b \text{ or } -2a.$$
$$\therefore x=-b \text{ or } -a.$$

11.
$$\frac{1}{a-x}-\frac{1}{a+x}=\frac{3+x^2}{a^2-x^2}.$$
$$a+x-a+x=3+x^2,$$
$$x^2-2x=-3,$$
$$x^2-2x+1=-2,$$
$$x-1=\pm\sqrt{-2}.$$
$$\therefore x=1\pm\sqrt{-2}.$$

12.
$$\frac{x^2+2ab(a^2+b^2)}{a^2+b^2}=2x.$$
$$x^2+2ab(a^2+b^2)=2x(a^2+b^2),$$
$$x^2-2x(a^2+b^2)=-2a^3b-2ab^3,$$
$$x^2-(\ )+(a^2+b^2)^2=a^4-2a^3b+2a^2b^2-2ab^3+b^4,$$
$$x^2-(\ )+(a^2+b^2)^2=(a^2+b^2)(a-b)^2,$$
$$x-(a^2+b^2)=\pm(a-b)\sqrt{a^2+b^2}.$$
$$\therefore x=a^2+b^2\pm(a-b)\sqrt{a^2+b^2}.$$

13. $$\frac{(2x-a)^2}{2x-a+2b}=b.$$

$$4x^2-4ax+a^2=2bx-ab+2b^2,$$
$$4x^2-2x(2a+b)=-a^2-ab+2b^2,$$
$$16x^2-(\ )+(2a+b)^2=9b^2,$$
$$4x-(2a+b)=\pm 3b,$$
$$4x=2a-2b,$$
$$\text{or } 2a+4b.$$
$$\therefore x=\frac{a-b}{2}\text{ or }\frac{a+2b}{2}.$$

14. $$x^2+ax=a+x.$$

$$x^2+ax-x=a,$$
$$x^2+(a-1)x=a,$$
$$4x^2+(\ )+(a-1)^2=a^2+2a+1,$$
$$2x+(a-1)=\pm(a+1),$$
$$2x=-(a-1)\pm(a+1).$$
$$\therefore x=1\text{ or }-a.$$

15. $$x^2+ax=bx+ab.$$

$$x^2+(a-b)x=ab,$$
$$4x^2+(\ )+(a-b)^2=(a+b)^2,$$
$$2x+(a-b)=\pm(a+b),$$
$$2x=-2a\text{ or }2b.$$
$$\therefore x=-a\text{ or }b.$$

16. $$\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}.$$

$$x^2b+a^2b=ax^2+ab^2,$$
$$x^2b-x^2a=ab^2-a^2b,$$
$$x^2(b-a)=ab(b-a).$$

Divide by $(b-a)$,

$$x^2=ab.$$
$$\therefore x=\pm\sqrt{ab}.$$

17.

$$\frac{1}{x}+\frac{1}{x+b}=\frac{1}{a}+\frac{1}{a+b}.$$

$$a^2x+a^2b+abx+ab^2+a^2x+abx=ax^2+abx+bx^2+b^2x+ax^2+abx,$$
$$2ax^2+bx^2-2a^2x+b^2x=a^2b+ab^2,$$
$$(2a+b)x^2-(2a^2-b^2)x=a^2b+ab^2,$$
$$4x^2(2a+b)^2-(\ )+(2a^2-b^2)^2=4a^4+8a^3b+8a^2b^2+4ab^3+b^4,$$
$$2x(2a+b)-(2a^2-b^2)=\pm(2a^2+2ab+b^2),$$
$$2x(2a+b)=(4a^2+2ab)\text{ or }-(2ab+2b^2).$$
$$\therefore x=a\text{ or }-\frac{b(a+b)}{2a+b}.$$

18. $$\frac{a}{3}+\frac{5x}{4}-\frac{x^2}{3a}=0.$$

$$4a^2+15ax-4x^2=0,$$
$$4x^2-15ax=4a^2,$$
$$64x^2-(\ )+225a^2=289a^2,$$
$$8x-15a=\pm 17a,$$
$$8x=32a\text{ or }-2a.$$
$$\therefore x=4a\text{ or }-\frac{a}{4}.$$

19. $$\frac{x+3}{x-3}=a+\frac{x-3}{x+3}.$$

$$\frac{x+3}{x-3}-\frac{x-3}{x+3}=a,$$
$$x^2+6x+9-x^2+6x-9=ax^2-9a,$$
$$ax^2-12x=9a,$$
$$a^2x^2-12ax=9a^2,$$
$$a^2x^2-(\ )+36=9a^2+36,$$
$$ax-6=\pm\sqrt{9(a^2+4)},$$
$$ax=6\pm 3\sqrt{a^2+4}.$$
$$\therefore x=\frac{6\pm 3\sqrt{a^2+4}}{a}.$$

20.

$$mx^2 - 1 = \frac{x(m^3 - n^2)}{mn}.$$

$$m^2nx^2 - mn = x(m^3 - n^2),$$

$$m^2nx^2 - (m^3 - n^2)x = mn,$$

$$4m^4n^2x^2 - (\ ) + (m^3 - n^2)^2 = m^6 + 2m^3n^2 + n^4,$$

$$2m^2nx - (m^3 - n^2) = \pm(m^3 + n^2),$$

$$2m^2nx = 2m^3 \text{ or } -2n^2.$$

$$\therefore x = \frac{m}{n} \text{ or } -\frac{n}{m^2}.$$

21.

$$(ax - b)(bx - a) = c^2,$$

$$abx^2 - b^2x - a^2x + ab = c^2,$$

$$abx^2 - (a^2 + b^2)x = c^2 - ab,$$

$$4a^2b^2x^2 - 4ab(a^2 + b^2)x = 4abc^2 - 4a^2b^2,$$

$$4a^2b^2x^2 - (\ ) + (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4abc^2,$$

$$2abx - (a^2 + b^2) = \pm\sqrt{(a^2 - b^2)^2 + 4abc^2}.$$

$$\therefore x = \frac{a^2 + b^2 \pm \sqrt{(a^2 - b^2)^2 + 4abc^2}}{2ab}.$$

22. $$\frac{ax + b}{bx + a} = \frac{mx + n}{nx + m}.$$

$$anx^2 + bnx + amx + bm$$

$$= bmx^2 + bnx + amx + an,$$

$$anx^2 - bmx^2 = an - bm,$$

$$x^2(an - bm) = an - bm,$$

$$x^2 = 1.$$

$$\therefore x = \pm 1.$$

23. $$\frac{m}{m + x} + \frac{m}{m - x} = c.$$

$$m^2 - mx + m^2 + mx = cm^2 - cx^2,$$

$$cx^2 = cm^2 - 2m^2,$$

$$x\sqrt{c} = \pm m\sqrt{c - 2}.$$

$$\therefore x = \pm m\sqrt{\frac{c-2}{c}}.$$

24.

$$\frac{(a-1)^2x^2 + 2(3a - 1)x}{4a - 1} = 1.$$

$$(a-1)^2x^2 + 2(3a - 1)x = 4a - 1,$$

$$4(a-1)^4x^2 + (\ ) + (6a - 2)^2 = 16a^3,$$

$$2(a-1)^2x + (6a - 2) = \pm 4a\sqrt{a},$$

$$2(a-1)^2x = 2 - 6a \pm 4a\sqrt{a}.$$

$$\therefore x = \frac{1 - 3a \pm 2a\sqrt{a}}{(a-1)^2}.$$

25.

$$\frac{(a^2-b^2)(x^2+1)}{a^2+b^2}=2x.$$

$$\begin{aligned}
a^2x^2-b^2x^2+a^2-b^2 &= 2a^2x+2b^2x,\\
a^2x^2-b^2x^2-2a^2x-2b^2x &= b^2-a^2,\\
x^2(a^2-b^2)-2x(a^2+b^2) &= b^2-a^2,\\
x^2(a^2-b^2)^2-2x(a^4-b^4) &= -a^4+2a^2b^2-b^4,\\
x^2(a^2-b^2)^2-(\ )+(a^2+b^2)^2 &= 4a^2b^2,\\
x(a^2-b^2)-(a^2+b^2) &= \pm 2ab,\\
x(a^2-b^2) &= a^2+b^2\pm 2ab\\
&= (a\pm b)^2.\\
\therefore x &= \frac{a+b}{a-b} \text{ or } \frac{a-b}{a+b}.
\end{aligned}$$

26.

$$\frac{x^2-4mnx}{(m+n)^2}=(m-n)^2.$$

$$\begin{aligned}
x^2-4mnx &= m^4-2m^2n^2+n^4,\\
x^2-(\ )+4m^2n^2 &= m^4+2m^2n^2+n^4,\\
x-2mn &= \pm(m^2+n^2),\\
x &= 2mn\pm(m^2+n^2)\\
&= (m+n)^2 \text{ or } -(m-n)^2.
\end{aligned}$$

27.

$$x^2+\frac{a-b}{ab^2}=\frac{14a^2-5ab-10b^2}{18a^2b^2}+\frac{(2a-3b)x}{2ab}.$$

$$\begin{aligned}
18a^2b^2x^2-(18a^2b-27ab^2)x &= -4a^2+13ab-10b^2,\\
144a^2b^2x^2-(144a^2b-216ab^2)x &= -32a^2+104ab-80b^2,\\
144a^2b^2x^2-(\ )+(6a-9b)^2 &= 4a^2-4ab+b^2,\\
12abx-(6a-9b) &= \pm(2a-b),\\
12abx &= 8a-10b \text{ or } 4a-8b.\\
\therefore x &= \frac{4a-5b}{6ab} \text{ or } \frac{a-2b}{3ab}.
\end{aligned}$$

28.

$$abx^2+\frac{b^2x}{c}=\frac{6a^2+ab-2b^2}{c^2}-\frac{3a^2x}{c}.$$

$$\begin{aligned}
abc^2x^2+b^2cx &= 6a^2+ab-2b^2-3a^2cx,\\
abc^2x^2+(3a^2c+b^2c)x &= 6a^2+ab-2b^2,\\
4a^2b^2c^2x^2+4abcx(3a^2+b^2) &= 24a^3b+4a^2b^2-8ab^3,\\
4a^2b^2c^2x^2+(\ )+(3a^2+b^2)^2 &= 9a^4+24a^3b+10a^2b^2-8ab^3+b^4,\\
2abcx+3a^2+b^2 &= \pm(3a^2+4ab-b^2).\\
\therefore x &= \frac{2a-b}{ac} \text{ or } -\frac{3a+2b}{bc}.
\end{aligned}$$

29.

$$\frac{x^2}{3m-2a}-\frac{m^2-4a^2}{4a-6m}=\frac{x}{2}.$$

$$\frac{2x^2}{3m-2a}-\frac{4a^2-m^2}{3m-2a}=x,$$

$$2x^2-4a^2+m^2=3mx-2ax,$$

$$2x^2+(2a-3m)x=4a^2-m^2,$$

$$16x^2+(\ )+(2a-3m)^2=(36a^2-12am+m^2),$$

$$4x+(2a-3m)=\pm(6a-m),$$

$$4x=-8a+4m \text{ or } 4a+2m.$$

$$\therefore x=m-2a \text{ or } a+\frac{m}{2}.$$

30.

$$6x+\frac{(a+b)^2}{x}=5(a-b)+\frac{25ab}{6x}.$$

$$36x^2+6(a+b)^2=30x(a-b)+25ab,$$

$$36x^2-30x(a-b)=25ab-6(a+b)^2,$$

$$36x^2-(\ )+\frac{25}{4}(a-b)^2=\frac{a^2+2ab+b^2}{4},$$

$$6x-\frac{5}{2}(a-b)=\pm\frac{a+b}{2},$$

$$6x=\frac{6a-4b}{2} \text{ or } \frac{4a-6b}{2}.$$

$$\therefore x=\frac{3a-2b}{6} \text{ or } \frac{2a-3b}{6}.$$

31.

$$\tfrac{8}{9}(x^2+a^2+ab)=\tfrac{1}{9}x(20a+4b).$$

$$8x^2+8a^2+8ab=20ax+4bx,$$

$$8x^2-(20a+4b)x=-8a^2-8ab,$$

$$16x^2-2(20a+4b)x=-16a^2-16ab,$$

$$16x^2-(\ )+(5a+b)^2=9a^2-6ab+b^2,$$

$$4x-(5a+b)=\pm(3a-b),$$

$$4x=(5a+b)\pm(3a-b).$$

$$\therefore x=2a \text{ or } \frac{a+b}{2}.$$

32.

$$x^2-(b-a)c=ax-bx+cx.$$

$$x^2+bx-ax-cx=(b-a)c,$$

$$x^2+(b-a-c)x=(b-a)c,$$

$$4x^2+(\ )+(b-a-c)^2=a^2+b^2+c^2-2ab-2ac+2bc,$$

$$2x+(b-a-c)=\pm(a-b-c),$$

$$2x=2a-2b \text{ or } 2c.$$

$$\therefore x=a-b \text{ or } c.$$

33.

$$x^2 - 2mx = (n - p + m)(n - p - m).$$
$$x^2 - 2mx = n^2 - 2np + p^2 - m^2,$$
$$x^2 - (\ ) + m^2 = n^2 - 2np + p^2,$$
$$x - m = \pm (n - p).$$
$$\therefore x = m \pm (n - p).$$

34.

$$x^2 \quad (m + n)x = \tfrac{1}{4}(p + q + m + n)(p + q - m - n).$$
$$4x^2 - 4(m + n)x = (p + q + m + n)(p + q - m - n),$$
$$4x^2 - (\ ) + (m + n)^2 = p^2 + 2pq + q^2,$$
$$2x - (m + n) = \pm (p + q),$$
$$2x = m + n \pm (p + q).$$
$$\therefore x = \frac{m + n \pm (p + q)}{2}.$$

35.

$$mnx^2 - (m+n)(mn+1)x + (m+n)^2 = 0.$$
$$mnx^2 - (m^2n + mn^2 + m + n)x = -m^2 - 2mn - n^2,$$
$$4m^2n^2x^2 - 4mn(m^2n + mn^2 + m + n)x = -4m^3n - 8m^2n^2 - 4mn^3,$$
$$4m^2n^2x^2 - (\ ) + (m^2n + mn^2 + m + n)^2 = m^4n^2 + 2m^3n^3 - 2m^3n - 4m^2n^2$$
$$+ m^2n^4 - 2mn^3 + m^2 + 2mn + n^2,$$
$$2mnx - (m^2n + mn^2 + m + n) = \pm (m^2n + mn^2 - m - n),$$
$$2mnx = 2m^2n + 2mn^2 \text{ or } 2m + 2n.$$
$$\therefore x = m + n \text{ or } \frac{m + n}{mn}.$$

36.

$$\frac{2b - x - 2a}{bx} + \frac{4b - 7a}{ax - bx} = \frac{x - 4a}{ab - b^2}.$$
$$4ab - ax - 2a^2 - 2b^2 + bx + 4b^2 - 7ab = x^2 - 4ax,$$
$$x^2 - 3ax - bx = 2b^2 - 3ab - 2a^2,$$
$$x^2 - (3a + b)x = 2b^2 - 3ab - 2a^2,$$
$$4x^2 - 4(3a + b)x = 8b^2 - 12ab - 8a^2,$$
$$4x^2 - (\ ) + (3a + b)^2 = a^2 - 6ab + 9b^2,$$
$$2x - (3a + b) = \pm (a - 3b),$$
$$2x = 4a - 2b \text{ or } 2a + 4b.$$
$$\therefore x = 2a - b \text{ or } a + 2b.$$

37.

$$2x^2(a^2-b^2)-(3a^2+b^2)(x-1)=(3b^2+a^2)(x+1).$$
$$2x^2(a^2-b^2)-3a^2x-b^2x+3a^2+b^2=3b^2x+a^2x+3b^2+a^2,$$
$$2x^2(a^2-b^2)-4a^2x-4b^2x=2b^2-2a^2.$$
Divide by 2, $x^2(a^2-b^2)-2(a^2+b^2)x=b^2-a^2,$
$$x^2(a^2-b^2)^2-2(a^4-b^4)x=(b^2-a^2)(a^2-b^2),$$
$$x^2(a^2-b^2)^2-(\ )+(a^2+b^2)^2=4a^2b^2,$$
$$x(a^2-b^2)-(a^2+b^2)=\pm 2ab,$$
$$x(a^2-b^2)=(a+b)^2 \text{ or } (a-b)^2.$$
$$\therefore x=\frac{a+b}{a-b} \text{ or } \frac{a-b}{a+b}.$$

38.

$$\frac{a-2b-x}{a^2-4b^2}-\frac{5b-x}{ax+2bx}+\frac{2a-x-19b}{2bx-ax}=0.$$
$$\frac{a-2b-x}{(a-2b)(a+2b)}-\frac{5b-x}{(a+2b)x}-\frac{2a-x-19b}{(a-2b)x}=0,$$
$$ax-2bx-x^2-5ab+ax+10b^2-2bx-2a^2$$
$$+ax+15ab+2bx+38b^2=0,$$
$$3ax-2bx-x^2+10ab+48b^2-2a^2=0.$$
$$x^2-(3a-2b)x=-2a^2+10ab+48b^2,$$
$$4x^2-(\ )+(3a-2b)^2=a^2+28ab+196b^2,$$
$$2x-(3a-2b)=\pm(a+14b),$$
$$2x=4a+12b \text{ or } 2a-16b.$$
$$\therefore x=2a+6b \text{ or } a-8b.$$

39.

$$\frac{x+13a+3b}{5a-3b-x}-1=\frac{a-2b}{x+2b}$$
$$x^2+13ax+5bx+26ab+6b^2$$
$$-5ax+5bx+x^2+6b^2-10ab=5a^2-13ab-ax+6b^2+2bx,$$
$$2x^2+(9a+8b)x=5a^2-29ab-6b^2,$$
$$16x^2+8(9a+8b)x=40a^2-232ab-48b^2,$$
$$16x^2+(\ )+(9a+8b)^2=121a^2-88ab+16b^2,$$
$$4x+(9a+8b)=\pm(11a-4b),$$
$$4x=2a-12b \text{ or } -(20a+4b).$$
$$\therefore x=\frac{a}{2}-3b \text{ or } -(5a+b).$$

40.

$$\frac{x+3b}{8a^2-12ab}-\frac{3b}{9b^2-4a^2}-\frac{a+3b}{(2a+3b)(x-3b)}=0.$$

$$\frac{x+3b}{4a(2a-3b)}+\frac{3b}{(2a-3b)(2a+3b)}-\frac{a+3b}{(2a+3b)(x-3b)}=0,$$

$$(x^2-9b^2)(2a+3b)+12abx-36ab^2-4a(2a^2+3ab-9b^2)=0.$$

$$(2a+3b)^2x^2+12abx=8a^3+12a^2b+18ab^2+27b^3,$$

$$4(2a+3b)^2x^2+(\ )+(12ab)^2=64a^4+192a^3b+432a^2b^2$$
$$+432ab^3+324b^4,$$

$$2(2a+3b)x+12ab=\pm(8a^2+12ab+18b^2),$$

$$(2a+3b)x=4a^2+9b^2 \text{ or } -(4a^2+12ab+9b^2).$$

$$\therefore\ x=\frac{4a^2+9b^2}{2a+3b} \text{ or } -(2a+3b).$$

41.

$$nx^2+px-px^2-mx+m-n=0.$$

$$nx^2-px^2+px-mx=n-m,$$

$$x^2(n-p)+x(p-m)=n-m,$$

$$4x^2(n-p)^2+(\ )+(p-m)^2=4n^2-4mn-4pn+4pm+p^2-2pm+m^2,$$

$$2x(n-p)+(p-m)=\pm(2n-p-m),$$

$$2x(n-p)=m-p+2n-p-m,$$
$$\text{or } m-p-2n+p+m.$$

$$\therefore\ x=1 \text{ or } \frac{m-n}{n-p}.$$

42.

$$(a+b+c)x^2-(2a+b+c)x+a=0.$$

$$(a+b+c)x^2-(2a+b+c)x=-a,$$

$$4x^2(a+b+c)^2-(\ )+(2a+b+c)^2=b^2+2bc+c^2,$$

$$2x(a+b+c)-(2a+b+c)=\pm(b+c),$$

$$2x(a+b+c)=(2a+b+c)\pm(b+c),$$

$$2x(a+b+c)=2a+2b+2c \text{ or } 2a.$$

$$\therefore\ x=1 \text{ or } \frac{a}{a+b+c}.$$

43.

$$(ax-b)(c-d)=(a-b)(cx-d)\,x.$$

$$acx-bc-adx+bd=acx^2-adx-bcx^2+bdx,$$

$$bcx^2-acx^2+acx-bdx=bc-bd,$$

$$(bc-ac)x^2+(ac-bd)x=bc-bd,$$

$$4(bc-ac)^2x^2+(\ )+(ac-bd)^2=4b^2c^2-4abc^2-4b^2cd$$
$$+2abcd+a^2c^2+b^2d^2,$$

$$2(bc-ac)x+(ac-bd)=\pm(2bc-ac-bd),$$

$$2(bc-ac)x=-(ac-bd)\pm(2bc-ac-bd),$$

$$x=\frac{-ac+bd+2bc-ac-bd}{2(bc-ac)}$$

$$\text{or }\ \frac{-ac+bd-2bc+ac+bd}{2(bc-ac)}.$$

$$\therefore\ x=1\text{ or }\frac{b(c-d)}{c(a-b)}.$$

44.

$$\frac{2x+1}{b}-\frac{1}{x}\left(\frac{1}{b}-\frac{2}{a}\right)=\frac{3x+1}{a}.$$

$$\frac{2x+1}{b}-\frac{a-2b}{abx}=\frac{3x+1}{a},$$

$$2ax^2+ax-a+2b=3bx^2+bx,$$

$$2ax^2-3bx^2+ax-bx=a-2b,$$

$$x^2(2a-3b)+x(a-b)=a-2b,$$

$$4x^2(2a-3b)^2+4x(a-b)(2a+3b)=(4a-8b)(2a-3b),$$

$$4x^2(2a-3b)^2+(\ )+(a-b)^2=9a^2-30ab+25b^2,$$

$$2x(2a-3b)+(a-b)=\pm(3a-5b),$$

$$2x(2a-3b)=-a+b\pm(3a-5b),$$

$$2x(2a-3b)=2a-4b\text{ or }-4a+6b.$$

$$\therefore\ x=\frac{a-2b}{2a-3b}\text{ or }-1.$$

45.

$$\frac{1}{2x^2+x-1}+\frac{1}{2x^2-3x+1}=\frac{a}{2bx-b}-\frac{2bx+b}{ax^2-a}.$$

$$\frac{1}{(2x-1)(x+1)}+\frac{1}{(2x-1)(x-1)}=\frac{a}{b(2x-1)}-\frac{2bx+b}{a(x-1)(x+1)}.$$

$$\text{L.C.D.}=ab(x-1)(x+1)(2x-1).$$

Simplify, $abx-ab+abx+ab=a^2x^2-a^2-4b^2x^2+b^2,$

$$2abx=a^2x^2-a^2-4b^2x^2+b^2,$$

$$4b^2x^2-a^2x^2+2abx=b^2-a^2,$$

$$x^2(4b^2-a^2)+x(2ab)=b^2-a^2,$$

$$4x^2(4b^2-a^2)^2+4x(2ab)(4b^2-a^2)=16b^4-20b^2a^2+4a^4,$$

$$4x^2(4b^2-a^2)^2+(\ )+(2ab)^2=16b^4-16b^2a^2+4a^4,$$

$$2x(4b^2-a^2)+2ab=\pm(4b^2-2a^2),$$

$$2x(4b^2-a^2)=4b^2-2ab-2a^2$$

$$\text{or } 2a^2-2ab-4b^2,$$

$$x=\frac{2b^2-ab-a^2}{4b^2-a^2}\text{ or }\frac{a^2-ab-2b^2}{4b^2-a^2}.$$

$$\therefore x=\frac{b-a}{2b-a}\text{ or }-\frac{b+a}{2b+a}.$$

Exercise LXXXVI.

1. $(x+1)(x-2)(x^2+x-2)=0.$

$(x+1)(x-2)(x-1)(x+2)=0.$

$\therefore x=-1,\ 2,\ 1,\ -2.$

2. $(x^2-3x+2)(x^2-x-12)=0.$

$(x-2)(x-1)(x-4)(x+3)=0.$

$\therefore x=2,\ 1,\ 4,\ -3.$

3. $(x+1)(x-2)(x+3)=-6.$

$x^3+2x^2-5x-6=-6,$

$x^3+2x^2-5x=0.$

$x(x^2+2x-5)=0;$

which is satisfied if $x=0,$

or if $x^2+2x-5=0.$

By solving $x^2+2x-5=0,$

$x=-1\pm\sqrt{6}.$

$\therefore x=0,\ -1\pm\sqrt{6}.$

4. $2x^3+4x^2-70x=0.$

$2x(x^2+2x-35)=0,$

$2x(x+7)(x-5)=0;$

which is satisfied if $x=0,$

$x+7=0,$

or if $x-5=0.$

$\therefore x=0,\ -7,\ 5.$

5. $(x^2-x-6)(x^2-x-20)=0.$

$(x-3)(x+2)(x-5)(x+4)=0.$

$\therefore x=3,\ -2,\ 5,\ -4.$

6.

$$x(x+1)(x+2) = a(a+1)(a+2).$$
$$x^3 + 3x^2 + 2x = a^3 + 3a^2 + 2a,$$
$$x^3 + 3x^2 + 2x - a^3 - 3a^2 - 2a = 0,$$
$$(x^3 - a^3) + (3x^2 - 3a^2) + (2x - 2a) = 0,$$
$$(x^2 + ax + a^2)(x - a) + (3x + 3a)(x - a) + 2(x - a) = 0,$$
$$(x^2 + ax + a^2 + 3x + 3a + 2)(x - a) = 0.$$
$$\therefore x - a = 0,$$
$$\text{and } x = a.$$
$$\text{Or, } x^2 + ax + a^2 + 3x + 3a + 2 = 0.$$
$$x^2 + ax + 3x = -a^2 - 3a - 2,$$
$$x^2 + x(a+3) = -a^2 - 3a - 2,$$
$$4x^2 + (\ ) + (a+3)^2 = 1 - 6a - 3a^2,$$
$$2x + (a+3) = \pm\sqrt{1 - 6a - 3a^2}.$$
$$\therefore x = -\frac{a+3}{2} \pm \tfrac{1}{2}\sqrt{1 - 6a - 3a^2}.$$

7.
$$x^3 - x^2 - x + 1 = 0.$$
$$(x^2 - 1)(x - 1) = 0,$$
$$(x+1)(x-1)(x-1) = 0.$$
$$\therefore x = 1,\ 1,\ -1.$$

8.
$$8x^3 - 1 = 0.$$
$$(2x-1)(4x^2+2x+1) = 0.$$
From the first factor,
$$x = \tfrac{1}{2},$$
$$\text{or}\quad 4x^2 + 2x = -1.$$
$$16x^2 + (\ ) + 1 = -3,$$
$$4x + 1 = \pm\sqrt{-3}.$$
$$\therefore x = \tfrac{1}{4}(-1 \pm \sqrt{-3}).$$

9.
$$8x^3 + 1 = 0.$$
$$(2x+1)(4x^2-2x+1) = 0.$$
From the first factor,
$$x = -\tfrac{1}{2},$$
$$\text{or}\quad 4x^2 - 2x = -1.$$
$$4x^2 - 2x = -1,$$
$$16x^2 - (\ ) + 1 = -3,$$
$$4x - 1 = \pm\sqrt{-3}.$$
$$\therefore x = \tfrac{1}{4}(1 \pm \sqrt{-3}).$$

10.
$$x^6 - 1 = 0.$$
$$(x^3 + 1)(x^3 - 1) = 0,$$
$$(x+1)(x^2 - x + 1)$$
$$(x-1)(x^2 + x + 1) = 0,$$
$$\text{and}\quad x = -1,\ 1.$$
$$\text{From}\quad x^2 - x + 1 = 0,$$
$$x^2 - x = -1,$$
$$4x^2 - (\ ) + 1 = -3,$$
$$2x - 1 = \pm\sqrt{-3},$$
$$\text{and}\quad x = \frac{1 \pm \sqrt{-3}}{2}.$$
$$\text{From}\quad x^2 + x + 1 = 0,$$
$$x^2 + x = -1,$$
$$4x^2 + (\ ) + 1 = -3,$$
$$2x + 1 = \pm\sqrt{-3},$$
$$\text{and}\quad x = \frac{-1 \pm \sqrt{-3}}{2}.$$
$$\therefore x = 1,\ -1,\ \frac{1 \pm \sqrt{-3}}{2},$$
$$\text{and}\quad \frac{-1 \pm \sqrt{-3}}{2}.$$

11. $$x(x-a)(x^2-b^2)=0.$$
$$x(x-a)(x+b)(x-b)=0.$$
$$\therefore\ x=0,\ a,\ \pm b.$$

12. $$n(x^3+1)+(x+1)=0.$$
$$(x+1)(nx^2-nx+n+1)=0.$$
$$(n+1)(x+1)(x^2-x+1)=0.$$
If $$x+1=0,$$
$$x=-1;$$
or if $$nx^2-nx+n+1=0.$$
By solving, $$x=\tfrac{1}{2}\pm\frac{1}{2n}\sqrt{-3n^2-4n}.$$

Exercise LXXXVII.

1. $$(x-2)(x-1)=0.$$
$$x^2-3x+2=0.$$

2. $$(x-7)(x+3)=0.$$
$$x^2-4x-21=0.$$

3. $$\left(x-\frac{1}{2}\right)\left(x-\frac{1}{3}\right)=0.$$
$$(2x-1)(3x-1)=0,$$
$$6x^2-5x+1=0.$$

4. $$\left(x-\frac{2}{3}\right)\left(x+\frac{3}{2}\right)=0.$$
$$x^2+\frac{5x}{6}-1=0,$$
$$6x^2+5x-6=0.$$

5 $$\left(x+5\right)\left(x+\frac{1}{2}\right)=0,$$
or $$x^2+\frac{11x}{2}+\frac{5}{2}=0,$$
or $$2x^2+11x+5=0.$$

6. $$\left(x+\frac{7}{9}\right)\left(x-\frac{9}{7}\right)=0.$$
$$(9x+7)(7x-9)=0,$$
$$63x^2-32x-63=0.$$

7. $$\left(x-3\right)\left(x+3\right)\left(x-\frac{3}{4}\right)\left(x+\frac{3}{4}\right)=0.$$
$$x^4-\frac{153x^2}{16}+\frac{81}{16}=0,$$
$$16x^4-153x^2+81=0.$$

8. $$(x-0)(x-1)(x-2)(x-3)=0.$$
$$x^4-6x^3+11x^2-6x=0.$$

9. $$\left(x-0\right)\left(x+\frac{1}{2}\right)\left(x-\frac{3}{2}\right)\left(x+1\right)=0.$$
$$4x^4-7x^2-3x=0.$$

10. $\{x-(a-2b)\}\{x-(3a+2b)\}=0.$
$(x-a+2b)(x-3a-2b)=0,$
$x^2-4ax+3a^2-4ab-4b^2=0.$

11. $\{x-(2a-b)\}\{x-(b-3a)\}=0.$
$(x-2a+b)(x-b+3a)=0,$
$x^2+ax-6a^2+5ab-b^2=0.$

12. $\{x-(a^2+a)\}\{x-(1-a)=0.$
$(x-a^2-a)(x-1+a)=0,$
$x^2-a^2x-x-a^3+a=0.$

13. $x^2-7x+12=0.$

In this equation p is 7 and q is 12.

$\therefore\ \sqrt{p^2-4q}=\sqrt{49-48}$
$=\sqrt{1}.$

$\therefore$ roots are rational, and both positive.

14. $x^2-7x-30=0.$

In this equation p is -7 and q is -30.

$\therefore\ \sqrt{p^2-4q}=\sqrt{49+120}$
$=13.$

$\therefore$ roots are rational, and of opposite signs.

15. $x^2+4x-5=0.$

In this equation p is 4 and q is -5.

$\therefore\ \sqrt{p^2-4q}=\sqrt{16+20}=6.$

$\therefore$ roots are rational, and of opposite signs.

16. $5x^2+8=0.$

In this equation p is 0 and q is 8.

$\therefore\ \sqrt{p^2-4q}=\sqrt{0-32}$
$=\sqrt{-32}.$

$\therefore$ roots are imaginary.

17. $x^2+4x+1=0.$

In this equation p is 4 and q is 1.

$\therefore\ \sqrt{p^2-4q}=\sqrt{16-4}$
$=\sqrt{12}.$

$\therefore$ roots are surds, and negative.

18. $x^2-2x+9=0.$

In this equation p is -2 and q is 9.

$\therefore\ \sqrt{p^2-4q}=\sqrt{4-36}$
$=\sqrt{-32}.$

$\therefore$ roots are imaginary.

19. $3x^2-4x-4=0.$

$x^2-\frac{4}{3}x-\frac{4}{3}=0.$

Here $p=-\frac{4}{3},\quad q=-\frac{4}{3}.$

$\therefore\ \sqrt{p^2-4q}=\sqrt{\frac{16}{9}+\frac{16}{3}}$
$=\sqrt{\frac{64}{9}}=\frac{8}{3}.$

$\therefore$ roots are rational, and of opposite signs.

20. $x^2+4x+4=0.$

In this equation p is 4 and q is 4.

$\therefore\ \sqrt{p^2-4q}=\sqrt{16-16}$
$=\sqrt{0}=0.$

$\therefore$ roots are rational, equal in value, and both negative.

Exercise LXXXVIII.

1.

Let
$$4 + 6x - x^2 = m.$$
$$x^2 - 6x - 4 = -m,$$
$$x^2 - 6x = 4 - m,$$
$$x^2 - (\ ) + 9 = 13 - m,$$
$$x - 3 = \pm\sqrt{13 - m}.$$
$$\therefore x = 3 \pm \sqrt{13 - m}.$$

Since $\sqrt{13 - m}$ cannot be negative, m cannot be greater than 13; that is, the maximum value is 13.

2.

Let
$$\frac{(x + a)^2}{x} = m.$$

Then
$$x^2 + 2ax + a^2 = mx,$$

and
$$4x^2 + 4x(2a - m) = -4a^2.$$
$$4x^2 + (\ ) + (2a - m)^2 = m^2 - 4am,$$
$$2x + (2a - m) = \pm\sqrt{m(m - 4a)},$$
$$2x = -(2a - m) \pm \sqrt{m(m - 4a)}.$$
$$\therefore x = -\frac{2a - m}{2} \pm \tfrac{1}{2}\sqrt{m(m - 4a)},$$
$$x = -\tfrac{1}{2}(2a - m) \mp \sqrt{m(m - 4a)}.$$

Since $\sqrt{m - 4a}$ cannot be negative, m cannot be less than $4a$. Hence $4a$ is the minimum value.

3.

Let
$$\frac{x^2 + 1}{x} = m.$$
$$x^2 + 1 = mx,$$
$$x^2 - mx = -1,$$
$$4x^2 - (\ ) + (m)^2 = m^2 - 4,$$
$$2x - m = \pm\sqrt{m^2 - 4}.$$

Since $\sqrt{m^2 - 4}$ cannot be negative, m cannot be between $+2$ and -2, but may have any other values. Hence $+\infty$ is the maximum and $-\infty$ is the minimum value.

4.

Let $(a-x)(x-b)=m.$

Then $ax-ab-x^2+bx=m,$

or $x^2-x(a+b)=-ab-m,$

$$4x^2-(\ )+(a+b)^2=(a-b)^2-4m,$$

$$2x-(a+b)=\pm\sqrt{(a-b)^2-4m}.$$

$$\therefore x=\tfrac{1}{2}(a+b)\pm\sqrt{(a-b)^2-4m}.$$

Now, for all possible values of x, $(a-b)^2-4m$ cannot be negative; that is, m cannot be greater than $\frac{(a-b)^2}{4}$; hence this is the maximum value.

5.

Let $\frac{x}{1+x^2}=m.$

$$x=m+m^2,$$

$$mx^2-x=-m,$$

$$4m^2x^2-(\ )+1=1-4m^2,$$

$$2mx-1=\pm\sqrt{1-4m^2}.$$

$$\therefore x=\frac{1}{2m}(1\pm\sqrt{1-4m^2}).$$

For all possible values of x, $1-4m^2$ cannot be negative; that is, m cannot be greater than $\frac{1}{2}$, and for this value $x=1$.

$\therefore \frac{1}{2}$ is the maximum value.

6.

Let $x^2+8x+20=m.$

$$x^2+8x=m-20,$$

$$x^2+(\ )+16=m-4,$$

$$x+4=\pm\sqrt{m-4}.$$

$$\therefore x=-4\pm\sqrt{m-4}.$$

For all possible values of x, m cannot be negative; that is, m cannot be less than 4.

$\therefore$ 4 is the minimum value.

7.

Let
$$x^2 - 2x + 9 = m.$$
$$x^2 - 2x = m - 9,$$
$$x^2 - (\ ) + 1 = m - 8,$$
$$x - 1 = \pm\sqrt{m-8}.$$
$$\therefore x = 1 \pm \sqrt{m-8}.$$

For all possible values of x, $m - 8$ cannot be negative; that is, m cannot be less than 8, and for this value $x = 1$.

$\therefore$ 8 is the minimum value.

8.

Let
$$\frac{x^2}{(x+a)(x-b)} = m.$$
$$x^2 = mx^2 + amx - bmx - abm,$$
$$x^2 - mx^2 - amx + bmx = -abm,$$
$$x^2(1-m) - x(am - bm) = -abm,$$
$$4(1-m)^2x^2 - (\ ) + (am - bm)^2 = a^2m^2 + 2abm^2 + b^2m^2 - 4abm,$$
$$4(1-m)^2x^2 - (\ ) + (am - bm)^2 = (am + bm)^2 - 4abm,$$
$$2x(1-m) - (am - bm) = \pm\sqrt{(am+bm)^2 - 4abm},$$
$$2x(1-m) - (am - bm) = \pm\sqrt{m^2(a+b)^2 - 4abm}.$$
$$\therefore x = \tfrac{1}{2}\left\{\frac{am - bm}{1-m} \pm \frac{1}{1-m}\sqrt{m^2(a+b)^2 - 4abm}\right\}.$$

For all possible values of x, $m^2(a+b)^2 - 4abm$ cannot be negative; that is, $m^2(a+b)^2$ cannot be less than $-4abm$, and for this value $x = \dfrac{am - bm}{2(1-m)}$.

$\therefore \dfrac{4ab}{(a+b)^2}$ is the minimum value.

9.

Let
$$\frac{x}{a + x^2} = m.$$
$$x = am + mx^2,$$
$$mx^2 - x = -am,$$
$$4m^2x^2 - (\ ) + 1 = 1 - 4am^2,$$
$$2mx - 1 = \pm\sqrt{1 - 4am^2}.$$
$$\therefore x = \frac{1}{2m}(1 \pm \sqrt{1 - 4am^2}).$$

For all possible values of x, $1 - 4am^2$ cannot be negative; that is, m cannot be greater than $+\tfrac{1}{2}\sqrt{\dfrac{1}{a}}$, and for this value $x = \dfrac{1}{2m}$.

$\therefore +\tfrac{1}{2}\sqrt{\dfrac{1}{a}}$ is the maximum value.

10. Divide a line 20 in. long into two parts so that the sum of the squares on these two parts may be the least possible.

Let $x =$ number of inches in first part,
and $20 - x =$ number of inches in second part.

$$x^2 + (20 - x)^2 = m,$$
$$x^2 + 400 - 40x + x^2 = m,$$
$$2x^2 - 40x = m - 400,$$
$$4x^2 - (\ ) + 400 = 2m - 400,$$
$$2x - 20 = \pm\sqrt{2m - 400}.$$

Then, as $\sqrt{2m - 400}$ cannot be a negative expression, $2m$ cannot be less than 400.

$\therefore$ 200 is the minimum value.

For this value,
$$2x^2 - 40x = -200,$$
$$x^2 - 20x = -100,$$
$$x^2 - (\ ) + 100 = 0,$$
$$x - 10 = 0.$$
$$\therefore x = 10,$$
$$20 - x = 10.$$

11. Divide a line 20 in. long into two parts so that the rectangle contained by the parts may be the greatest possible.

Let $x =$ one part,
then $20 - x =$ the other part.

$$20x - x^2 = m,$$
$$x^2 - 20x = -m,$$
$$x^2 - (\ ) + 100 = 100 - m,$$
$$x - 10 = \pm\sqrt{100 - m}.$$
$$\therefore x = 10 \pm \sqrt{100 - m}.$$

For all possible values of x, $100 - m$ cannot be negative; that is, m cannot be greater than 100, and for this value $x = 10$.

$\therefore$ 100 is the maximum value.

Substitute value of m,
$$x = 10 \pm \sqrt{100 - 100},$$
$$x = 10,$$
$$20 - x = 10.$$

12. **Find the fraction which has the greatest excess over its square.**

Let $x =$ the fraction,
then $x^2 =$ the square of the fraction.

$$x - x^2 = m,$$
$$x^2 - x = -m,$$
$$4x^2 - (\ ) + 1 = 1 - 4m,$$
$$2x - 1 = \pm\sqrt{1-4m}.$$
$$\therefore x = \tfrac{1}{2}(1 \pm \sqrt{1-4m}).$$

For all possible values of x, $1 - 4m$ cannot be negative; that is, m cannot be greater than $\frac{1}{4}$.

$\therefore \frac{1}{4}$ is the maximum value.

For this value, $x = \frac{1}{2}$.

Exercise LXXXIX.

1. $$x^6 + 7x^3 = 8.$$
$$4x^6 + (\ ) + 49 = 81,$$
$$2x^3 + 7 = \pm 9,$$
$$2x^3 = -7 \pm 9,$$
$$x^3 = -8 \text{ or } 1.$$

Since $x^3 = -8$,
$$x^3 + 8 = 0,$$
or $(x+2)(x^2-2x+4) = 0.$

Whence $x + 2 = 0$,
and $x = -2$,
or $$x^2 - 2x + 4 = 0.$$
$$x^2 - 2x = -4,$$
$$x^2 - (\ ) + 1 = -3,$$
$$x - 1 = \pm\sqrt{-3}.$$
$$\therefore x = 1 \pm \sqrt{-3}.$$

Since $x^3 = 1$,
$$x^3 - 1 = 0,$$
or $(x-1)(x^2+x+1) = 0.$

Whence $x - 1 = 0$,
and $x = 1$,
or $$x^2 + x + 1 = 0,$$
$$x^2 + x = -1,$$
$$4x^2 + (\ ) + 1 = -3,$$
$$2x + 1 = \pm\sqrt{-3}.$$
$$\therefore x = \tfrac{1}{2}(-1 \pm \sqrt{-3}).$$

$\therefore x = -2,\ 1,\ 1 \pm \sqrt{-3},$
and $\frac{1}{2}(-1 \pm \sqrt{-3})$.

2. $$x^4 - 5x^2 + 4 = 0.$$
$$x^4 - 5x^2 = -4,$$
$$4x^4 - (\ ) + 25 = 9,$$
$$2x^2 - 5 = \pm 3,$$
$$2x^2 = 5 \pm 3,$$
$$x^2 = 4 \text{ or } 1.$$
$$\therefore x = \pm 2, \pm 1.$$

3. $$37x^2 - 9 = 4x^4.$$
$$4x^4 - 37x^2 = -9,$$
$$64x^4 - (\ ) + (37)^2 = 1225,$$
$$8x^2 - 37 = \pm 35,$$
$$8x^2 = 72 \text{ or } 2,$$
$$x^2 = 9 \text{ or } \tfrac{1}{4}.$$
$$\therefore x = \pm 3 \text{ or } \pm \tfrac{1}{2}.$$

4. $$16x^8 = 17x^4 - 1.$$
$$16x^8 - 17x^4 = -1,$$
$$1024x^8 - (\ ) + (17)^2 = 225,$$
$$32x^4 - 17 = \pm 15,$$
$$32x^4 = 32 \text{ or } 2,$$
$$x^4 = 1 \text{ or } \tfrac{1}{16}.$$

Since $x^4 = 1$,
$$x^4 - 1 = 0,$$
or $(x^2+1)(x+1)(x-1) = 0.$
$$\therefore x = \pm\sqrt{-1},\ -1, \text{ or } 1.$$

Since $x^4 = \frac{1}{16}$,
$$x^4 - \tfrac{1}{16} = 0,$$
or $(x^2+\frac{1}{4})(x+\frac{1}{2})(x-\frac{1}{2}) = 0.$
$$\therefore x = \pm\tfrac{1}{2}\sqrt{-1},\ -\tfrac{1}{2}, \text{ or } \tfrac{1}{2}.$$

$\therefore$ the roots are
$\pm 1, \pm\sqrt{-1}, \pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{-1}.$

5. $32x^{10}-33x^5+1=0.$

$$32x^{10}-33x^5=-1,$$
$$4096x^{10}-(\ )+(33)^2=961,$$
$$64x^5-33=\pm 31,$$
$$64x^5=64 \text{ or } 2,$$
$$x^5=1 \text{ or } \tfrac{1}{32}.$$
$$\therefore x=1 \text{ or } \tfrac{1}{2}.$$

Other roots may be found by methods given later.

6. $(x^2-2)^2=\frac{1}{4}(x^2+12).$

$$x^4-4x^2+4=\frac{x^2+12}{4},$$
$$4x^4-16x^2+16=x^2+12,$$
$$4x^4-17x^2=-4,$$
$$64x^4-(\ )+(17)^2=225,$$
$$8x^2-17=\pm 15,$$
$$8x^2=2 \text{ or } 32,$$
$$x^2=\tfrac{1}{4} \text{ or } 4.$$
$$\therefore x=\pm\tfrac{1}{2} \text{ or } \pm 2.$$

7. $x^{4n}-\frac{5x^{2n}}{3}-\frac{25}{12}=0.$

$$12x^{4n}-20x^{2n}-25=0,$$
$$12x^{4n}-20x^{2n}=25,$$
$$36x^{4n}-(\ )+25=100,$$
$$6x^{2n}-5=\pm 10,$$
$$6x^{2n}=15 \text{ or } -5,$$
$$x^{2n}=\tfrac{5}{2} \text{ or } -\tfrac{5}{6}.$$
$$\therefore x=\pm\sqrt[2n]{\tfrac{5}{2}} \text{ or } \pm\sqrt[2n]{-\tfrac{5}{6}}.$$

8. $(x^2-9)^2=3+11(x^2-2).$

$$x^4-18x^2+81=11x^2-19,$$
$$x^4-29x^2=-100,$$
$$4x^4-(\ )+(29)^2=441,$$
$$2x^2-29=\pm 21,$$
$$x^2=25 \text{ or } 4.$$
$$\therefore x=\pm 5 \text{ or } \pm 2.$$

9. $x^6+14x^3+24=0.$

$$x^6+14x^3=-24,$$
$$x^6+(\ )+49=25,$$
$$x^3+7=\pm 5,$$
$$x^3=-2 \text{ or } -12.$$
$$\therefore x=\sqrt[3]{-2} \text{ or } \sqrt[3]{-12}.$$

10. $19x^4+216x^7=x.$

$$216x^7+19x^4-x=0,$$
$$x(216x^6+19x^3-1)=0,$$
$$x(27x^3-1)(8x^3+1)=0,$$
$$x(3x-1)(9x^2+3x+1)$$
$$(2x+1)(4x^2-2x+1)=0.$$
$$\therefore x=0,\ \tfrac{1}{3},\ -\tfrac{1}{2}.$$

From

$$9x^2+3x+1=0,$$
$$9x^2+3x=-1,$$
$$9x^2+(\ )+\tfrac{1}{4}=-\tfrac{3}{4},$$
$$3x+\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{-3},$$
$$3x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{6}(-1\pm\sqrt{-3}).$$

From

$$4x^2-2x+1=0,$$
$$4x^2-2x=-1,$$
$$4x^2-2x+\tfrac{1}{4}=-\tfrac{3}{4},$$
$$2x-\tfrac{1}{2}=\pm\tfrac{1}{2}\sqrt{-3},$$
$$2x=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$
$$\therefore x=\tfrac{1}{4}(1\pm\sqrt{-3}).$$

$\therefore$ roots are $0,\ \frac{1}{3},\ -\frac{1}{2},$
$\frac{1}{6}(-1\pm\sqrt{-3}),\ \frac{1}{4}(1\pm\sqrt{-3}).$

11. $x^8+22x^4+21=0.$

$$x^8+22x^4=-21,$$
$$x^8+(\ )+121=100,$$
$$x^4+11=\pm 10,$$
$$x^4=-1 \text{ or } -21.$$
$$\therefore x=\pm\sqrt[4]{-1}$$
$$\text{or } \pm\sqrt[4]{-21}.$$

That is, the roots are imaginary.

12. $x^{2m}+3x^m-4=0.$

$$4x^{2m}+12x^m-16=0,$$
$$4x^{2m}+(\ )+(3)^2=25,$$
$$2x^m+3=\pm 5,$$
$$2x^m=2 \text{ or } -8.$$
$$\therefore x=1 \text{ or } \sqrt[m]{-4}.$$

13.

$$4x^4 - 20x^3 + 23x^2 + 5x = 6.$$
$$4x^4 - 20x^3 + 23x^2 + 5x - 6 = 0.$$

$$\begin{array}{r|l}
 & 4x^4 - 20x^3 + 23x^2 + 5x - 6 \;\underline{|\, 2x^2 - 5x - \tfrac{1}{2}} \\
 & 4x^4 \\
4x^2 - 5x & -20x^3 + 23x^2 \\
 & \underline{-20x^3 + 25x^2} \\
4x^2 - 10x - \tfrac{1}{2} & -2x^2 + 5x - 6 \\
 & \underline{-2x^2 + 5x + \tfrac{1}{4}} \\
 & \qquad\qquad -\tfrac{25}{4}
\end{array}$$

If $\frac{25}{4}$ were added to both members the square would be complete, and the equation would read

$$4x^4 - 20x^3 + 23x^3 + 5x + \tfrac{1}{4} = \tfrac{25}{4}.$$

Extract the root, $2x^2 - 5x - \frac{1}{2} = \pm \frac{5}{2}$,

$$2x^2 - 5x = 3 \text{ or } -2,$$
$$16x^2 - (\;) + 25 = 49 \text{ or } 9,$$
$$4x - 5 = \pm 7 \text{ or } \pm 3,$$
$$4x = 12, \text{ or } -2, \text{ or } 8, \text{ or } 2.$$
$$\therefore x = 3, -\tfrac{1}{2}, 2, \tfrac{1}{2}.$$

14.

$$\frac{1}{x^{2n}} + \frac{3}{x^n} - 20 = 0.$$
$$1 + 3x^n - 20x^{2n} = 0,$$
$$20x^{2n} - 3x^n = 1,$$
$$1600x^{2n} - (\;) + 9 = 89,$$
$$40x^n - 3 = \pm\sqrt{89}.$$
$$\therefore x = \sqrt[n]{\tfrac{3}{40} \pm \tfrac{1}{40}\sqrt{89}}.$$

15.

$$x^4 - 4x^3 - 10x^2 + 28x - 15 = 0.$$

Extract root of left side,

$$\begin{array}{r|l}
 & x^4 - 4x^3 - 10x^2 + 28x - 15 \;\underline{|\, x^2 - 2x - 7} \\
 & x^4 \\
2x^2 - 2x & -4x^3 - 10x^2 \\
 & \underline{-4x^3 + 10x^2} \\
2x^2 - 4x - 7 & -14x^2 + 28x - 15 \\
 & \underline{-14x^2 + 28x + 49} \\
 & \qquad\qquad -64
\end{array}$$

Add 64 to both sides to complete the square,

$$x^4 - 4x^3 - 10x^2 + 28x + 49 = 64.$$

Extract the root, $x^2 - 2x - 7 = \pm 8$,

$$x^2 - 2x = 15 \text{ or } -1,$$
$$4x^2 - (\;) + 4 = 64 \text{ or } 0,$$
$$2x - 2 = \pm 8 \text{ or } \pm 0,$$
$$2x = 10, -6, 2, 2.$$
$$\therefore x = 5, -3, 1, 1.$$

16.

$$x^4 - 2x^3 - 13x^2 + 14x = -24.$$

Extract root of left side,

$$\begin{array}{r|l}
 & x^4 - 2x^3 - 13x^2 + 14x + 24 \;\underline{|\, x^2 - x - 7} \\
 & x^4 \\
2x^2 - 4 & -2x^3 - 13x^2 \\
 & \underline{-2x^3 + \quad x^2} \\
 & \quad -14x^2 + 14x + 24 \\
 & \quad \underline{-14x^2 + 14x + 49} \\
 & \qquad\qquad -25
\end{array}$$

Add 25 to both sides to complete the square,

$$x^4 - 2x^3 - 13x^2 + 14x + 49 = 25.$$

Extract the root,
$$x^2 - x - 7 = \pm 5,$$
$$x^2 - x = 12 \text{ or } 2,$$
$$4x^2 - (\;) + 1 = 49 \text{ or } 9,$$
$$2x - 1 = \pm 7 \text{ or } \pm 3.$$
$$\therefore x = 4, -3, 2, -1.$$

17.

$$108x^4 = 20x(9x^2 - 1) - 51x^2 + 7.$$
$$108x^4 - 108x^3 + 51x^2 + 20x = 7.$$

Multiply by 12, and add 16 to both sides,

$$1296x^4 - 2160x^3 + 612x^2 + 240x + 16 = 100.$$

$$\begin{array}{r|l}
 & 1296x^4 - 2160x^3 + 612x^2 + 240x + 16 \;\underline{|\, 36x^2 - 30x - 4} \\
 & 1296x^4 \\
72x^2 - 30x & -2160x^3 + 612x^2 \\
 & \underline{-2160x^3 + 900x^2} \\
72x^2 - 60x - 4 & -288x^2 + 240x + 16 \\
 & \underline{-288x^2 + 240x + 16}
\end{array}$$

$$36x^2 - 30x - 4 = \pm 10,$$
$$36x^2 - 30x = 14 \text{ or } -6,$$
$$144x^2 - (\;) + 25 = 81 \text{ or } 1,$$
$$12x - 5 = \pm 9 \text{ or } \pm 1,$$
$$12x = 14, -4, 6, 4.$$
$$\therefore x = 1\tfrac{1}{6}, -\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{3}.$$

18.

$$(x^2 - 1)(x^2 - 2) + (x^2 - 3)(x^2 - 4) = x^4 + 5.$$

Simplify, $x^4 - 3x^2 + 2 + x^4 - 7x^2 + 12 = x^4 + 5.$

Transpose and combine, $x^4 - 10x^2 = -9.$

Complete the square, $x^4 - (\;) + 25 = 16.$

Extract the root,
$$x^2 - 5 = \pm 4,$$
$$x^2 = 9 \text{ or } 1.$$
$$\therefore x = \pm 3 \text{ or } \pm 1.$$

Exercise XC.

1. The sum of the squares of three consecutive numbers is 365. Find the numbers.

Let $x =$ first number,

$x + 1 =$ second number,

and $x + 2 =$ third number.

$$\therefore x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = 365,$$
$$3x^2 + 6x = 360,$$
$$x^2 + 2x = 120,$$
$$x^2 + (\ ) + 1 = 121,$$
$$x + 1 = \pm 11.$$
$$\therefore x = 10 \text{ or } -12.$$

Hence, the numbers are 10, 11, 12.

2. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.

Let $x =$ first number,

and $x + 1 =$ second number.

$3x^2 + 3x =$ three times product,

$8x + 4 =$ four times sum.

$$\therefore 3x^2 + 3x - (8x + 4) = 8,$$
$$3x^2 - 5x = 12,$$
$$36x^2 - (\ ) + 25 = 169,$$
$$6x - 5 = \pm 13,$$
$$6x = 18 \text{ or } -8.$$
$$\therefore x = 3 \text{ or } -\tfrac{4}{3}.$$

Hence, the numbers are 3, 4.

3. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.

Let $x =$ first number.

Then $x + 1 =$ second number,

and $x + 2 =$ third number.

$$\therefore x(x + 1)(x + 2) = 3(x + 1),$$
$$x^3 + 3x^2 + 2x = 3x + 3,$$
$$x^3 + 3x^2 - x - 3 = 0,$$
$$(x + 1)(x - 1)(x + 3) = 0.$$
$$\therefore x = 1, -1, -3.$$

Hence, the numbers are 1, 2, 3.

4. A boy bought a number of apples for 16 cents. Had he bought 4 more for the same money he would have paid $\frac{1}{3}$ of a cent less for each apple. How many did he buy?

Let $x =$ number of apples bought.

Then $\frac{16}{x} =$ number of cents one apple costs,

and $\frac{16}{x+4} =$ number of cents one apple costs when he gets four more.

$$\therefore \frac{16}{x} - \frac{16}{x+4} = \frac{1}{3},$$

$$48x + 192 - 48x = x^2 + 4x,$$

$$x^2 + 4x = 192,$$

$$x^2 + (\ ) + 4 = 196,$$

$$x + 2 = \pm 14.$$

$$\therefore x = 12 \text{ or } -16.$$

Hence, $12 =$ number of apples bought.

5. For building 108 rods of stone-wall, 6 days less would have been required if 3 rods more a day had been built. How many rods a day were built?

Let $x =$ number of rods built in a day,

$\frac{108}{x} =$ number of days in which the whole wall was built,

$\frac{108}{x+3} =$ number of days it would have taken to build the whole wall if 3 rods more a day had been built.

Then
$$\frac{108}{x} - \frac{108}{x+3} = 6.$$

$$108x + 324 - 108x = 6x^2 + 18x,$$

$$6x^2 + 18x = 324,$$

$$x^2 + 3x = 54,$$

$$4x^2 + (\ ) + 9 = 225,$$

$$2x + 3 = \pm 15,$$

$$2x = 12 \text{ or } -18.$$

$$\therefore x = 6 \text{ or } -9.$$

Hence, $6 =$ number of rods built in a day.

6. A merchant bought some pieces of silk for \$900. Had he bought three pieces more for the same money, he would have paid \$15 less for each piece. How many did he buy?

Let $x =$ number of pieces bought.

Then $\frac{900}{x} =$ number of dollars each piece cost,

and $\frac{900}{x+3} =$ number of dollars each piece would have cost if he had received three more for \$900.

Then
$$\frac{900}{x} - \frac{900}{x+3} = 15,$$
$$900x + 2700 - 900x = 15x^2 + 45x,$$
$$15x^2 + 45x = 2700,$$
$$x^2 + 3x = 180,$$
$$4x^2 + (\ ) + 9 = 729,$$
$$2x + 3 = \pm 27.$$
$$\therefore x = 12 \text{ or } -15.$$

Hence, $12 =$ number of pieces bought.

7. A merchant bought some pieces of cloth for \$168.75. He sold the cloth for \$12 a piece, and gained as much as 1 piece cost him. How much did he pay for each piece?

Let $x =$ number of pieces,

$12x =$ number of dollars received for all,

$\frac{168.75}{x} =$ number of dollars paid for one piece.

Then $12x - 168.75 =$ number of dollars gained.

$$\therefore 12x - 168.75 = \frac{168.75}{x},$$
$$12x^2 - 168.75x = 168.75.$$

Multiply by $\frac{4}{3}$, $16x^2 - 225x = 225,$
$$1024x^2 - (\ ) + (225)^2 = 65025,$$
$$32x - 225 = \pm 255,$$
$$32x = 480 \text{ or } -30.$$
$$\therefore x = 15 \text{ or } -\frac{15}{16},$$

and
$$\frac{168.75}{15} = 11.25.$$

Hence, one piece cost \$11.25.

8. Find the price of eggs per score when 10 more in $62\frac{1}{2}$ cents' worth lowers the price $31\frac{1}{4}$ cents per hundred.

Let x = number of eggs at $62\frac{1}{2}$ cents.

Then $\frac{62.5}{x}$ = cost of one egg in cents,

and $\frac{62.5}{x+10}$ = cost of one egg in cents, if he had received ten more.

$$\therefore \frac{6250}{x} - \frac{6250}{x+10} = \text{difference in price per hundred.}$$

$$\therefore \frac{6250}{x} - \frac{6250}{x+10} = \frac{125}{4}.$$

Divide by 125, $\frac{50}{x} - \frac{50}{x+10} = \frac{1}{4}$,

$$200x + 2000 - 200x = x^2 + 10x,$$
$$x^2 + 10x = 2000,$$
$$x^2 + (\ ) + 25 = 2025,$$
$$x + 5 = \pm 45.$$
$$\therefore x = 40.$$

Hence, one egg cost $\frac{62.5}{40}$ cents, and 20 eggs cost $\frac{62.5}{40} \times 20 = 31\frac{1}{4}$ cents.

9. The area of a square may be doubled by increasing its length by 6 inches and its breadth by 4 inches. Determine its side.

Let x = the side of the square.

$$(x+4)(x+6) = 2x^2,$$
$$x^2 + 10x + 24 = 2x^2,$$
$$x^2 - 10x = 24,$$
$$x^2 - (\ ) + 25 = 49,$$
$$x - 5 = \pm 7.$$
$$\therefore x = 12 \text{ or } -2.$$

Hence, the side of the square is 12 inches.

10. The length of a rectangular field exceeds the breadth by 1 yard, and the area is 3 acres. Determine its dimensions.

Let x = number of yards in breadth,

$x + 1$ = number of yards in length,

and $x(x+1)$ = number of square yards in area.

But area is $3A$, or 14,520 square yards.

$$\therefore x^2 + x = 14{,}520.$$
$$4x^2 + (\ ) + 1 = 58{,}081,$$
$$2x + 1 = \pm 241.$$
$$\therefore x = 120 \text{ or } -121.$$

Hence, the field is 121 yards long by 120 broad.

11. There are three lines of which two are each $\frac{4}{7}$ of the third, and the sum of the squares described on them is equal to a square yard. Determine the lengths of the lines in inches.

Let $\qquad x =$ number of inches in third line,

and $\qquad \frac{4x}{7} =$ number of inches in each of the others.

Then $x^2 + \frac{16x^2}{49} + \frac{16x^2}{49} =$ the sum of the squares.

1 square yard = 1296 square inches.

$$\therefore x^2 + \frac{16x^2}{49} + \frac{16x^2}{49} = 1296,$$

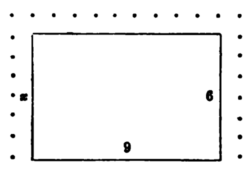
$$\frac{81x^2}{49} = 1296,$$

$$\frac{9x}{7} = \pm 36.$$

$$\therefore x = \pm 28.$$

Hence, the lengths are 16, 16, and 28 inches.

12. A grass plot 9 yards long and 6 yards broad has a path round it. The area of the path is equal to that of the plot. Determine the width of the path.



Let $\qquad x =$ number of yards in width of path.

Then $(9 + 2x)2 + 6 \times 2 =$ entire length of path in yards.

Also, $[(9+2x)2+6\times 2]x$

or $\qquad (30 + 4x)x =$ area of path in square yards,

and $\qquad 9 \times 6 =$ area of grass plot in square yards.

But area of path equals area of grass plot.

$$\therefore (30 + 4x)x = 54.$$

$$4x^2 + 30x = 54,$$

$$16x^2 + (\ ) + 225 = 441,$$

$$4x + 15 = \pm 21.$$

$$\therefore x = 1\tfrac{1}{2} \text{ or } -9.$$

Hence, the width of the path is $1\frac{1}{2}$ yards.

13. Find the radius of a circle the area of which would be doubled by increasing its radius by 1 inch.

Let $x =$ radius of circle,
and $x + 1 =$ radius increased.

The ratio of the circles is the same as the ratio of the squares on the radii.

$$\therefore 2x^2 = x^2 + 2x + 1,$$
$$x^2 - 2x = 1,$$
$$x^2 - (\ ) + 1 = 2,$$
$$x - 1 = \pm\sqrt{2},$$
$$x = 1 \pm \sqrt{2},$$
$$x = 2.4142.$$

14. Divide a line 20 inches long into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.

Let $x =$ one part.
Then $20 - x =$ the other part.

$$\therefore 20(20 - x) = x^2,$$
$$400 - 20x = x^2,$$
$$x^2 + 20x = 400,$$
$$x^2 + (\ ) + (10)^2 = 500,$$
$$x + 10 = \pm\sqrt{500},$$
$$x = -10 \pm 22.36,$$
$$x = 12.36.$$

Hence, one part is 12.36 inches, and the other is 7.64 inches.

15. A can do some work in 9 hours less time than B can do it, and together they can do it in 20 hours. How long will it take each alone to do it?

Let $x =$ number of hours it takes B.
Then $x - 9 =$ number of hours it takes A,
and $\frac{1}{x} =$ part B could do in 1 hour.
$\frac{1}{x-9} =$ part A could do in 1 hour.

$$\therefore \frac{1}{x} + \frac{1}{x-9} = \frac{1}{20}.$$
$$20x - 180 + 20x = x^2 - 9x,$$
$$x^2 - 49x = -180,$$
$$4x^2 - (\ ) + (49)^2 = 1681,$$
$$2x - 49 = \pm 41,$$
$$2x = 90 \text{ or } 8.$$
$$\therefore x = 45 \text{ or } 4.$$

Hence, B can do the work in 45 hours and A in 36 hours.

16. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 2 hours 55 minutes. How long will it take each pipe alone to fill the vessel?

Let $\quad x =$ number of hours it takes first pipe,

$x - 2 =$ number of hours it takes second pipe.

2 hours 55 minutes equals $\frac{35}{12}$ hours.

$$\therefore \frac{1}{x} + \frac{1}{x-2} = \frac{12}{35}.$$

$$35x - 70 + 35x = 12x^2 - 24x,$$

$$12x^2 - 94x = -70,$$

$$144x^2 - (\ ) + (47)^2 = 1369.$$

Extract the root, $\quad 12x - 47 = \pm 37,$

$$12x = 84 \text{ or } 10.$$

$$\therefore x = 7 \text{ or } \tfrac{5}{6}.$$

Hence, one pipe will fill it in 7 hours, the other in 5 hours.

17. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 1 hour 52 minutes 30 seconds. How long will it take each pipe alone to fill the vessel?

Let $\quad x =$ number of hours it takes first pipe.

Then $\quad x + 2 =$ number of hours it takes second pipe.

$\frac{1}{x} =$ part first pipe fills in 1 hour,

and $\quad \frac{1}{x+2} =$ part second pipe fills in 1 hour.

1 hour 52 minutes 30 seconds equals $\frac{15}{8}$ hours.

$$\therefore \frac{1}{x} + \frac{1}{x+2} = \frac{8}{15}.$$

$$15x + 30 + 15x = 8x^2 + 16x,$$

$$8x^2 - 14x = 30,$$

$$64x^2 - 112x = 240,$$

$$64x^2 - (\ ) + 49 = 289,$$

$$8x - 7 = \pm 17,$$

$$8x = 24 \text{ or } -10.$$

$$\therefore x = 3 \text{ or } -1\tfrac{1}{4}.$$

Hence, one pipe will fill it in 3 hours, the other in 5 hours.

18. An iron bar weighs 36 pounds. If it had been 1 foot longer, each foot would have weighed $\frac{1}{2}$ a pound less. Find the length and the weight per foot.

Let $x =$ number of feet in length.

Then $\frac{36}{x} =$ weight in pounds per foot,

and $\frac{36}{x} - \frac{1}{2} =$ weight per foot if it had been 1 foot longer.

But $\frac{36}{x+1} =$ weight per foot if it had been 1 foot longer.

$$\therefore \frac{36}{x} - \frac{1}{2} = \frac{36}{x+1},$$
$$72x + 72 - x^2 - x = 72x,$$
$$x^2 + x = 72,$$
$$4x^2 + (\ ) + 1 = 289,$$
$$2x + 1 = \pm 17.$$
$$\therefore x = 8 \text{ or } -9,$$
$$\frac{36}{x} = 4\tfrac{1}{2}.$$

Hence, the bar is 8 feet long, and weighs $4\frac{1}{2}$ pounds per foot.

19. A number is expressed by two digits, one of which is the square of the other, and when 54 is added its digits are interchanged. Find the number.

Let $x =$ digit in tens' place,

Then $x^2 =$ digit in units' place,

$10x + x^2 =$ number,

and $10x^2 + x =$ number with digits reversed.

$$\therefore 10x + x^2 + 54 = 10x^2 + x,$$
$$-9x^2 + 9x = -54,$$
$$x^2 - x = 6,$$
$$4x^2 - (\ ) + 1 = 25,$$
$$2x - 1 = \pm 5.$$
$$\therefore x = 3.$$

Hence, the number is 39.

20. Divide 35 into two parts so that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{12}$.

Let $x =$ one part.

Then $35 - x =$ the other part.

$$\therefore \frac{x}{35-x} + \frac{35-x}{x} = \frac{25}{12},$$
$$12x^2 + 14700 - 840x + 12x^2 = 875x - 25x^2$$
$$49x^2 - 1715x = -14700,$$
$$x^2 - 35x = -300,$$
$$4x^2 - (\ ) + (35)^2 = 25,$$
$$2x - 35 = \pm 5.$$
$$\therefore x = 20 \text{ or } 15.$$

Hence, the parts are 20 and 15.

21. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour 40 minutes. If the current of the river is 2 miles per hour, determine their rate of rowing in still water.

Let $x =$ rate in still water,

$x + 2 =$ rate down stream.

1 hour 40 minutes equals $\frac{5}{3}$ hours.

$\frac{3\frac{1}{2}}{x+2} =$ number of hours going down stream,

$\frac{3\frac{1}{2}}{x-2} =$ number of hours going up stream.

$$\therefore \frac{7}{2(x+2)} + \frac{7}{2(x-2)} = \frac{5}{3},$$

$$21x - 42 + 21x + 42 = 10x^2 - 40,$$

$$10x^2 - 42x = 40,$$

$$400x^2 - (\ ) + (42)^2 = 3364,$$

$$20x - 42 = \pm 58,$$

$$20x = 100 \text{ or } -16.$$

$$\therefore x = 5 \text{ or } -\tfrac{4}{5}.$$

Hence, the rate of rowing in still water is 5 miles an hour.

22. A detachment from an army was marching in regular column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.

Let $x =$ number of men in front,

and $x + 5 =$ number of men in depth.

Then $x^2 + 5x =$ number of men in all.

But $x + 845 =$ number of men in front,

and $5 =$ number of men in depth.

Then $5x + 4225 =$ number of men in all.

$$\therefore x^2 + 5x = 5x + 4225,$$

$$x^2 = 4225.$$

$$\therefore x = \pm 65.$$

Hence, the whole number of men is 4550.

23. A jockey sold a horse for \$144, and gained as much per cent as the horse cost. What did the horse cost?

Let $x =$ number of dollars the horse cost.

Then $\frac{x}{100} =$ gain per cent,

$\frac{x}{100}$ of $x =$ whole gain,

and $x + \frac{x^2}{100} =$ amount received.

$$\therefore x + \frac{x^2}{100} = 144,$$

$$x^2 + 100x = 14400,$$

$$x^2 + (\ ) + 2500 = 16900,$$

$$x + 50 = \pm 130.$$

$$\therefore x = 80 \text{ or } -180.$$

Hence, the horse cost \$80.

24. **A merchant expended a certain sum of money in goods, which he sold again for \$24, and lost as much per cent as the goods cost him. How much did he pay for the goods?**

Let $x =$ number of dollars paid for goods.

Then $\frac{x}{100} =$ per cent lost,

and $\frac{x}{100}$ of $x =$ whole loss.

$$\therefore x - \frac{x^2}{100} = 24,$$

$$100x - x^2 = 2400,$$

$$x^2 - 100x = -2400,$$

$$x^2 - (\ ) + (50)^2 = 100,$$

$$x - 50 = \pm 10,$$

$$x = 60 \text{ or } 40.$$

Hence, the goods cost either \$60 or \$40.

25. **A broker bought a number of bank shares (\$100 each), when they were at a certain per cent *discount*, for \$7500; and afterwards when they were at the same per cent *premium*, sold all but 60 for \$5000. How many shares did he buy, and at what price?**

Let $\qquad x =$ number of shares bought,

$\frac{7500}{x} =$ number of dollars each share cost,

and $\qquad 100 - \frac{7500}{x} =$ number of dollars discount on each share.

Then

$$\frac{100 - \frac{7500}{x}}{100} \text{ or } \frac{100x - 7500}{100x} = \text{rate of discount.}$$

Also, $\qquad x - 60 =$ number of shares sold.

Then $\qquad \frac{5000}{x-60} =$ number of dollars received for each share,

and $\qquad \frac{5000}{x-60} - 100 =$ number of dollars premium on each share.

$$\frac{\frac{5000}{x-60} - 100}{100} \text{ or } \frac{11000 - 100x}{100x - 6000} = \text{rate per cent of premium.}$$

But rate per cent discount was equal to rate per cent premium.

$$\therefore \frac{100x - 7500}{100x} = \frac{11000 - 100x}{100x - 6000},$$

$$x^2 - 135x + 4500 = 110x - x^2,$$

$$2x^2 - 245x = -4500,$$

$$16x^2 - (\ ) + (245)^2 = 24025.$$

Extract the root, $\qquad 4x - 245 = \pm 155,$

$$4x = 400 \text{ or } 90.$$

$$\therefore x = 100 \text{ or } 22\tfrac{1}{2}.$$

Hence, the broker bought 100 shares at 75.

26. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length is equal to the sum of its width and thickness; also, the number of cubic yards in its volume added to the number of linear yards in its edges is $\frac{5}{3}$ of the number of square yards in its surface. Determine its dimensions.

Let $\qquad 3x =$ number of yards in width,

$2x =$ number of yards in thickness,

and $\qquad 5x =$ number of yards in length.

$$30x^3 + 40x = \tfrac{5}{3}(62x^2),$$

$$90x^3 - 310x^2 = -120x.$$

Divide by $10x$, $\qquad 9x^2 - 31x = -12,$

$$9x^2 - (\ ) + (\tfrac{31}{6})^2 = \tfrac{529}{36},$$

$$3x - \tfrac{31}{6} = \pm \tfrac{23}{6},$$

$$3x = 9 \text{ or } \tfrac{4}{3}.$$

$$\therefore x = 3 \text{ or } \tfrac{4}{9}.$$

Hence, the dimensions are $15 \times 9 \times 6$ yards,

or $2\frac{2}{9} \times 1\frac{1}{3} \times \frac{8}{9}$ yards.

27. **If a carriage-wheel $16\frac{1}{2}$ feet round took 1 second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?**

Let x = number of seconds it takes the wheel to revolve;

$\frac{3600}{x}$ = number of revolutions it makes per hour,

$\frac{59400}{x}$ or $16\frac{1}{2} \times \frac{3600}{x}$ = number of feet it goes per hour,

$\frac{59400}{x+1}$ = number of feet it would go, if it took one second more to revolve,

Then $\frac{59400}{x} - \frac{59400}{x+1} = 9900$, number of feet in $1\frac{7}{8}$ miles.

$$59400x + 59400 - 59400x = 9900x^2 + 9900x,$$
$$9900x^2 + 9900x = 59400,$$
$$x^2 + x = 6,$$
$$x^2 + (\ ) + \tfrac{1}{4} = \tfrac{25}{4},$$
$$x + \tfrac{1}{2} = \pm \tfrac{5}{2}.$$
$$\therefore x = 2 \text{ or } -3.$$
$$\tfrac{59400}{2} = 29700.$$

Since 29700 feet equal $5\frac{5}{8}$ miles, the carriage is travelling at the rate of $5\frac{5}{8}$ miles per hour.

Exercise XCI.

1.
$$\left.\begin{aligned} x + y &= 13 \\ xy &= 36 \end{aligned}\right\} \begin{matrix}(1)\\(2)\end{matrix}$$

Square (1),
$$x^2 + 2xy + y^2 = 169 \quad (3)$$
$$(2)\times 4 \text{ is} \quad 4xy = 144 \quad (4)$$
$$\text{Subt.,} \quad x^2 - 2xy + y^2 = 25$$

Extract root, $x - y = \pm 5$ (5)

Add (1) and (5), $2x = 18$ or 8.
$$\therefore x = 9 \text{ or } 4.$$

Subtract (5) from (1),
$$2y = 8 \text{ or } 18.$$
$$\therefore y = 4 \text{ or } 9,$$

2.
$$x + y = 29 \quad (1)$$
$$xy = 100 \quad (2)$$

Square (1),
$$x^2 + 2xy + y^2 = 841 \quad (2)$$
$$(2)\times 4 \text{ is} \quad 4xy = 400 \quad (4)$$
$$\text{Subt.,} \quad x^2 - 2xy + y^2 = 441$$

Extract root, $x - y = \pm 21$ (5)

Add (1) and (5), $2x = 50$ or 8.
$$\therefore x = 25 \text{ or } 4.$$

Subtract (5) from (1),
$$2y = 8 \text{ or } 50.$$
$$\therefore y = 4 \text{ or } 25.$$

3.

$$x - y = 19 \qquad (1)$$
$$xy = 66 \qquad (2)$$

Square (1),

$$x^2 - 2xy + y^2 = 361 \qquad (3)$$

(2)×4 is $\quad 4xy \quad = 264 \qquad (4)$

Add, $x^2 + 2xy + y^2 = 625$

Extract root, $x + y = \pm 25 \qquad (5)$

Add (5) and (1), $2x = 44$ or -6.

$\therefore x = 22$ or -3.

Subtract (1) from (5),

$2y = 6$ or -44.

$\therefore y = 3$ or -22.

4.

$$x - y = 45 \qquad (1)$$
$$xy = 250 \qquad (2)$$

Square (1),

$$x^2 - 2xy + y^2 = 2025 \qquad (3)$$

(2)×4 is $\quad 4xy \quad = 1000 \qquad (4)$

Add, $x^2 + 2xy + y^2 = 3025$

Extract root, $x + y = \pm 55 \qquad (5)$

Add (5) and (1), $2x = 100$ or -10.

$\therefore x = 50$ or -5.

Subtract (1) from (5),

$2y = 10$ or -100.

$\therefore y = 5$ or -50.

5.

$$x - y = 10 \qquad (1)$$
$$x^2 + y^2 = 178 \qquad (2)$$

Square (1),

$$x^2 - 2xy + y^2 = 100 \qquad (3)$$

(2) is $\quad x^2 \quad + y^2 = 178$

Subt., $\quad 2xy \quad = 78 \qquad (4)$

(2) is $\quad x^2 \quad + y^2 = 178$

Add, $x^2 + 2xy + y^2 = 256$

Extract root, $x + y = \pm 16 \qquad (5)$

Add (5) and (1), $2x = 26$ or -6.

$\therefore x = 13$ or -3.

Subtract (1) from (5),

$2y = 6$ or -26.

$\therefore y = 3$ or -13.

6.

$$x - y = 14 \qquad (1)$$
$$x^2 + y^2 = 436 \qquad (2)$$

Square (1),

$$x^2 - 2xy + y^2 = 196 \qquad (3)$$

Subtract (2) from (3),

$$-2xy = -240 \qquad (4)$$

Subtract (4) from (2),

$$x^2 + 2xy + y^2 = 676.$$

Extract root, $x + y = \pm 26 \qquad (5)$

Add (1) and (5), $2x = 40$ or -12.

$\therefore x = 20$ or -6.

Subtract (1) from (5),

$2y = 12$ or -40.

$\therefore y = 6$ or -20.

7.

$$x + y = 12 \qquad (1)$$
$$x^2 + y^2 = 104 \qquad (2)$$

Square (1), $\quad x^2 + 2xy + y^2 = 144 \qquad (3)$

Subtract (2) from (3), $\quad 2xy = 40 \qquad (4)$

Subtract (4) from (2), $\quad x^2 - 2xy + y^2 = 64.$

Extract root, $\quad x - y = \pm 8 \qquad (5)$

Add (1) and (5), $\quad 2x = 20$ or 4.

$\therefore x = 10$ or 2.

Subtract (5) from (1), $\quad 2y = 4$ or 20.

$\therefore y = 2$ or 10.

8. $$\frac{1}{x}+\frac{1}{y}=\frac{3}{4} \quad (1)$$
$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16} \quad (2)$$

Square (1),
$$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{9}{16} \quad (3)$$

Subtract (2) from (3),
$$\frac{2}{xy}=\frac{4}{16} \quad (4)$$

Subtract (4) from (2),
$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{16}.$$

Extract root, $\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{4}$ (5)

Add (1) and (5), $\frac{2}{x}=1 \text{ or } \frac{1}{2}$.

$\therefore x=2 \text{ or } 4.$

Subtract (5) from (1),
$$\frac{2}{y}=\frac{1}{2} \text{ or } 1.$$
$\therefore y=4 \text{ or } 2.$

9. $$\frac{1}{x}+\frac{1}{y}=5 \quad (1)$$
$$\frac{1}{x^2}+\frac{1}{y^2}=13 \quad (2)$$

Square (1),
$$\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=25 \quad (3)$$

Subtract (2) from (3),
$$\frac{2}{xy}=12 \quad (4)$$

Subtract (4) from (2),
$$\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=1.$$

Extract root, $\frac{1}{x}-\frac{1}{y}=\pm 1$ (5)

Add (1) and (5), $\frac{2}{x}=6 \text{ or } 4.$

$\therefore x=\frac{1}{3} \text{ or } \frac{1}{2}.$

Subtract (5) from (1),
$$\frac{2}{y}=4 \text{ or } 6.$$
$\therefore y=\frac{1}{2} \text{ or } \frac{1}{3}.$

10. $$7x^2-8xy=159 \quad (1)$$
$$5x+2y=7 \quad (2)$$
$$2y=7-5x.$$
$$\therefore y=\frac{7-5x}{2}.$$

Substitute in (1),
$$7x^2-8x\left(\frac{7-5x}{2}\right)=159,$$
$$14x^2-56x+40x^2=318,$$
$$54x^2-56x=318.$$

Divide by 6,
$$9x^2-\tfrac{28}{3}x=53.$$

Complete the square,
$$9x^2-(\;)+(\tfrac{14}{9})^2=\tfrac{4489}{81}.$$

Extract root,
$$3x-\tfrac{14}{9}=\pm\tfrac{67}{9},$$
$$3x=\tfrac{81}{9} \text{ or } -\tfrac{53}{9}.$$
$$\therefore x=3 \text{ or } -1\tfrac{26}{27}.$$

Substitute value of x in (2).
$$\therefore y=-4 \text{ or } 8\tfrac{11}{27}.$$

11. $$x+y=49 \quad (1)$$
$$x^2+y^2=1681 \quad (2)$$

Square (1),
$$x^2+2xy+y^2=2401 \quad (3)$$

Subtract (2) from (3),
$$2xy=720 \quad (4)$$

Subtract (4) from (2),
$$x^2-2xy+y^2=961.$$

Extract root, $x-y=\pm 31$ (5)

Add (5) and (1), $2x=80 \text{ or } 18.$

$\therefore x=40 \text{ or } 9.$

Substitute value of x in (1).

$\therefore y=9 \text{ or } 40.$

12.
$$x^3+y^3=341 \quad (1)$$
$$x+y=11 \quad (2)$$
Divide (1) by (2),
$$x^2-xy+y^2=31 \quad (3)$$
Sq. (2), $x^2+2xy+y^2=121$ (4)

Subt., $-3xy=-90$
$$\therefore -xy=-30 \quad (5)$$
Add (3) and (5),
$$x^2-2xy+y^2=1.$$
Extract root, $x-y=\pm 1$ (6)

Add (2) and (6), $2x=12$ or 10.
$$\therefore x=6 \text{ or } 5.$$
Subtract (6) from (2),
$$2y=10 \text{ or } 12.$$
$$\therefore y=5 \text{ or } 6.$$

13.
$$x^3+y^3=1008 \quad (1)$$
$$x+y=12 \quad (2)$$
Divide (1) by (2),
$$x^2-xy+y^2=84 \quad (3)$$
Sq. (2), $x^2+2xy+y^2=144$ (4)

Subt., $-3xy=-60$
$$\therefore -xy=-20 \quad (5)$$
Add (3) and (5),
$$x^2-2xy+y^2=64.$$
Extract root, $x-y=\pm 8$ (6)

Add (2) and (6), $2x=20$ or 4.
$$\therefore x=10 \text{ or } 2.$$
Subtract (6) from (2),
$$2y=4 \text{ or } 20.$$
$$\therefore y=2 \text{ or } 10.$$

14.
$$x^3-y^3=98 \quad (1)$$
$$x-y=2 \quad (2)$$
Divide (1) by (2),
$$x^2+xy+y^2=49 \quad (3)$$
Sq. (2), $x^2-2xy+y^2=4$ (4)

Subt., $3xy=45$
$$\therefore xy=15 \quad (5)$$
Add (3) and (5),
$$x^2+2xy+y^2=64.$$
Extract root, $x+y=\pm 8$ (6)

Add (2) and (6), $2x=10$ or -6.
$$\therefore x=5 \text{ or } -3.$$
Subtract (2) from (6),
$$2y=6 \text{ or } -10.$$
$$\therefore y=3 \text{ or } -5.$$

15.
$$x^3-y^3=279 \quad (1)$$
$$x-y=3 \quad (2)$$
Divide (1) by (2),
$$x^2+xy+y^2=93 \quad (3)$$
Sq. (2), $x^2-2xy+y^2=9$ (4)

Subt., $3xy=84$
$$\therefore xy=28 \quad (5)$$
Add (5) and (3),
$$x^2+2xy+y^2=121.$$
Extract root, $x+y=\pm 11$ (6)

Add (6) and (2), $2x=14$ or -8.
$$\therefore x=7 \text{ or } -4.$$
Subtract (2) from (6),
$$2y=8 \text{ or } -14.$$
$$\therefore y=4 \text{ or } -7.$$

16.
$$x-3y=1 \quad (1)$$
$$xy+y^2=5 \quad (2)$$
Transpose (1), $x=1+3y.$

Substitute in (2),
$$y(1+3y)+y^2=5,$$
$$y+3y^2+y^2=5,$$
$$4y^2+y=5,$$
$$4y^2+(\ )+\tfrac{1}{16}=\tfrac{81}{16},$$
$$2y+\tfrac{1}{4}=\pm\tfrac{9}{4}.$$
$$\therefore y=1 \text{ or } -1\tfrac{1}{4}.$$
Substitute value of y in (1),
$$x=4 \text{ or } -2\tfrac{3}{4}.$$

17.
$$4y=5x+1 \quad (1)$$
$$2xy=33-x^2 \quad (2)$$
$$y=\frac{5x+1}{4}.$$
Substitute value of y in (2),
$$\frac{10x^2+2x}{4}=33-x^2,$$
$$14x^2+2x=132.$$
Divide by 2,
$$7x^2+x=66,$$
$$196x^2+(\ )+1=1849.$$
Extract root,
$$14x+1=\pm 43,$$
$$14x=42 \text{ or } -44.$$
$$\therefore x=3 \text{ or } -3\tfrac{1}{7}.$$
Substitute value of x in (1).
$$\therefore y=4 \text{ or } -3\tfrac{19}{28}.$$

18. $$\frac{1}{x}-\frac{1}{y}=3 \qquad (1)$$

$$\frac{1}{x^2}-\frac{1}{y^2}=21 \qquad (2)$$

Divide (2) by (1),

$$\frac{1}{x}+\frac{1}{y}=7 \qquad (3)$$

Add (3) and (1), $\frac{2}{x}=10.$

$$\therefore x=\tfrac{1}{5}.$$

Subtract (1) from (3),

$$\frac{2}{y}=4.$$

$$\therefore y=\tfrac{1}{2}.$$

19. $$\frac{1}{x}-\frac{1}{y}=2\tfrac{1}{2} \qquad (1)$$

$$\frac{1}{x^2}-\frac{1}{y^2}=8\tfrac{3}{4} \qquad (2)$$

Divide (2) by (1),

$$\frac{1}{x}+\frac{1}{y}=\frac{7}{2} \qquad (3)$$

Add (1) and (3), $\frac{2}{x}=\frac{12}{2}.$

$$\therefore x=\tfrac{1}{3}.$$

Subtract (1) from (3),

$$\frac{2}{y}=1.$$

$$\therefore y=2.$$

20.

$$x^2-2xy-y^2=1 \qquad (1)$$

$$x+y=2 \qquad (2)$$

Square (2), $x^2+2xy+y^2=4 \qquad (3)$

Add (3) and (1), $2x^2=5,$

$$x^2=2\tfrac{1}{2}.$$

$$\therefore x=\pm\sqrt{2\tfrac{1}{2}}.$$

Substitute value of x in (2), $y=2\mp\sqrt{2\tfrac{1}{2}}.$

Exercise XCII.

$$x^2+xy+2y^2=74 \qquad (1)$$

$$2x^2+2xy+y^2=73 \qquad (2)$$

Add, $3x^2+3xy+3y^2=147$

Divide by 3, $x^2+xy+y^2=49 \qquad (3)$

Subtract (3) from (1), $y^2=25.$

$$\therefore y=\pm 5.$$

Substitute value of y in (3), $x^2\pm 5x+25=49,$

$$x^2\pm 5x=24,$$

$$4x^2\pm 20x+25=121.$$

Extract the root, $2x\pm 5=\pm 11,$

$$2x=\pm 6 \text{ or } \pm 16.$$

$$\therefore x=\pm 3 \text{ or } \pm 8.$$

2.

$$x^2 + xy + 4y^2 = 6 \qquad (1)$$
$$3x^2 + 8y^2 = 14 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 + vx^2 + 4v^2x^2 = 6.$

$$\therefore x^2 = \frac{6}{1 + v + 4v^2} \qquad (3)$$

From (2), $3x^2 + 8v^2x^2 = 14.$

$$\therefore x^2 = \frac{14}{3 + 8v^2} \qquad (4)$$

Equate values of x^2,

$$\frac{6}{1 + v + 4v^2} = \frac{14}{3 + 8v^2},$$
$$18 + 48v^2 = 14 + 14v + 56v^2,$$
$$8v^2 + 14v = 4 \qquad (5)$$
$$64v^2 + (\ ) + 49 = 81,$$
$$8v + 7 = \pm 9,$$
$$8v = 2 \text{ or } -16.$$
$$\therefore v = \tfrac{1}{4} \text{ or } -2.$$

Substitute values of v in (4), $x^2 = \dfrac{14}{3 + \frac{1}{2}}$ or $\dfrac{14}{3 + 32}.$

Then $x^2 = 4 \text{ or } \tfrac{2}{5}.$

$$\therefore x = \pm 2, \pm\sqrt{\tfrac{2}{5}}.$$

From (2), $y = \pm\tfrac{1}{2}, \pm 2\sqrt{\tfrac{2}{5}}.$

3.

$$x^2 - xy + y^2 = 21 \qquad (1)$$
$$y^2 - 2xy = -15 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 - vx^2 + v^2x^2 = 21.$

$$\therefore x^2 = \frac{21}{1 - v + v^2} \qquad (3)$$

From (2), $v^2x^2 - 2vx^2 = -15.$

$$\therefore x^2 = \frac{-15}{v^2 - 2v} \qquad (4)$$

Equate values of x^2, $\dfrac{21}{1 - v + v^2} = \dfrac{-15}{v^2 - 2v} \qquad (5)$

$$21v^2 - 42v = -15 + 15v - 15v^2,$$
$$36v^2 - 57v = -15,$$
$$5184v^2 - (\ ) + (57)^2 = 1089,$$
$$72v - 57 = \pm 33.$$
$$\therefore v = \tfrac{5}{4} \text{ or } \tfrac{1}{3}.$$

Substitute values of v in (4), $x^2 = 16 \text{ or } 27.$

$$\therefore x = \pm 4 \text{ or } \pm 3\sqrt{3}.$$
$$\therefore y = \pm 5 \text{ or } \pm\sqrt{3}.$$

4.

$$x^2 - 4y^2 - 9 = 0.$$
$$xy + 2y^2 - 3 = 0.$$

$$x^2 - 4y^2 = 9 \qquad (1)$$

Transpose, $$xy + 2y^2 = 3 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $$x^2 - 4v^2x^2 = 9.$$
$$\therefore x^2 = \frac{9}{1-4v^2} \qquad (3)$$

From (2), $$x^2v + 2x^2v^2 = 3.$$
$$\therefore x^2 = \frac{3}{v+2v^2} \qquad (4)$$

Equate values of x^2, $$\frac{9}{1-4v^2} = \frac{3}{v+2v^2},$$
$$30v^2 + 9v = 3,$$
$$10v^2 + 3v = 1,$$
$$400v^2 + (\ ) + 9 = 49,$$
$$20v + 3 = \pm 7,$$
$$20v = 4 \text{ or } -10.$$
$$\therefore v = \tfrac{1}{5} \text{ or } -\tfrac{1}{2}.$$

Substitue values of v in (3), $x^2 = \frac{75}{7}$ or ∞.
$$\therefore x = \pm 5\sqrt{\tfrac{3}{7}}.$$
$$\therefore y = \pm\sqrt{\tfrac{3}{7}}.$$

5.

$$x^2 - xy = 35 \qquad (1)$$
$$xy + y^2 = 18 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $$x^2 - vx^2 = 35.$$
$$\therefore x^2 = \frac{35}{1-v} \qquad (3)$$

From (2), $$vx^2 + v^2x^2 = 18.$$
$$\therefore x^2 = \frac{18}{v+v^2} \qquad (4)$$

Equate values of x^2, $$\frac{35}{1-v} = \frac{18}{v+v^2},$$
$$35v^2 + 53v = 18,$$
$$4900v^2 + (\ ) + (53)^2 = 5329,$$
$$70v + 53 = \pm 73,$$
$$70v = 20 \text{ or } -126.$$
$$\therefore v = \tfrac{2}{7} \text{ or } -\tfrac{9}{5}.$$

Substitute values of v in (3), $x^2 = 49$ or $\frac{25}{2}$.
$$\therefore x = \pm 7 \text{ or } \pm 5\sqrt{\tfrac{1}{2}}.$$
$$\therefore y = \pm 2 \text{ or } \mp 9\sqrt{\tfrac{1}{2}}.$$

6.

$$x^2 + xy + 2y^2 = 44 \qquad (1)$$
$$2x^2 - xy + y^2 = 16 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 + vx^2 + 2v^2x^2 = 44.$

$$\therefore x^2 = \frac{44}{1 + v + 2v^2} \qquad (3)$$

From (2), $2x^2 - vx^2 + v^2x^2 = 16.$

$$\therefore x^2 = \frac{16}{2 - v + v^2}.$$

Equate values of x^2, $\dfrac{44}{1 + v + 2v^2} = \dfrac{16}{2 - v + v^2}$,

$$88 - 44v + 44v^2 = 16 + 16v + 32v^2,$$
$$12v^2 - 60v = -72,$$
$$v^2 - 5v = -6,$$
$$4v^2 - (\ ) + 25 = 1,$$
$$2v - 5 = \pm 1.$$
$$\therefore v = 3 \text{ or } 2.$$

Substitute values of v in (3), $x^2 = 2$ or 4.

From (3). $\therefore x = \pm\sqrt{2}$ or ± 2.

$$\therefore y = \pm 3\sqrt{2} \text{ or } \pm 4.$$

7.

$$x^2 + xy = 15 \qquad (1)$$
$$xy - y^2 = 2 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 + vx^2 = 15.$

$$\therefore x^2 = \frac{15}{1+v} \qquad (3)$$

From (2), $vx^2 - v^2x^2 = 2.$

$$\therefore x^2 = \frac{2}{v - v^2} \qquad (4)$$

Equate values of x^2, $\dfrac{15}{1+v} = \dfrac{2}{v - v^2}$,

$$15v - 15v^2 = 2 + 2v,$$
$$15v^2 - 13v = -2,$$
$$900v^2 - (\ ) + 169 = 49.$$

Extract the root, $30v - 13 = \pm 7,$

$$30v = 20 \text{ or } 6.$$
$$\therefore v = \tfrac{2}{3} \text{ or } \tfrac{1}{5}.$$

Substitute values of v in (3), $x^2 = 9$ or $\frac{25}{2}$.

$$\therefore x = \pm 3 \text{ or } \pm 5\sqrt{\tfrac{1}{2}}.$$
$$\therefore y = \pm 2 \text{ or } \pm\sqrt{\tfrac{1}{2}}.$$

8.

$$x^2 - xy + y^2 = 7 \qquad (1)$$
$$3x^2 + 13xy + 8y^2 = 162 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 - vx + v^2x^2 = 7.$

$$\therefore x^2 = \frac{7}{1 - v + v^2} \qquad (3)$$

From (2), $3x^2 + 13v + 8v^2x^2 = 162.$

$$\therefore x^2 = \frac{162}{3 + 13v + 8v^2}.$$

Equate values of x^2, $\dfrac{7}{1 - v + v^2} = \dfrac{162}{3 + 13v + 8v^2} \qquad (4)$

$$\therefore 106v^2 - 253v = -141,$$
$$44944v^2 - (\;) + (253)^2 = 4225.$$

Extract the root, $212v - 253 = \pm 65.$

$$\therefore v = \tfrac{3}{2} \text{ or } \tfrac{47}{53}.$$

Substitute values of v in (3), $x^2 = 4$ or $\tfrac{2809}{361}$.

$$\therefore x = \pm 2 \text{ or } \pm 2\tfrac{15}{19}.$$
$$\therefore y = \pm 3 \text{ or } \pm 2\tfrac{9}{19}.$$

9.

$$2x^2 + 3xy + y^2 = 70 \qquad (1)$$
$$6x^2 + xy - y^2 = 50 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $2x^2 + 3vx^2 + v^2x^2 = 70.$

$$\therefore x^2 = \frac{70}{2 + 3v + v^2} \qquad (3)$$

From (2), $6x^2 + vx^2 - v^2x^2 = 50.$

$$\therefore x^2 = \frac{50}{6 + v - v^2} \qquad (4)$$

Equate values of x^2, $\dfrac{70}{2 + 3v + v^2} = \dfrac{50}{6 + v - v^2},$

$$420 + 70v - 70v^2 = 100 + 150v + 50v^2,$$
$$12v^2 + 8v = 32,$$
$$36v^2 + (\;) + (2)^2 = 100,$$
$$6v + 2 = \pm 10.$$
$$\therefore v = 1\tfrac{1}{3} \text{ or } -2.$$

Substitute value of v in (3), $x^2 = 9$ or ∞.

$$\therefore x = \pm 3.$$
$$\therefore y = \pm 4.$$

10.

$$x^2 - xy - y^2 = 5 \qquad (1)$$

$$2x^2 + 3xy + y^2 = 28 \qquad (2)$$

Substitute vx for y in both equations.

From (1), $x^2 - vx^2 - v^2x^2 = 5.$

$$\therefore x^2 = \frac{5}{1 - v - v^2} \qquad (3)$$

From (2), $2x^2 + 3vx^2 + v^2x^2 = 28.$

$$\therefore x^2 = \frac{28}{2 + 3v + v^2}.$$

Equate values of x^2, $\dfrac{5}{1 - v - v^2} = \dfrac{28}{2 + 3v + v^2} \qquad (4)$

$$10 + 15v + 5v^2 = 28 - 28v - 28v^2,$$

$$33v^2 + 43v = 18,$$

$$4356v^2 + (\ ) + (43)^2 = 4225,$$

$$66v + 43 = \pm 65.$$

$$\therefore v = \tfrac{1}{3} \text{ or } -\tfrac{18}{11}.$$

Substitute values of v in (3), $x^2 = 9$ or $-121.$

$$\therefore x = \pm 3 \text{ or } \pm 11\sqrt{-1}.$$

$$\therefore y = \pm 1 \text{ or } \mp 18\sqrt{-1}.$$

Exercise XCIII.

1.

$$4xy = 96 - x^2y^2 \qquad (1)$$

$$x + y = 6 \qquad (2)$$

Let $x = (u + v),$

and $y = (u - v).$

From (2), $u + v + u - v = 6,$

$$2u = 6,$$

$$u = 3.$$

From (1), $4(u^2 - v^2) = 96 - u^4 + 2u^2v^2 - v^4,$

$$4u^2 - 4v^2 = 96 - u^4 + 2u^2v^2 - v^4.$$

Substitute value of u, $v^4 - 22v^2 = -21,$

$$4v^4 - (\ ) + (22)^2 = 400,$$

$$2v^2 - 22 = \pm 20,$$

$$2v^2 = 22 \pm 20,$$

$$v^2 = 21 \text{ or } 1.$$

$$\therefore v = \pm\sqrt{21} \text{ or } \pm 1.$$

$$\therefore x = u + v = 3 \pm \sqrt{21},\ 4,\ 2,$$

and $y = u - v = 3 \mp \sqrt{21},\ 2,\ 4.$

2.

$$x^2 + y^2 = 18 - x - y \qquad (1)$$
$$xy = 6 \qquad (2)$$

Put $u + v$ for x, and $u - v$ for y.

(1) becomes $\quad (u+v)^2 + (u-v)^2 = 18 - 2u,$
$$2u^2 + 2v^2 + 2u = 18,$$
$$u^2 + v^2 + u = 9 \qquad (3)$$

(2) becomes $\quad (u+v)(u-v) = 6,$

or
$$u^2 - v^2 = 6 \qquad (4)$$

Add (3) and (4), $\quad 2u^2 + u = 15.$

Complete the square, $16u^2 + (\ ) + 1 = 121,$
$$4u + 1 = \pm 11.$$
$$\therefore u = 2\tfrac{1}{2} \text{ or } -3.$$

Substitute value of u in (4), $\quad -v^2 = 6 - \frac{25}{4} \text{ or } 6 - 9.$
$$\therefore v = \pm \tfrac{1}{2} \text{ or } \pm\sqrt{3}.$$
$$\therefore x = u + v = 3,\ 2, \text{ or } -3 \pm \sqrt{3},$$
$$\text{and} \quad y = u - v = 2,\ 3, \text{ or } -3 \mp \sqrt{3}.$$

3.

$$2(x^2 + y^2) = 5xy \qquad (1)$$
$$4(x - y) = xy \qquad (2)$$

Put $u + v$ for x, and $u - v$ for y,
$$2(2u^2 + 2v^2) = 5(u^2 - v^2) \qquad (3)$$
$$4(2v) = u^2 - v^2 \qquad (4)$$

Transpose and combine, $\quad 9v^2 - u^2 = 0 \qquad (5)$
$$\underline{+u^2 - v^2 = 8v} \qquad (6)$$

Add (5) and (6), $\quad 8v^2 = 8v$
$$8v^2 - 8v = 0.$$
$$\therefore v = 0 \text{ or } 1.$$

Substitute value of v in (6), $\quad u^2 = 8v + v^2,$
$$u^2 = 0 \text{ or } 9,$$
$$u = 0 \text{ or } \pm 3.$$
$$\therefore x = u + v = 0,\ 4,\ -2.$$
$$\text{and } y = u - v = 0,\ 2,\ -4.$$

4.

$$4(x+y)=3xy \qquad (1)$$
$$x+y+x^2+y^2=26 \qquad (2)$$

Put $u+v$ for x, and $u-v$ for y.

(1) becomes $\quad 8u=3u^2-3v^2.$

$$\therefore 8u-3u^2+3v^2=0 \qquad (3)$$

(2) becomes $\quad 2u+2v^2+2u^2=26 \qquad (4)$

Multiply (4) by 3, $\quad 6u+6u^2+6v^2=78$

Multiply (3) by 2, $\quad 16u-6u^2+6v^2=\ \ 0$

Subtract, $\quad 12u^2-10u=78$

Complete the square, $\quad 144u^2-(\ )+25=961,$

$$12u-5=\pm 31,$$
$$12u=36 \text{ or } -26.$$
$$\therefore u=3 \text{ or } -2\tfrac{1}{6}.$$

Substitute value of u in (3), $\quad 3v^2=3.$

$$\therefore v=\pm 1.$$

Substitute $-2\frac{1}{6}$ for u in (3), $\quad 3v^2=\frac{1131}{36}.$

$$\therefore v=\pm\tfrac{1}{6}\sqrt{377}.$$
$$\therefore x=u+v=4,\ 2, \text{ or } \tfrac{1}{6}(-13\pm\sqrt{377}),$$
$$\text{and}\quad y=u-v=2,\ 4, \text{ or } \tfrac{1}{6}(-13\mp\sqrt{377}).$$

5.

$$4x^2+xy+4y^2=58 \qquad (1)$$
$$5x^2+5y^2=65 \qquad (2)$$

Multiply (1) by 5, $\quad 20x^2+5xy+20y^2=290 \qquad (3)$

Multiply (2) by 4, $\quad 20x^2\qquad\ +20y^2=260 \qquad (4)$

Subtract, $\quad 5xy\qquad = \ 30$

$$\therefore xy=6 \qquad (5)$$

Divide (2) by 5, $\quad x^2+y^2=13 \qquad (6)$

Substitute $u+v$ for x, and $u-v$ for y in (5) and (6),

$$u^2-v^2=6 \qquad (7)$$
$$2u^2+2v^2=13 \qquad (8)$$

Multiply (7) by 2, $\quad 2u^2-2v^2=12 \qquad (9)$

Add, $\quad 4u^2\qquad =25$

$$\therefore u=\pm\tfrac{5}{2}.$$

Subtract (9) from (8), $\quad 4v^2=1,$

$$v^2=\tfrac{1}{4}.$$
$$\therefore v=\pm\tfrac{1}{2}.$$
$$\therefore x=u+v=\pm\tfrac{5}{2}\pm\tfrac{1}{2}=\pm 3 \text{ or } \pm 2,$$
$$\text{and}\quad y=u-v=\pm\tfrac{5}{2}\mp\tfrac{1}{2}=\pm 2 \text{ or } \pm 3.$$

6.

$$xy(x+y)=30 \qquad (1)$$
$$x^3+y^3=35 \qquad (2)$$

Substitute $u+v$ for x, and $u-v$ for y.

(1) becomes $(u+v)(u-v)\{(u+v)+(u-v)\}=30,$

or $2u^3-2uv^2=30 \qquad (3)$

(2) becomes $(u+v)^3+(u-v)^3=35,$

or $2u^3+6uv^2=35 \qquad (4)$

Multiply (3) by 3, $6u^3-6uv^2=\ 90$

(4) is $2u^3+6uv^2=\ 35$

Add, $8u^3 \qquad =125$

$$2u=5,$$
$$u=\tfrac{5}{2}.$$

Substitute value of u in (3), $\dfrac{250}{8}-\dfrac{10v^2}{2}=30,$

$$250-40v^2=240,$$
$$40v^2=10.$$
$$\therefore\ v=\pm\tfrac{1}{2}.$$
$$\therefore\ x=u+v=3 \text{ or } 2,$$
$$\text{and}\quad y=u-v=2 \text{ or } 3.$$

Exercise XCIV.

1. $x-y=7 \qquad (1)$

$x^2+xy+y^2=13 \qquad (2)$

Square (1),

$x^2-2xy+y^2=49 \qquad (3)$

Subtract (2) from (3),

$-3xy=36.$

Divide by -3, $xy=-12 \qquad (4)$

Add (4) and (2),

$x^2+2xy+y^2=1.$

Extract root, $x+y=\pm 1 \qquad (5)$

Add (5) and (1), $2x=8$ or 6.

$\therefore\ x=4$ or 3.

Substitute value of x in (1),

$y=-3$ or -4.

2. $x^2+xy=35 \qquad (1)$

$xy-y^2=6 \qquad (2)$

Substitute vx for y.

From (1), $x^2=\dfrac{35}{1+v} \qquad (3)$

From (2), $x^2=\dfrac{6}{v-v^2} \qquad (4)$

$$\therefore\ \frac{35}{1+v}=\frac{6}{v-v^2},$$
$$35v^2-29v=-6,$$
$$4900v^2-(\ )+(29)^2=1,$$
$$70v-29=\pm 1,$$
$$70v=30 \text{ or } 28,$$
$$\therefore\ v=\tfrac{3}{7} \text{ or } \tfrac{2}{5}.$$

Substitute value of v in (3),

$$x^2=\tfrac{49}{2} \text{ or } 25.$$
$$x=\pm 7\sqrt{\tfrac{1}{2}} \text{ or } \pm 5.$$
$$y=vx=\pm 3\sqrt{\tfrac{1}{2}} \text{ or } \pm 2.$$

3.

$$xy - 12 = 0 \quad (1)$$
$$x - 2y = 5 \quad (2)$$

Transpose in (1), $xy = 12,$

$$y = \frac{12}{x}.$$

Substitute value of y in (2),

$$x - \frac{24}{x} = 5.$$

Simplify, $x^2 - 24 = 5x.$

Transpose, $x^2 - 5x = 24.$

Complete the square,

$$4x^2 - (\ ) + 25 = 121,$$
$$2x - 5 = \pm 11,$$
$$2x = 16 \text{ or } -6.$$
$$\therefore x = 8 \text{ or } -3.$$

Substitute value of x in (2),

$$y = 1\tfrac{1}{2} \text{ or } -4.$$

4.

$$xy - 7 = 0 \quad (1)$$
$$x^2 + y^2 = 50 \quad (2)$$

Transpose in (1), $xy = 7$ (3)

Multiply (3) by 2,

$$2xy = 14 \quad (4)$$

Add (4) and (2),

$$x^2 + 2xy + y^2 = 64.$$
$$\therefore x + y = \pm 8 \quad (5)$$

Subtract (4) from (2),

$$x^2 - 2xy + y^2 = 36.$$
$$\therefore x - y = \pm 6 \quad (6)$$

Add (5) and (6), $2x = \pm 14 \text{ or } \pm 2.$

$$\therefore x = \pm 7 \text{ or } \pm 1.$$

Subtract (6) from (5),

$$2y = \pm 2 \text{ or } \pm 14.$$
$$\therefore y = \pm 1 \text{ or } \pm 7.$$

5.

$$2x - 5y = 9 \quad (1)$$
$$x^2 - xy + y^2 = 7 \quad (2)$$

From (1), $x = \frac{9 + 5y}{2}$ (3)

Substitute value of x in (2),

$$\left(\frac{9 + 5y}{2}\right)^2 - y\left(\frac{9 + 5y}{2}\right) + y^2 = 7.$$

Simplify, $19y^2 + 72y = -53.$

Complete the square,

$$1444y^2 + (\ ) + (72)^2 = 1156,$$
$$38y + 72 = \pm 34,$$
$$38y = -38 \text{ or } -106.$$
$$\therefore y = -1 \text{ or } -2\tfrac{15}{19}.$$

Substitute value of y in (1), $2x = 9 + (-5) \text{ or } 9 + (-13\tfrac{18}{19}).$

$$\therefore x = 2 \text{ or } -2\tfrac{9}{19}.$$

6.

$$x - y = 9 \quad (1)$$
$$xy + 8 = 0 \quad (2)$$

Transpose (2), $xy = -8$ (3)

Square (1), $x^2 - 2xy + y^2 = 81$

Multiply (3) by 4, $4xy = -32$

Add, $x^2 + 2xy + y^2 = 49$

Extract root, $x + y = \pm 7$ (4)

Add (1) and (4), $2x = 16 \text{ or } 2.$

$$\therefore x = 8 \text{ or } 1.$$

Subtract (1) from (4), $2y = -2 \text{ or } -16.$

$$\therefore y = -1 \text{ or } -8.$$

7. $$5x - 7y = 0 \qquad (1)$$

$$5x^2 - \frac{13xy}{4} = 4 - 7y^2 \qquad (2)$$

From (1), $$x = \frac{7y}{5} \qquad (3)$$

Simplify (2),

$$20x^2 - 13xy + 28y^2 = 16 \qquad (4)$$

Substitute value of x in (4),

$$\frac{980y^2}{25} - \frac{91y^2}{5} + 28y^2 = 16,$$

$$1225y^2 = 400,$$

$$35y = \pm 20.$$

$$\therefore y = \pm \tfrac{4}{7}.$$

Substitute value of y in (1),

$$5x \mp \tfrac{28}{7} = 0.$$

$$\therefore x = \pm \tfrac{4}{5}.$$

8. $$x - y = 1 \qquad (1)$$

$$x^2 + y^2 = 8\tfrac{1}{2} \qquad (2)$$

Square (1),

$$x^2 - 2xy + y^2 = 1 \qquad (3)$$

(2) is $$x^2 + y^2 = 8\tfrac{1}{2}$$

Subt., $$-2xy = -7\tfrac{1}{2} \qquad (4)$$

Subtract (4) from (2),

$$x^2 + 2xy + y^2 = 16.$$

Extract root, $$x + y = \pm 4 \qquad (5)$$

Add (5) and (1), $$2x = 5 \text{ or } -3.$$

$$\therefore x = 2\tfrac{1}{2} \text{ or } -1\tfrac{1}{2}.$$

Subtract (5) from (1),

$$-2y = -3 \text{ or } 5.$$

$$\therefore y = 1\tfrac{1}{2} \text{ or } -2\tfrac{1}{2}.$$

9. $$x^2 + 4xy = 3 \qquad (1)$$

$$4xy + y^2 = 2\tfrac{1}{4} \qquad (2)$$

Substitute vx for y,

$$x^2 + 4vx^2 = 3 \qquad (3)$$

$$4vx^2 + v^2x^2 = 2\tfrac{1}{4} \qquad (4)$$

From (3) and (4),

$$x^2 = \frac{3}{1+4v} \qquad (5)$$

$$x^2 = \frac{9}{16v + 4v^2} \qquad (6)$$

$$\therefore \frac{3}{1+4v} = \frac{9}{16v + 4v^2}.$$

$$48v + 12v^2 = 9 + 36v,$$

$$12v^2 + 12v = 9,$$

$$4v^2 + 4v = 3,$$

$$4v^2 + (\ ) + 1 = 4,$$

$$2v + 1 = \pm 2.$$

$$\therefore v = \tfrac{1}{2} \text{ or } -1\tfrac{1}{2}.$$

Substitute values of v in (5),

$$x = \pm 1 \text{ or } \pm\sqrt{-\tfrac{3}{5}},$$

$$\text{and } y = \pm\tfrac{1}{2} \text{ or } \mp\tfrac{3}{2}\sqrt{-\tfrac{3}{5}}.$$

10. $$x^2 - xy + y^2 = 48 \qquad (1)$$

$$x - y - 8 = 0 \qquad (2)$$

(1) is $$x^2 - xy + y^2 = 48$$

Sq.(2), $$x^2 - 2xy + y^2 = 64$$

Subt., $$xy = -16$$

Multiply by 3,

$$3xy = -48 \qquad (3)$$

Add (3) and (1),

$$x^2 + 2xy + y^2 = 0.$$

Extract root,

$$x + y = 0 \qquad (4)$$

Add (4) and (2),

$$2x = 8.$$

$$\therefore x = 4.$$

Subtract (2) from (4),

$$2y = -8.$$

$$\therefore y = -4.$$

11.

$$x^2 + 3xy + y^2 = 1 \qquad (1)$$

$$3x^2 + xy + 3y^2 = 13 \qquad (2)$$

Subtract (1) from (2), $2x^2 - 2xy + 2y^2 = 12.$

Divide by 2, $x^2 - xy + y^2 = 6 \qquad (3)$

(1) is $x^2 + 3xy + y^2 = 1$

Subtract, $-4xy = 5 \qquad (4)$

Add $4 \times$ (1) to (4), $4x^2 + 8xy + 4y^2 = 9 \qquad (5)$

Extract the root, $2x + 2y = \pm 3.$

Divide by 2, $x + y = \pm \frac{3}{2} \qquad (6)$

Add $\frac{1}{4}$ of (4) to (3), $x^2 - 2xy + y^2 = \frac{29}{4},$

$$x - y = \pm \tfrac{1}{2}\sqrt{29} \qquad (7)$$

Add (6) and (7), $2x = \pm \frac{3}{2} \pm \frac{1}{2}\sqrt{29}.$

$$\therefore x = \tfrac{1}{4}(\pm 3 \pm \sqrt{29}).$$

Subtract (7) from (6), $2y = \pm \frac{3}{2} \mp \frac{1}{2}\sqrt{29}.$

$$\therefore y = \tfrac{1}{4}(\pm 3 \mp \sqrt{29}.)$$

12.

$$x^2 - 2xy + 3y^2 = 1\tfrac{2}{9} \qquad (1)$$

$$x^2 + xy - y^2 = \tfrac{1}{9} \qquad (2)$$

Substitute vx for y.

From (1), $x^2 - 2vx^2 + 3v^2x^2 = \frac{11}{9}.$

From (2), $x^2 + vx^2 - v^2x^2 = \frac{1}{9}.$

Whence $x^2 = \dfrac{11}{9 - 18v + 27v^2} \qquad (3)$

and $x^2 = \dfrac{1}{9 + 9v - 9v^2} \qquad (4)$

$$\therefore \frac{11}{9 - 18v + 27v^2} = \frac{1}{9 + 9v - 9v^2}.$$

$$99 + 99v - 99v^2 = 9 - 18v + 27v^2,$$

$$-126v^2 + 117v = -90.$$

Divide by -9, $14v^2 - 13v = 10.$

Complete the square, $784v^2 - (\ ) + 169 = 729.$

Extract the root, $28v - 13 = \pm 27,$

$$28v = 40 \text{ or } -14.$$

$$\therefore v = \tfrac{10}{7} \text{ or } -\tfrac{1}{2}.$$

Substitute value of v in (4), $x^2 = \frac{49}{171}$ or $\frac{4}{9}.$

$$\therefore x = \pm \tfrac{7}{3}\sqrt{\tfrac{1}{19}} \text{ or } \pm \tfrac{2}{3},$$

and $y = \pm \frac{10}{3}\sqrt{\frac{1}{19}}$ or $\mp \frac{1}{3}.$

13.
$$x + y = a \qquad (1)$$
$$4xy = a^2 - 4b^2 \qquad (2)$$

Square (1),
$$x^2 + 2xy + y^2 = a^2 \qquad (3)$$
(1) is
$$4xy \qquad = a^2 - 4b^2$$

Subt., $x^2 - 2xy + y^2 = \qquad 4b^2 \quad (4)$

Extract root, $x - y = \pm 2b \quad (5)$

Add (5) and (1), $2x = a \pm 2b$.
$$\therefore x = \frac{a}{2} \pm b.$$

Subtract (5) from (1),
$$2y = a \mp 2b.$$
$$\therefore y = \frac{a}{2} \mp b.$$

14.
$$x - y = 1 \qquad (1)$$
$$\frac{x}{y} + \frac{y}{x} = 2\tfrac{1}{6}. \qquad (2)$$

In (1), $x = y + 1.$

Substitute in (2),
$$\frac{y+1}{y} + \frac{y}{y+1} = \frac{13}{6},$$
$$6y^2 + 12y + 6 + 6y^2 = 13y^2 + 13y,$$
$$y^2 + y = 6.$$

Complete the square,
$$4y^2 + (\ ) + 1 = 25,$$
$$2y + 1 = \pm 5.$$
$$\therefore y = 2 \text{ or } -3.$$
$$\therefore x = 3 \text{ or } -2.$$

15.
$$x^2 + 9xy = 340 \qquad (1)$$
$$7xy - y^2 = 171 \qquad (2)$$

Subtract (2) from (1), $x^2 + 2xy + y^2 = 169,$
$$x + y = \pm 13 \qquad (3)$$
$$\therefore x = 13 - y \text{ or } -(13 + y).$$

Substitute in (1) the first value of x,
$$(13 - y)^2 + 9(13 - y)y = 340,$$
$$169 - 26y + y^2 + 117y - 9y^2 = 340,$$
$$8y^2 - 91y = 171.$$

Complete the square,
$$256y^2 - (\ ) + (91)^2 = 2809,$$
$$16y - 91 = \pm 53.$$
$$\therefore y = 9 \text{ or } 2\tfrac{3}{8} \qquad (4)$$

Substitute in (1) the second value of x,
$$(13 + y)^2 - 9(13 + y)y = 340,$$
$$169 + 26y + y^2 - 117y - 9y^2 = 340,$$
$$8y^2 + 91y = -171.$$

Whence, $y = -9 \text{ or } -2\tfrac{3}{8}.$

Substitute values of y in (3), $x = \pm 4 \text{ or } \pm 10\tfrac{5}{8}.$

16.
$$x + y = 6 \quad (1)$$
$$x^3 + y^3 = 72 \quad (2)$$

Divide (2) by (1),
$$x^2 - xy + y^2 = 12 \quad (3)$$
Sq. (1), $x^2 + 2xy + y^2 = 36 \quad (4)$

Subtract, $-3xy = -24$
$$\therefore xy = 8 \quad (5)$$

Subtract (5) from (3),
$$x^2 - 2xy + y^2 = 4.$$

Extract root, $x - y = \pm 2 \quad (6)$

Add (6) and (1), $2x = 8$ or 4.
$$\therefore x = 4 \text{ or } 2.$$

Subtract (6) from (1),
$$2y = 4 \text{ or } 8.$$
$$\therefore y = 2 \text{ or } 4.$$

17. $3xy + 2x + y = 485 \quad (1)$
$$3x - 2y = 0 \quad (2)$$

In (2), $x = \dfrac{2y}{3} \quad (3)$

Substitute value of x in (1),
$$\frac{6y^2}{3} + \frac{4y}{3} + y = 485,$$
$$6y^2 + 7y = 1455.$$

Complete the square,
$$144y^2 + (\ ) + 49 = 34969,$$
$$12y + 7 = \pm 187.$$
$$12y = 180 \text{ or } -194.$$
$$\therefore y = 15 \text{ or } -16\tfrac{1}{6}.$$

Substitute value of y in (3),
$$x = 10 \text{ or } -10\tfrac{7}{9}.$$

18.
$$x - y = 1 \quad (1)$$
$$x^3 - y^3 = 19 \quad (2)$$

Divide (2) by (1),
$$x^2 + xy + y^2 = 19 \quad (3)$$
Sq. (1), $x^2 - 2xy + y^2 = 1$

Subt., $3xy = 18$
$$\therefore xy = 6 \quad (4)$$

Add (4) and (3),
$$x^2 + 2xy + y^2 = 25.$$

Extract root, $x + y = \pm 5 \quad (5)$

Add (5) and (1), $2x = 6$ or -4.
$$\therefore x = 3 \text{ or } -2.$$

Subtract (1) from (5),
$$2y = 4 \text{ or } -6.$$
$$\therefore y = 2 \text{ or } -3.$$

19.
$$x^3 + y^3 = 2728 \quad (1)$$
$$x^2 - xy + y^2 = 124 \quad (2)$$

Divide (1) by (2),
$$x + y = 22 \quad (3)$$

Square (3),
$$x^2 + 2xy + y^2 = 484 \quad (4)$$

Subtract (4) from (2),
$$-3xy = -360.$$

Divide by 3, $-xy = -120 \quad (5)$

Add (5) and (2),
$$x^2 - 2xy + y^2 = 4.$$

Extract root, $x - y = \pm 2 \quad (6)$

Add (3) and (6), $2x = 24$ or 20.
$$\therefore x = 12 \text{ or } 10.$$

Subtract (6) from (3),
$$2y = 20 \text{ or } 24.$$
$$\therefore y = 10 \text{ or } 12.$$

20.

$$x + y = a \quad (1)$$
$$x^2 + y^2 = b^2 \quad (2)$$

Square (1), $x^2 + 2xy + y^2 = a^2 \quad (3)$

Subtract (2), $x^2 + y^2 = b^2$
$$2xy = a^2 - b^2 \quad (4)$$

Subtract (4) from (2), $x^2 - 2xy + y^2 = 2b^2 - a^2 \quad (5)$

Extract root, $x - y = \pm\sqrt{2b^2 - a^2} \quad (6)$

From (1) and (6), $2x = a \pm \sqrt{2b^2 - a^2}.$
$$\therefore x = \tfrac{1}{2}(a \pm \sqrt{2b^2 - a^2}).$$
$$2y = a \mp \sqrt{2b^2 - a^2}.$$
$$\therefore y = \tfrac{1}{2}(a \mp \sqrt{2b^2 - a^2}).$$

21.

$$x^2 - y^2 = 0 \qquad (1)$$
$$3x^2 - 4xy + 5y^2 = 9 \qquad (2)$$

From (1), $x^2 = y^2$.

$\therefore y = \pm x$.

Hence, in (2), $3x^2 \pm 4x^2 + 5x^2 = 9$,

$12x^2$ or $4x^2 = 9$.

$\therefore x = \pm \frac{1}{2}\sqrt{3}$ or $\pm \frac{3}{2}$,

and $y = \pm \frac{1}{2}\sqrt{3}$ or $\pm \frac{3}{2}$.

22.

$$\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{10}{3} \qquad (1)$$
$$x^2 + y^2 = 45 \qquad (2)$$

From (1), $3(x^2+2xy+y^2)+3(x^2-2xy+y^2) = 10x^2 - 10y^2$,

$3x^2 + 6xy + 3y^2 + 3x^2 - 6xy + 3y^2 = 10x^2 - 10y^2$,

$-4x^2 + 16y^2 = 0$,

$-x^2 + 4y^2 = 0 \qquad (3)$

Add (2) and (3), $5y^2 = 45$.

$\therefore y = \pm 3$.

Substitute values of y in (2), $x^2 + 9 = 45$.

$\therefore x = \pm 6$.

23.

$$\frac{1}{x} + \frac{1}{y} = 5$$
$$\frac{1}{x+1} + \frac{1}{y+1} = \frac{17}{12}$$

Clear of fractions and unite, $x + y = 5xy \qquad (1)$

$5x + 5y = 7 - 17xy \qquad (2)$

Divide (2) by (1), $5 = \dfrac{7 - 17xy}{5xy} \qquad (3)$

$25xy = 7 - 17xy \qquad (4)$

$42xy = 7$,

$xy = \frac{1}{6} \qquad (5)$

From (1), $x + y = \frac{5}{6} \qquad (6)$

Square (6), $x^2 + 2xy + y^2 = \frac{25}{36}$

Multiply (5) by 4, $4xy = \frac{24}{36}$

Subtract, $x^2 - 2xy + y^2 = \frac{1}{36}$

$x - y = \pm \frac{1}{6} \qquad (7)$

Add (7) and (6), $2x = 1$ or $\frac{2}{3}$.

$\therefore x = \frac{1}{2}$ or $\frac{1}{3}$.

Subtract (7) from (6). $2y = \frac{2}{3}$ or 1.

$\therefore y = \frac{1}{3}$ or $\frac{1}{2}$.

24.

$$x^2 - xy + y^2 = 7 \qquad (1)$$
$$x^4 + x^2y^2 + y^4 = 133 \qquad (2)$$

Divide (2) by (1), $\quad x^2 + xy + y^2 = 19 \qquad (3)$

Subtract (1) from (3), $\quad 2xy = 12.$

$$\therefore xy = 6 \qquad (4)$$

Add (4) to (3), and subtract (4) from (1),

$$x^2 + 2xy + y^2 = 25,$$
$$x^2 - 2xy + y^2 = 1.$$

Whence $\quad x + y = \pm 5,$

$$x - y = \pm 1.$$
$$\therefore x = \pm 3 \text{ or } \pm 2,$$
$$\text{and } y = \pm 2 \text{ or } \pm 3.$$

25.

$$x + y = 4 \qquad (1)$$
$$x^4 + y^4 = 82 \qquad (2)$$

Put $u + v$ for x, and $u - v$ for y.

(1) becomes $\quad 2u = 4.$

$$\therefore u = 2.$$

(2) becomes $\quad u^4 + 6u^2v^2 + v^4 = 41 \qquad (3)$

Substitute 2 for u in (3), $\quad 16 + 24v^2 + v^4 = 41,$

$$v^4 + 24v^2 = 25.$$

Complete the square, $\quad v^4 + (\ ) + 144 = 169,$

$$v^2 + 12 = \pm 13,$$
$$v^2 = 1 \text{ or } -25.$$
$$\therefore v = \pm 1 \text{ or } \pm\sqrt{-25}.$$
$$\therefore x = 3,\ 1, \text{ or } 2 \pm \sqrt{-25},$$
$$\text{and} \quad y = 1,\ 3, \text{ or } 2 \mp \sqrt{-25}.$$

26.

$$x^3 - y^3 = a^3 \qquad (1)$$
$$x - y = a \qquad (2)$$

Divide (1) by (2), $\quad x^2 + xy + y^2 = a^2 \qquad (3)$

Square (2), $\quad x^2 - 2xy + y^2 = a^2$

Subtract, $\quad 3xy = 0$

$$xy = 0 \qquad (4)$$

Add (3) and (4), $\quad x^2 + 2xy + y^2 = a^2.$

Extract the root, $\quad x + y = \pm a \qquad (5)$

Subtract (2) from (5), $\quad 2y = 0 \text{ or } -2a.$

$$\therefore y = 0 \text{ or } -a.$$

Add (2) and (5), $\quad 2x = 2a \text{ or } 0.$

$$\therefore x = a \text{ or } 0.$$

27.

$$\left.\begin{aligned} x^2 - xy &= a^2 + b^2 \quad (1)\\ xy - y^2 &= 2ab \quad (2)\end{aligned}\right\}$$

Subtract (2) from (1), $x^2 - 2xy + y^2 = a^2 - 2ab + b^2.$

Extract root, $x - y = \pm(a - b).$

(1) is $x(x - y) = a^2 + b^2.$

Substitute value of $x - y$ in (1), $\pm x(a - b) = a^2 + b^2.$

$$\therefore x = \pm\frac{a^2 + b^2}{a - b}.$$

(2) is $y(x - y) = 2ab.$

Substitute value of $(x - y)$ in (2),

$$\pm y(a - b) = 2ab.$$

$$\therefore y = \pm\frac{2ab}{a - b}.$$

28.

$$\left.\begin{aligned} x^2 - y^2 &= 4ab \quad (1)\\ xy &= a^2 - b^2 \quad (2)\end{aligned}\right\}$$

In (2), $y = \dfrac{a^2 - b^2}{x}.$

Substitute value of y in (1), $x^2 - \dfrac{(a^2-b^2)^2}{x^2} = 4ab,$

$$x^4 - a^4 + 2a^2b^2 - b^4 = 4abx^2,$$

$$x^4 - 4abx^2 = a^4 - 2a^2b^2 + b^4.$$

Complete the square, $x^4 - (\ ) + 4a^2b^2 = a^4 + 2a^2b^2 + b^4.$

Extract root, $x^2 - 2ab = \pm(a^2 + b^2),$

$$x^2 = \pm(a^2 + 2ab + b^2).$$

$$\therefore x = \pm(a + b).$$

Substitute value of x in (1), $(a + b)^2 - y^2 = 4ab,$

$$y^2 = a^2 - 2ab + b^2.$$

$$\therefore y = \pm(a - b).$$

29.

$xy = 0$ (1)

$x^2 + y^2 = 16$ (2)

Multiply (1) by 2, $2xy = 0$ (3)

Add (3) and (2),

$x^2 + 2xy + y^2 = 16$ (4)

Extract root, $x + y = \pm 4$ (5)

Multiply (1) by 4, $4xy = 0$ (6)

Subtract (6) from (4),

$x^2 - 2xy + y^2 = 16$

Extract root, $x - y = \pm 4$ (7)

Add (5) and (7), $2x = \pm 8$ or 0.

$\therefore x = \pm 4$ or 0.

Subtract (7) from (5),

$2y = 0$ or ± 8.

$\therefore y = 0$ or ± 4.

30.

$$\left.\begin{aligned} x^2 + xy + y^2 &= 37 \quad (1)\\ x^4 + x^2y^2 + y^4 &= 481 \quad (2)\end{aligned}\right\}$$

Divide (2) by (1),

$x^2 - xy + y^2 = 13$ (3)

(1) is $x^2 + xy + y^2 = 37$

Subt., $-2xy = -24$

$\therefore -xy = -12$ (4)

Add (3) and (4),

$x^2 - 2xy + y^2 = 1.$

Extract root, $x - y = \pm 1$ (5)

Subtract (4) from (1),

$x^2 + 2xy + y^2 = 49.$

Extract root, $x + y = \pm 7$ (6)

Add (5) and (6), $2x = \pm 8$ or ± 6.

$\therefore x = \pm 4$ or ± 3.

Subt. (5) fr. (6), $2y = \pm 6$ or ± 8.

$\therefore y = \pm 3$ or ± 4.

31.

$$x^2 = ax + by \qquad (1)$$
$$y^2 = ay + bx \qquad (2)$$

If $x = 0$, y must equal 0.

If $x = y$, and does not equal 0, then $x = a + b$, and $y = a + b$.

If x does not equal y, subtract (2) from (1), and divide by $x - y$.

$$x + y = a - b \qquad (3)$$

Add (1) and (2), $\quad x^2 + y^2 = a(x + y) + b(x + y)$.

Substitute $a - b$ for $x + y$, $\quad = a(a - b) + b(a - b)$.

That is, $\quad x^2 \quad + y^2 = a^2 \quad - b^2 \qquad (4)$

Square (3), $\quad x^2 + 2xy + y^2 = a^2 - 2ab + b^2 \qquad (5)$

Subtract (4) from (5), $\quad 2xy = -2ab + 2b^2 \qquad (6)$

Subtract (6) from (4), $\quad x^2 - 2xy + y^2 = a^2 + 2ab - 3b^2$.

Extract root, $\quad x - y = \pm\sqrt{a^2 + 2ab - 3b^2} \qquad (7)$

Add (7) and (3), $\quad 2x = a - b \pm \sqrt{a^2 + 2ab - 3b^2}$.

$$\therefore x = \tfrac{1}{2}(a - b \pm \sqrt{a^2 + 2ab - 3b^2}).$$

Subtract (7) from (3), $\quad 2y = a - b \mp \sqrt{a^2 + 2ab - 3b^2}$.

$$\therefore y = \tfrac{1}{2}(a - b \mp \sqrt{a^2 + 2ab - 3b^2}).$$

32.

$$x - y - 2 = 0 \qquad (1)$$
$$15(x^2 - y^2) = 16xy \qquad (2)$$

Transpose (1), $\quad x - y = 2 \qquad (3)$

Divide (2) by (3), $\quad 15(x + y) = 8xy$,

$$15x + 15y - 8xy = 0 \qquad (4)$$

From (1), $\quad x = y + 2$.

Substitute value of x in (4),

$$15y + 30 + 15y - 8y^2 - 16y = 0,$$
$$8y^2 - 14y = 30.$$

Complete the square, $\quad 64y^2 - (\;) + (7)^2 = 289$,

$$8y - 7 = \pm 17,$$
$$8y = 24 \text{ or } -10.$$
$$\therefore y = 3 \text{ or } -1\tfrac{1}{4}.$$

Substitute value of y in (1). $\quad \therefore x = 5 \text{ or } \tfrac{3}{4}$.

33.

$$\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{89}{40} \qquad (1)$$

$$6x=20y+9 \qquad (2)$$

Simplify (1), $\quad 9x^2-169y^2=0,$

$$9x^2=169y^2 \qquad (3)$$

Extract the root, $\quad 3x=\pm 13y,$

$$3x\mp 13y=0.$$

Multiply by 2, $\quad 6x\mp 26y=0 \qquad (4)$

Transpose in (2), $\quad 6x-20y=9 \qquad (5)$

Subtract (4) from (5), $\quad 6y=9,$

$$\text{or } -46y=9.$$

$$\therefore y=1\tfrac{1}{2} \text{ or } -\tfrac{9}{46}.$$

Substitute values of y in (2), $\quad x=6\tfrac{1}{2} \text{ or } -2\tfrac{7}{46}.$

34.

$$\frac{x}{a}+\frac{y}{b}=1 \qquad (1)$$

$$\frac{a}{x}+\frac{b}{y}=4 \qquad (2)$$

Simplify (1), $\quad bx+ay=ab \qquad (3)$

Simplify (2), $\quad bx+ay=4xy \qquad (4)$

$$\therefore 4xy=ab,$$

$$\text{and } y=\frac{ab}{4x}.$$

Substitute value of y in (3), $\quad bx+\dfrac{a^2b}{4x}=ab.$

Simplify, $\quad 4x^2+a^2=4ax.$

Transpose, $\quad 4x^2-4ax=-a^2.$

Complete the square, $\quad 4x^2-(\;)+a^2=0.$

Extract the root, $\quad 2x-a=0.$

$$\therefore x=\frac{a}{2}.$$

Substitute value of x in (3), $\quad y=\dfrac{b}{2}.$

35.

$$x^2 + y^2 = 7 + xy \qquad (1)$$

$$x^3 + y^3 = 6xy - 1 \qquad (2)$$

Transpose xy in (1), $\qquad x^2 - xy + y^2 = 7 \qquad (3)$

Divide (2) by (3), $\qquad x + y = \dfrac{6xy - 1}{7}.$

Simplify, $\qquad 7x + 7y = 6xy - 1 \qquad (4)$

Put $u + v$ for x, and $u - v$ for y, in (4),

$$7(u + v) + 7(u - v) = 6(u^2 - v^2) - 1,$$

$$6u^2 - 6v^2 - 14u = 1 \qquad (5)$$

Put $u + v$ for x, and $u - v$ for y, in (3),

$$(u + v)^2 - (u^2 - v^2) + (u - v)^2 = 7,$$

$$u^2 + 3v^2 = 7 \qquad (6)$$

Multiply (6) by 2, $\qquad 2u^2 + 6v^2 = 14 \qquad (7)$

Add (5) and (7), $\qquad 8u^2 - 14u = 15.$

Complete the square, $\qquad 256u^2 - (\ ) + (14)^2 = 676.$

Extract the root, $\qquad 16u - 14 = \pm 26,$

$$16u = 40 \text{ or } -12,$$

$$u = \tfrac{5}{2} \text{ or } -\tfrac{3}{4}.$$

Substitute $\frac{5}{2}$ for u in (6), $\qquad \frac{25}{4} + 3v^2 = 7,$

$$3v^2 = \tfrac{3}{4},$$

$$v^2 = \tfrac{1}{4}.$$

Extract the root, $\qquad v = \pm \frac{1}{2}.$

Substitute $-\frac{3}{4}$ for u in (6), $\qquad \frac{9}{16} + 3v^2 = 7,$

$$3v^2 = \tfrac{103}{16},$$

$$v^2 = \tfrac{103}{48}.$$

Extract the root, $\qquad v = \pm \frac{1}{4}\sqrt{\frac{103}{3}}.$

Since $x = u + v$, substitute $\frac{5}{2}$ for u and $\pm \frac{1}{2}$ for v,

$$x = \tfrac{5}{2} + (\pm \tfrac{1}{2}),$$

$$x = 3 \text{ or } 2.$$

Substitute value of $-\frac{3}{4}$ for u, and $\pm \frac{1}{4}\sqrt{\frac{103}{3}}$ for v,

$$x = -\tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{\tfrac{103}{3}},$$

$$x = \tfrac{1}{4}\left(-3 \pm \sqrt{\tfrac{103}{3}}\right).$$

Since $y = u - v$, substitute $\frac{5}{2}$ for u, and $\pm \frac{1}{2}$ for v,

$$y = \tfrac{5}{2} - (\pm \tfrac{1}{2}),$$

$$y = 2 \text{ or } 3.$$

Substitute $-\frac{3}{4}$ for u, and $\pm \frac{1}{4}\sqrt{\frac{103}{3}}$ for v,

$$y = -\tfrac{3}{4} - \left(\pm\sqrt{\tfrac{103}{3}}\right),$$

$$y = \tfrac{1}{4}\left(-3 \mp \sqrt{\tfrac{103}{3}}\right).$$

36.

$$x^5 - y^5 = 3093 \quad (1)$$
$$x - y = 3 \quad (2)$$

Let $x = u + v$, and $y = u - v$.

From (2), $u + v - u + v = 3,$

$$2v = 3.$$
$$\therefore v = \tfrac{3}{2}.$$

From (1), $u^5+5u^4v+10u^3v^2+10u^2v^3+5uv^4+v^5$
$-(u^5-5u^4v+10u^3v^2-10u^2v^3+5uv^4-v^5) = 3093.$

Transpose and combine, $10u^4v+20u^2v^3+2v^5 = 3093.$

Substitute value of v, $\dfrac{30u^4}{2} + \dfrac{540u^2}{8} + \dfrac{243}{16} = 3093.$

Simplify, $240u^4 + 1080u^2 = 49245.$

Divide by 15, $16u^4 + 72u^2 = 3283.$

Complete the square, $16u^4 + (\ ) + 81 = 3364.$

Extract the root, $4u^2 + 9 = \pm 58,$

$$u^2 = \tfrac{49}{4} \text{ or } -\tfrac{67}{4}.$$
$$\therefore u = \pm\tfrac{7}{2} \text{ or } \pm\tfrac{1}{2}\sqrt{-67},$$
$$x = u + v = 5, -2, \text{ or } \tfrac{1}{2}(3 \pm \sqrt{-67}),$$
$$y = u - v = 2, -5, \text{ or } \tfrac{1}{2}(-3 \pm \sqrt{-67}).$$

37.

$$\tfrac{3}{5}(x-1) - \tfrac{2}{3}(x+1)(y-1) = -11 \quad (1)$$
$$\tfrac{1}{3}(y+2) = \tfrac{1}{4}(x+2) \quad (2)$$

From (1), $9x - 9 - 10xy - 10y + 10x + 10 = -165,$

$$\text{or } 19x - 10xy - 10y = -166 \quad (3)$$

From (2), $4y + 8 = 3x + 6.$

$$\therefore y = \frac{3x-2}{4} \quad (4)$$

Substitute value of y in (3),

$$19x - 10x\left(\frac{3x-2}{4}\right) - 10\left(\frac{3x-2}{4}\right) = -166,$$
$$76x - 30x^2 + 20x - 30x + 20 = -664,$$
$$-30x^2 + 66x = -684,$$
$$5x^2 - 11x = 144.$$

Complete the square, $100x^2 - (\ ) + (11)^2 = 2401,$

$$10x - 11 = \pm 49,$$
$$10x = 60 \text{ or } -38.$$
$$\therefore x = 6 \text{ or } -3\tfrac{4}{5}.$$

Substitute values of x in (4), $y = 4 \text{ or } -3\tfrac{7}{10}.$

38.

$$10x^2 + 15xy = 3ab - 2a^2 \qquad (1)$$
$$10y^2 + 15xy = 3ab - 2b^2 \qquad (2)$$

Let $\quad ux = y.$

(1) becomes $\quad 10x^2 + 15x^2u = 3ab - 2a^2,$

$$x^2 = \frac{3ab - 2a^2}{10 + 15u} \qquad (3)$$

(2) becomes $\quad 10x^2u^2 + 15x^2u = 3ab - 2b^2,$

$$x^2 = \frac{3ab - 2b^2}{10u^2 + 15u} \qquad (4)$$

Equate values of x^2, $\quad \dfrac{3ab - 2a^2}{10 + 15u} = \dfrac{3ab - 2b^2}{10u^2 + 15u}.$

Simplify,

$$30abu^2 - 20a^2u^2 - 30a^2u + 30b^2u = 30ab - 20b^2.$$

Divide by 10,

$$3abu^2 - 2a^2u^2 - 3a^2u + 3b^2u = 3ab - 2b^2,$$
$$\text{or } u^2(3ab - 2a^2) - 3u(a^2 - b^2) = 3ab - 2b^2.$$

Complete the square,

$$4u^2(3ab - 2a^2)^2 - (\ ) + 9(a^2 - b^2)^2 = 9a^4 - 24a^3b + 34a^2b^2 - 24ab^3 + 9b^4.$$

Extract the root,

$$2u(3ab - 2a^2) - 3(a^2 - b^2) = \pm(3a^2 - 4ab + 3b^2),$$
$$2u(3ab - 2a^2) = 6a^2 - 4ab \text{ or } 4ab - 6b^2.$$
$$\therefore u = \frac{3a - 2b}{3b - 2a} \text{ or } -\frac{b}{a}$$

Substitute value of $\dfrac{3a - 2b}{3b - 2a}$ for u in (3),

$$x^2 = \frac{(3b - 2a)^2}{25}.$$

Extract the root, $\quad x = \pm\left(\dfrac{3b - 2a}{5}\right).$

Substitute $-\dfrac{b}{a}$ for u in (3), $\quad x^2 = -\dfrac{a^2}{5}.$

$$\therefore x = \pm a\sqrt{-\tfrac{1}{5}}.$$

Since $ux = y$, $\quad y = \dfrac{3a - 2b}{3b - 2a} \times \pm\left(\dfrac{3b - 2a}{5}\right).$

$$\therefore y = \pm\frac{3a - 2b}{5},$$
$$\text{or } y = -\frac{b}{a} \times \left(\pm a\sqrt{-\tfrac{1}{5}}\right).$$
$$\therefore y = \mp b\sqrt{-\tfrac{1}{5}}.$$

EXERCISE XCV.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find the length and breadth.

Let x = length of rectangle,
and y = width of rectangle.

Then $(x+1)(y+1) = 48$ (1)

and $(x-1)(y-1) = 24$ (2)

Simplify (1), $xy + x + y + 1 = 48$ (3)

Simplify (2), $xy - x - y + 1 = 24$

Add, $2xy + 2 = 72$

$xy = 35$ (4)

Substitute value of xy in (3),

$35 + x + y + 1 = 48,$

$x + y = 12$ (5)

Square (5), $x^2 + 2xy + y^2 = 144$

Subtract $4 \times (4)$, $4xy = 140$

$x^2 - 2xy + y^2 = 4$

Extract the root, $x - y = \pm 2$ (6)

From (5) and (6), $x = 7$ or 5,

$y = 5$ or 7.

2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

Let x = digit in tens' place,
and y = digit in units' place.

$x^2 + y^2 = 25$ (1)

$xy = 12$ (2)

Multiply (2) by 2, $2xy = 24$ (3)

Add (3) and (1), $x^2 + 2xy + y^2 = 49.$

Extract the root, $x + y = \pm 7$ (4)

Subtract (3) from (1), $x^2 - 2xy + y^2 = 1.$

Extract the root, $x - y = \pm 1$ (5)

From (4) and (5), $2x = \pm 8$ or ± 6.

$\therefore x = \pm 4$ or ± 3,

$y = \pm 3$ or ± 4.

Hence, the required number is 43 or 34.

3. **The sum, the product, and the difference of the squares of two numbers are all equal. Find the numbers.**

Let $x + y =$ one number,
and $x - y =$ the other number.
Then $2x =$ the sum of the numbers,
$x^2 - y^2 =$ the product of the numbers,
and $4xy =$ the difference of the squares.

$$2x = x^2 - y^2 \quad (1)$$
$$x^2 - y^2 = 4xy \quad (2)$$

Transpose in (1), $x^2 - 2x \qquad - y^2 = 0 \quad (3)$
Transpose in (2), $x^2 \qquad - 4xy - y^2 = 0 \quad (4)$
Subtract, $2x - 4xy \qquad = 0$

$$1 - 2y = 0,$$
$$2y = 1.$$
$$\therefore y = \tfrac{1}{2}.$$

Substitute value of y in (1), $2x = x^2 - \frac{1}{4}$,
$$x^2 - 2x = \tfrac{1}{4}.$$
Complete the square, $x^2 - 2x + 1 = \frac{5}{4}$.
Extract the root, $x - 1 = \pm \frac{1}{2}\sqrt{5}$.
$$\therefore x = 1 \pm \tfrac{1}{2}\sqrt{5},$$
$$x + y = \tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2}(3 \pm \sqrt{5}),$$
$$x - y = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2}(1 \pm \sqrt{5}).$$

4. **The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?**

Let $x =$ greater number,
$y =$ lesser number,
and $x - y =$ difference of the numbers.

Then $x - y = \dfrac{3x}{8} \quad (1)$

and $x^2 + y^2 = 356 \quad (2)$

Simplify (1), $8x - 8y = 3x$.
$$\therefore x = \frac{8y}{5}. \quad (3)$$

Substitute value of x in (2), $\dfrac{64y^2}{25} + y^2 = 356$.

Simplify, $64y^2 + 25y^2 = 8900$,
$$89y^2 = 8900,$$
$$y^2 = 100.$$
Extract the root, $y = \pm 10$.
Substitute value of y in (3), $5x = \pm 80$.
$$\therefore x = \pm 16.$$

5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were interchanged the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

Let $\frac{x}{y} =$ one fraction,

and $\frac{x+1}{y+1} =$ the other fraction.

Then
$$\frac{x}{y} + \frac{x+1}{y+1} = \frac{17}{12} \qquad (1)$$

and
$$\frac{x+1}{y} + \frac{x}{y+1} = \frac{3}{2} \qquad (2)$$

Simplify (1), $12xy+12x+12xy+12y = 17y^2+17y.$

Simplify (2), $2xy+2y+2x+2+2xy = 3y^2+3y.$

Transpose and combine,
$$-17y^2 - 5y + 24xy + 12x = 0 \qquad (3)$$
$$-3y^2 - y + 4xy + 2x = -2 \qquad (4)$$

Multiply (4) by 6,
$$-18y^2 - 6y + 24xy + 12x = -12 \qquad (5)$$

Subtract (5) from (3), $y^2 + y = 12 \qquad (6)$

$$4y^2 + (\ ) + 1 = 49,$$
$$2y + 1 = \pm 7,$$
$$2y = 6 \text{ or } -8.$$
$$\therefore y = 3 \text{ or } -4.$$

Substitute 3 for y in (1), $\frac{x}{3} + \frac{x+1}{4} = \frac{17}{12}.$

Simplify, $4x + 3x + 3 = 17,$
$$7x = 14.$$
$$\therefore x = 2.$$

Hence, the fractions are $\frac{2}{3}$ and $\frac{3}{4}$.

6. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is $\frac{1}{2}$ mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours, by walking 1 mile more per hour than during the first half of the ascent. Find the distance to the top and the rates of walking.

Let $\quad 2x =$ distance,

and $\quad y =$ rate at first.

Then $\quad \dfrac{x}{y} =$ number of hours he was walking 1st half,

and $\quad \dfrac{x}{y-\frac{1}{2}} =$ number of hours he was walking 2d half.

Hence, $$\frac{x}{y} + \frac{x}{y-\frac{1}{2}} = 5\tfrac{1}{2}. \tag{1}$$

Also, $$\frac{2x}{y+1} = 3\tfrac{3}{4} \tag{2}$$

Clear (1) of fractions, $\quad 4xy - 2x + 4xy = 22y^2 - 11y,$

$$22y^2 - 8xy + 2x - 11y = 0 \tag{3}$$

Clear (2) of fractions, $\quad 8x = 15y + 15.$

$$\therefore x = \frac{15y+15}{8} \tag{4}$$

Substitute value of x in (3),

$$22y^2 - 8y\left(\frac{15y+15}{8}\right) + 2\left(\frac{15y+15}{8}\right) - 11y = 0,$$

$$176y^2 - 120y^2 - 120y + 30y + 30 - 88y = 0,$$

$$56y^2 - 178y = -30.$$

Complete the square, $\quad 3136y^2 - (\ ) + (89)^2 = 6241.$

Extract the root, $\quad 56y - 89 = \pm 79.$

$$\therefore y = 3.$$

Substitute value of y in (2), $\quad \dfrac{2x}{4} = \dfrac{15}{4},$

$$2x = 15.$$

Hence, the distance is 15 miles; and the rates of walking, 3, $2\frac{1}{2}$, and 4 miles.

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{5}{18}$ of their difference; and the difference of the squares of the numbers is 3960. Determine the numbers.

Let $\quad x =$ digit in ten's place,

and $\quad y =$ digit in unit's place.

Then $\quad 10x + y =$ first number,

$10y + x =$ second number,

$11x + 11y =$ sum of the numbers,

$9x - 9y =$ difference of the numbers.

$(10x + y)^2 - (x + 10y)^2 =$ difference of the squares.

$$\therefore 11x + 11y = \tfrac{5}{18}(9x - 9y) \tag{1}$$

and $$(10x + y)^2 - (x + 10y)^2 = 3960 \qquad (2)$$

Simplify (1), $$x + y = \frac{5x - 5y}{2},$$

$$7y - 3x = 0 \qquad (3)$$

$$\therefore x = \frac{7y}{3}.$$

Substitute value of x in (2),

$$\left(\frac{73y}{3}\right)^2 - \left(\frac{37y}{3}\right)^2 = 3960,$$

$$\frac{3960y^2}{9} = 3960,$$

$$y^2 = 9.$$

$$\therefore y = \pm 3.$$

From (3), $$3x = 7y.$$

$$\therefore x = \pm 7.$$

Hence, the numbers are 73 and 37.

8. The hypotenuse of a right triangle is 20, and the area of the triangle is 96. Determine the sides.

Let $x =$ longer side,

and $y =$ shorter side.

Since sum of squares on sides equals square on hypotenuse,

$$x^2 + y^2 = 400 \qquad (1)$$

Since area of triangle equals one-half product of sides,

$$\frac{xy}{2} = 96 \qquad (2)$$

$$xy = 192.$$

Multiply (2) by 2, $$2xy = 384 \qquad (3)$$

Add (1) and (3), $$x^2 + 2xy + y^2 = 784.$$

Extract the root, $$x + y = \pm 28 \qquad (4)$$

Subtract (3) from (1), $$x^2 - 2xy + y^2 = 16.$$

Extract the root, $$x - y = \pm 4 \qquad (5)$$

From (5) and (4), $$2x = \pm 32 \text{ or } \pm 24.$$

$$\therefore x = \pm 16 \text{ or } \pm 12.$$

$$2y = \pm 24 \text{ or } \pm 32.$$

$$\therefore y = \pm 12 \text{ or } \pm 16.$$

Hence, the sides are 16 and 12.

9. **Two boys run in opposite directions round a rectangular field the area of which is an acre; they start from one corner and meet 13 yards from the opposite corner; and the rate of one is $\frac{5}{6}$ of the rate of the other. Determine the dimensions of the field.**

Let $x =$ length of first side,

and $y =$ length of second side.

$x + y + 13 =$ number of yards one boy runs,

$x + y - 13 =$ number of yards the other boy runs.

$$x + y - 13 = \tfrac{5}{6}(x + y + 13).$$

$$\therefore 6x + 6y - 78 = 5x + 5y + 65,$$

and $x + y = 143$ (1)

$xy =$ area of field of one acre.

(Since 4840 sq. yds. = 1 acre),

$xy = 4840$ (2)

Square (1), $x^2 + 2xy + y^2 = 20449$

(2) × 4 is $4xy = 19360$

$$x^2 - 2xy + y^2 = 1089$$

$$x - y = \pm 33 \qquad (3)$$

From (1) and (3), $2x = 176$ or 110.

$\therefore x = 88$ or 55.

$2y = 110$ or 176.

$\therefore y = 55$ or 88.

Hence, the dimensions are 88 yds. by 55 yds.

10. **A, in running a race with B, to a post and back, met him 10 yards from the post. To make it a dead heat, B must have increased his rate from this point $41\frac{1}{4}$ yards per minute; and if, without changing his pace, he had turned back on meeting A, he would have come 4 seconds after him. How far was it to the post?**

Let $x =$ number of yards to the post.

Then $2x =$ number of yards to the post and back.

Let $y =$ number of yards A runs per minute.

Then $\dfrac{2x}{y} =$ number of minutes A is running the race.

B runs $(x - 10)$ yards while A is running $(x + 10)$ yards.

Hence, B runs $\dfrac{x - 10}{x + 10}$ of y yards $= \dfrac{xy - 10y}{x + 10}$ yards per minute.

A has $(x-10)$ yards to run when B meets him; and, as he runs y yards per minute, it will take him $\frac{x-10}{y}$ minutes to finish the race.

B has $(x+10)$ yards to run; and, if he increases his pace $41\frac{3}{7}$ yds. per min., he will be running at the rate of $\left(\frac{xy-10y}{x+10}+41\frac{3}{7}\right)$ yards per minute; and, as he has $(x+10)$ yards to run, it will take him $(x+10)\div\left(\frac{xy-10y}{x+10}+41\frac{3}{7}\right)$ minutes to finish the race. But this change of rate will make it a dead heat; therefore,

$$(x+10)\div\left(\frac{xy-10y}{x+10}+41\tfrac{3}{7}\right)=\frac{x-10}{y} \qquad (1)$$

Since 4 seconds $=\frac{1}{15}$ minute, B, without changing his rate, will be $\frac{1}{15}$ of a minute longer than A in running the $(x-10)$ yards which A has to run when he meets B; therefore,

$$(x-10)\div\left(\frac{xy-10y}{x+10}\right)-\frac{x-10}{y}=\frac{1}{15} \qquad (2)$$

Simplify (2),

$$\frac{x+10}{y}-\frac{x-10}{y}=\frac{1}{15} \qquad (3)$$

$$\therefore y=300.$$

Simplify (1),

$$(x+10)\div\left(\frac{7y(x-10)+290x+2900}{7(x+10)}\right)=\frac{x-10}{y},$$

$$\frac{7(x+10)^2}{7y(x-10)+290x+2900}=\frac{x-10}{y}.$$

Substitute 300 for y,

$$\frac{7(x+10)^2}{2390x-18100}=\frac{x-10}{300},$$

$$210x^2+4200x+21000=239x^2-4200x+18100,$$

$$29x^2-8400x=2900,$$

$$x^2-\frac{8400x}{29}=100,$$

$$x^2-(\quad)+\left(\tfrac{4200}{29}\right)^2=\left(\tfrac{4210}{29}\right)^2,$$

$$x-\tfrac{4200}{29}=\pm\tfrac{4210}{29}.$$

$$\therefore x=290 \text{ or } -\tfrac{10}{29}.$$

Hence, the distance to the post was 290 yards.

11. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet it would turn only 88 times more. Find the circumference of each.

Let x = circumference in feet of the fore wheel,

and y = circumference in feet of the hind wheel.

Then
$$\frac{5280}{x}-\frac{5280}{y}=132 \quad (1)$$
$$\frac{5280}{x+2}-\frac{5280}{y+2}=88 \quad (2)$$

Simplify (1), $5280y-5280x=132xy.$

Divide by 132, $40y-40x=xy \quad (3)$

Simplify (2), $5280y+10560-5280x-10560=88xy+176x+176y+352.$

Divide by 88, $60y-60x=xy+2x+2y+4.$

Transpose and combine, $58y-62x=xy+4 \quad (4)$

(3) is $40y-40x=xy$

Subtract, $18y-22x=4$

$$\therefore y=\frac{2+11x}{9}.$$

Substitute value of y in (3),

$$40\left(\frac{2+11x}{9}\right)-40x=\left(\frac{2+11x}{9}\right)x.$$

Simplify, $80+440x-360x=2x+11x^2,$

$11x^2-78x=80.$

Multiply by 11, $121x^2-858x=880.$

Complete the square, $121x^2-(\;)+(39)^2=2401.$

Extract the root, $11x-39=\pm 49,$

$11x=88 \text{ or } -10.$

$\therefore x=8 \text{ or } -\frac{10}{11}.$

Substitute 8 for x in (3). $\therefore y=10.$

12. A person has \$6500, which he divides into two parts and loans at *different rates* of interest, so that the two parts produce *equal* returns. If the first part had been loaned at the second rate of interest, it would have produced \$180; and if the second part had been loaned at the first rate of interest, it would have produced \$245. Find the rates of interest.

Let x = number of dollars in one part of the capital,

$6500 - x$ = number of dollars in the other part,

and y = return from each part.

Then $\frac{y}{x}$ = rate of interest on first part.

Also, $\frac{y}{6500 - x}$ = rate of interest on second part,

$x\left(\frac{y}{6500 - x}\right)$ = return of first part when loaned at second rate.

$$\therefore x\left(\frac{y}{6500 - x}\right) = 180 \qquad (1)$$

$(6500 - x)\frac{y}{x}$ = return of second part when loaned at first rate.

$$\therefore (6500 - x)\frac{y}{x} = 245 \qquad (2)$$

Simplify both equations and add,

$$\begin{array}{rll} xy = & 1170000 - 180x & (3) \\ -xy = & \qquad\quad + 245x - 6500y & (4) \\ \hline 0 = & 1170000 + \ \ 65x - 6500y & \end{array}$$

Transpose and divide by 65,

$$100y - x = 18000 \qquad (5)$$

From (3) $$y = \frac{1170000 - 180x}{x}.$$

Substitute value of y in (5),

$$100\left(\frac{1170000 - 180x}{x}\right) - x = 18000.$$

Simplify, $$x^2 + 36000x = 117000000,$$

$$x^2 + (\ ) + (18000)^2 = 441000000.$$

Extract the root, $$x + 18000 = \pm 21000.$$

$$\therefore x = 3000,$$

$$\text{and } 6500 - x = 3500.$$

From (5), $$y = 210.$$

$$\therefore \frac{y}{x} = 0.07,$$

$$\text{and } \frac{y}{6500 - x} = 0.06.$$

Hence, the rates of interest are 7% and 6%.

Exercise XCVI.

1. $2x + 11y = 49.$

Transpose, $2x = 49 - 11y.$

$\therefore x = 24 - 5y + \frac{1-y}{2}.$

Let $\frac{1-y}{2} = m,$

$1 - y = 2m.$

$\therefore y = 1 - 2m.$

Substitute value of y in original equation,

$2x + 11 - 22m = 49.$

$\therefore x = 19 + 11m.$

If $m = 0,\quad x = 19,\ y = 1.$

If $m = -1,\ x = 8,\quad y = 3.$

2. $7x + 3y = 40.$

Transpose, $3y = 40 - 7x.$

$\therefore y = 13 - 2x + \frac{1-x}{3}.$

Let $\frac{1-x}{3} = m,$

$1 - x = 3m.$

$\therefore x = 1 - 3m.$

Substitute value of x in original equation,

$7 - 21m + 3y = 40,$

$3y = 21m + 33.$

$\therefore y = 7m + 11.$

If $m = 0,\quad y = 11,\ x = 1.$

If $m = -1,\ y = 4,\quad x = 4.$

3. $5x + 7y = 53.$

Transpose, $5x = 53 - 7y.$

$\therefore x = 10 - y + \frac{3-2y}{5}$

$x - 10 + y = \frac{3-2y}{5}$

Multiply by 3,

$3x - 30 + 3y = \frac{9-6y}{5}$

$= 1 - y + \frac{4-y}{5}.$

Let $\frac{4-y}{5} = m,$

$4 - y = 5m.$

$\therefore y = 4 - 5m.$

From given equation,

$x = 5 + 7m.$

If $m = 0,\quad x = 5,\quad y = 4.$

4. $x + 10y = 29.$

Transpose, $x = 29 - 10y.$

If $y = 1,\ x = 19.$

If $y = 2,\ x = 9.$

If $y = 3,\ x = -1.$

$\therefore y$ can only $= 1$ or 2,

x can only $= 19$ or 9.

5. $3x + 8y = 61.$

$$3x = 61 - 8y.$$

$$\therefore x = 20 - 2y + \frac{1-2y}{3}.$$

$$x - 20 + 2y = \frac{1-2y}{3}.$$

Multiply by 2,

$$2x - 40 + 4y = \frac{2-4y}{3}$$

$$= -y + \frac{2-y}{3}.$$

Let $\frac{2-y}{3} = m,$

$$2 - y = 3m.$$

$$\therefore y = 2 - 3m.$$

Substitute in original equation,

$$3x + 16 - 24m = 61,$$

$$3x = 45 + 24m.$$

$$\therefore x = 15 + 8m.$$

If $m = 0,\quad x = 15,\ y = 2.$
If $m = -1,\ x = 7,\quad y = 5.$

6. $8x + 5y = 97.$

$$5y = 97 - 8x.$$

$$\therefore y = 19 - x + \frac{2-3x}{5}.$$

$$y - 19 + x = \frac{2-2x}{5}.$$

Multiply by 2,

$$2y - 38 + 2x = \frac{4-6x}{5}$$

$$= -x + \frac{4-x}{5}.$$

Let $\frac{4-x}{5} = m.$

$$\therefore x = 4 - 5m.$$

Substitute in original equation,

$$32 - 40m + 5y = 97,$$

$$5y = 65 + 40m.$$

$$\therefore y = 13 + 8m.$$

If $m = 0,\quad x = 4,\ y = 13.$
If $m = -1,\ x = 9,\ y = 5.$

7. $16x + 7y = 110.$

$$7y = 110 - 16x.$$

$$\therefore y = 15 - 2x + \frac{5-2x}{7}.$$

Transpose,

$$y + 2x - 15 = \frac{5-2x}{7}.$$

Multiply by 4,

$$4y + 8x - 60 = \frac{20-8x}{7}$$

$$= 2 - x + \frac{6-x}{7}.$$

Let $\frac{6-x}{7} = m,$

$$6 - x = 7m.$$

$$\therefore x = 6 - 7m.$$

Substitute in original equation,

$$96 - 112m + 7y = 110,$$

$$7y = 14 + 112m.$$

$$\therefore y = 2 + 16m.$$

If $m = 0,\ x = 6,\ y = 2.$

8. $7x + 10y = 206.$

$$7x = 206 - 10y.$$

$$\therefore x = 29 - y + \frac{3-3y}{7}.$$

$$x - 29 + y = \frac{3(1-y)}{7}.$$

Let $\frac{1-y}{7} = m.$

$$\therefore y = 1 - 7m.$$

Substitute in original equation,

$$7x + 10 - 70m = 206,$$

$$7x = 196 + 70m.$$

$$\therefore x = 28 + 10m.$$

If $m = 0,\quad x = 28,\ y = 1.$
If $m = -1,\ x = 18,\ y = 8.$
If $m = -2,\ x = 8,\quad y = 15.$

9. $12x - 7y = 1.$

Transpose, $7y = 12x - 1.$

$$\therefore y - x = \frac{5x - 1}{7}.$$

Multiply by 3,

$$3y - 3x = 2x + \frac{x - 3}{7}.$$

Let $\frac{x - 3}{7} = m.$

$$\therefore x = 7m + 3.$$

Substitute this value of x in original equation,

$$84m + 36 - 7y = 1,$$

$$7y = 35 + 84m.$$

$$\therefore y = 5 + 12m.$$

If $m = 0,\ x = 3,\ y = 5.$

11. $23y - 13x = 3.$

Transpose, $13x = 23y - 3.$

$$\therefore x - y = \frac{10y - 3}{13}.$$

Multiply by 4,

$$4x - 4y = 3y + \frac{y - 12}{13}.$$

Let $\frac{y - 12}{13} = m.$

$$\therefore y = 13m + 12.$$

Substitute this value of y in original equation,

$$23(13m + 12) - 13x = 3,$$

$$13x = 299m + 273.$$

$$\therefore x = 23m + 21.$$

If $m = 0,\ x = 21,\ y = 12.$

10. $5x - 17y = 23.$

$$5x = 23 + 17y.$$

$$\therefore x = 4 + 3y + \frac{3 + 2y}{5}.$$

$$x - 4 - 3y = \frac{3 + 2y}{5}.$$

Multiply by 3,

$$3x - 12 - 9y = 1 + y + \frac{4 + y}{5}.$$

Let $\frac{4 + y}{5} = m.$

Then $y = 5m - 4.$

Substitute this value of y in original equation,

$$5x - 17(5m - 4) = 23,$$

$$5x - 85m + 68 = 23,$$

$$5x = 85m - 45.$$

$$\therefore x = 17m - 9.$$

If $m = 1,\ x = 8,\ y = 1.$

12. $23x - 9y = 929.$

$$9y = 23x - 929.$$

$$\therefore y = 2x - 103 + \frac{5x - 2}{9}.$$

$$y - 2x + 103 = \frac{5x - 2}{9}.$$

Multiply by 2,

$$2y - 4x + 206 = x + \frac{x - 4}{9}.$$

Let $\frac{x - 4}{9} = m.$

Then $x - 4 = 9m.$

$$\therefore x = 9m + 4.$$

Substitute this value of y in original equation,

$$207m + 92 - 9y = 929,$$

$$9y = 207m - 837.$$

$$\therefore y = 23m - 93.$$

If $m = 5,\ x = 49,\ y = 22.$

13.

$$23x - 33y = 43.$$

$$23x = 33y + 43.$$

$$\therefore x = 1 + y + \frac{10(y+2)}{23}.$$

Let $\frac{y+2}{23} = m.$

Then $y = 23m - 2.$

Substitute this value of y in original equation,

$$23x - 33(23m - 2) = 43,$$

$$23x - 759m + 66 = 43,$$

$$23x = 759m - 23.$$

$$\therefore x = 33m - 1.$$

If $m = -1,\ x = 32,\ y = 21.$

14.

$$555x - 22y = 73.$$

$$22y = 555x - 73.$$

$$\therefore y = 25x - 3 + \frac{5x - 7}{22}$$

Transpose. $y - 25x + 3 = \frac{5x - 7}{22}.$

Multiply by 9, $9y - 225x + 27 = 2x + 2 + \frac{x - 19}{22}.$

Let $\frac{x - 19}{22} = m.$

Then $x - 19 = 22m.$

$$\therefore x = 19 + 22m.$$

Substitute value of x in original equation,

$$555(19 + 22m) - 22y = 73,$$

$$10545 + 12210m - 22y = 73,$$

$$22y = 10472 + 12210m.$$

$$\therefore y = 476 + 555m.$$

If $m = 0,\ x = 19,\ y = 476.$

15. How many fractions are there with denominators 12 and 18 whose sum is $\frac{25}{36}$?

Let $$\frac{x}{12}+\frac{y}{18}=\frac{25}{36}.$$

Simplify, $$3x+2y=25,$$
$$2y=25-3x.$$
$$\therefore y=12-x+\frac{1-x}{2}.$$

Let $$\frac{1-x}{2}=m.$$

Then $$1-x=2m.$$
$$\therefore x=1-2m.$$

Substitute value of x in original equation,
$$3-6m+2y=25.$$
$$\therefore y=11+3m.$$

If $m=0,\ \ x=1,\ y=11.$
If $m=-1,\ x=3,\ y=8.$
If $m=-2,\ x=5,\ y=5.$
If $m=-3,\ x=7,\ y=2.$

Hence, the pairs of fractions are
$$\tfrac{1}{12},\ \tfrac{11}{18};\ \ \tfrac{3}{12},\ \tfrac{8}{18};\ \ \tfrac{5}{12},\ \tfrac{5}{18};\ \ \tfrac{7}{12},\ \tfrac{2}{18}.$$

16. What is the least number which, when divided by 3 and 5, leaves remainders 2 and 3 respectively?

Let $$n=\text{number},$$
$$\frac{n-2}{3}=x \qquad (1)$$
$$\frac{n-3}{5}=y \qquad (2)$$

From (1) and (2), $$n=3x+2 \text{ and } 5y+3.$$
$$\therefore 3x+2=5y+3,$$
$$3x=5y+1 \qquad (3)$$
$$\therefore x=1+\frac{2y+1}{3}.$$

Transpose, $$x-1=\frac{2y+1}{3}.$$

Multiply by 2, $$2x-2=y+\frac{y+2}{3}.$$

Let $$\frac{y+2}{3}=m.$$

Then $$y=3m-2.$$
From (3), $$3x=15m-9.$$
$$\therefore x=5m-3.$$

If $m=1,\ x=2,\ y=1.$
But $$n=3x+2.$$
$$\therefore n=8.$$

17. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over; but when he counts them 5 at a time there are 4 over. How many are there in all?

Let $\frac{n-2}{3} = x,$

and $\frac{n-4}{5} = y.$

Then $n = 2 + 3x \text{ or } 4 + 5y.$

$$\therefore\ 2 + 3x = 4 + 5y,$$

$$3x = 2 + 5y \qquad (1)$$

$$x = y + \frac{2(1+y)}{3}.$$

Let $\frac{1+y}{3} = m.$

Then $y = 3m - 1.$

Substitute value of y in (1), $3x = 2 + 5(3m-1),$

$$3x = 15m - 3.$$

$$\therefore\ x = 5m - 1.$$

If $m = 4,\ x = 19,\ y = 11.$

Hence, the number of eggs is 59.

18. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 apiece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.

Let $x =$ number of pigs,

and $y =$ number of geese.

Then $40 - x - y =$ number of chickens.

$$5x + y + 10 - \frac{x}{4} - \frac{y}{4} = 40 \qquad (1)$$

$$\text{or } 20x + 4y + 40 - x - y = 160,$$

$$\text{or } 19x + 3y = 120,$$

$$3y = 120 - 19x \qquad (2)$$

$$y = 40 - 6x - \frac{x}{3}.$$

Let $\frac{x}{3} = m.$

$$\therefore\ x = 3m.$$

Substitute value of x in (2), $3y = 120 - 57m.$

$$\therefore\ y = 40 - 19m.$$

If $m = 1,\ x = 3,\ y = 21.$

If $m = 2,\ x = 6,\ y = 2.$

Hence, he bought 3 pigs, 21 geese, and 16 chickens; or 6 pigs, 2 geese, and 32 chickens.

19. Find the least multiple of 7 which, when divided by 2, 3, 4, 5, 6, leaves in each case 1 for a remainder.

Let $7x =$ least multiple of 7,
and $y =$ sum of quotients.
Then

$$\frac{7x-1}{2}+\frac{7x-1}{3}+\frac{7x-1}{4}+\frac{7x-1}{5}+\frac{7x-1}{6}=y.$$

Simplify,

$$210x-30+140x-20+105x-15$$
$$+84x-12+70x-10=60y,$$
$$609x-60y=87.$$

Divide by 3, $$203x-20y=29 \qquad (1)$$
$$-20y=-203x+29.$$
$$\therefore y=10x-1+\frac{3x-9}{20}.$$

Transpose, $$y-10x+1=\frac{3(x-3)}{20}.$$

Let $$\frac{x-3}{20}=m.$$

Then $$x-3=20m.$$
$$\therefore x=20m+3.$$

Substitute value of x in (1),

$$4060m+609-20y=29,$$
$$20y=-4060m-580 \qquad (2)$$
$$\therefore y=203m+29 \qquad (3)$$

If $m=2,\ x=43,\ y=435.$

Hence, the number is 301.

20. In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other of 9?

Let $7x =$ one part,
and $9y =$ the other part.

$$\therefore 7x+9y=100.$$
$$7x=100-9y.$$
$$\therefore x=14-y+\frac{2(1-y)}{7}.$$

Let $$\frac{1-y}{7}=m.$$

Then $$1-y=7m.$$
$$\therefore y=1-7m.$$

Substitute value of y in the original equation,

$$7x+9(1-7m)=100,$$
$$7x=100-9(1-7m),$$
$$7x=91+63m.$$
$$\therefore x=13+9m.$$

If $m=0,\quad x=13,\ y=1.$
If $m=-1,\ x=4,\quad y=8.$

Hence, the parts are 91 and 9, or 28 and 72.

21. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

$$18x - 5y = 70.$$

Let $$y = mx.$$

Substitute value of y in this equation,

$$18x - 5mx = 70.$$
$$x(18 - 5m) = 70.$$
$$\therefore x = \frac{70}{18 - 5m},$$

and $$y = \frac{70m}{18 - 5m}.$$

Now, if $m = 2$, $$x = \tfrac{70}{8} \text{ or } 8\tfrac{3}{4},$$
and $$y = \tfrac{140}{8} \text{ or } 17\tfrac{1}{2}.$$
And, if $m = 3$, $$x = \tfrac{70}{3} \text{ or } 23\tfrac{1}{3},$$
and $$y = \tfrac{210}{3} \text{ or } 70.$$

22. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

$$8x + 12y = 23 \quad (1)$$

Let $$x + y = m.$$

Transpose, $$x = m - y \quad (2)$$

Substitute value of x in (1),

$$8m - 8y + 12y = 23,$$
$$4y = 23 - 8m.$$
$$\therefore y = \frac{23 - 8m}{4}.$$

Substitute value of y in (1),

$$8x + 69 - 24m = 23,$$
$$8x = 24m - 46.$$
$$\therefore x = \frac{24m - 46}{8}.$$

Let $$m = 2.$$

Then $$x = \frac{48 - 46}{8} = \frac{1}{4},$$

and $$y = \frac{23 - 16}{4} = \frac{7}{4}.$$

23. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

Let $x =$ first part,

$y =$ second part,

and $70 - x - y =$ third part.

$$\frac{x}{6} + \frac{y}{7} + \frac{70 - x - y}{8} = 10 \quad (1)$$

Simplify,

$$24x + 24y + 1470 - 21x - 21y = 1680,$$

$$7x + 3y = 210 \quad (2)$$

$$3y = 210 - 7x.$$

$$\therefore y = 70 - 2x - \frac{x}{3}.$$

Let $\frac{x}{3} = m.$

$$\therefore x = 3m.$$

Substitute value of m in (2),

$$21m + 3y = 210,$$

$$3y = 210 - 21m.$$

$$\therefore y = 70 - 7m.$$

If $m = 2,\ 4,\ 6,\ 8,$

(the lowest values that will produce multiples of the numbers),

$$x = 6,\ 12,\ 18,\ 24,$$

$$y = 56,\ 42,\ 28,\ 14,$$

$$70 - x - y = 8,\ 16,\ 24,\ 32.$$

24. Divide 200 into three parts which shall give integral quotients when divided by 5, 7, 11, respectively, and the sum of the quotients shall be 20.

Let $x =$ first part,

and $y =$ second part.

Then $200 - x - y =$ third part,

$$\frac{x}{5} + \frac{y}{7} + \frac{200 - x - y}{11} = 20.$$

Simplify,

$$77x + 55y + 7000 - 35x - 35y = 7700,$$

$$42x + 20y = 700,$$

$$21x + 10y = 350 \quad (1)$$

$$\therefore y = 35 - 2x - \frac{x}{10}.$$

Let $\frac{x}{10} = m,$

$$x = 10m.$$

Substitute value of x in (1), $y = 35 - 21m.$

If $m = 1,\ x = 10,\ y = 14.$

$$200 - x - y = 176.$$

25. A number consisting of three digits, of which the middle one is 4, has the digits in the units' and hundreds' places interchanged by adding 792. Find the number.

Let $x =$ digit in hundreds' place,

and $y =$ digit in units' place.

$$\therefore\ 100x + 40 + y = \text{the number.}$$
$$100y + 40 + x = 792 + 100x + 40 + y.$$

Transpose and combine,

$$99y - 99x = 792.$$

Divide by 99, $$y - x = 8 \qquad (1)$$
$$y = x + 8.$$

Let $$x + 8 = m,$$
$$x = m - 8 \qquad (2)$$

and $$y = m.$$

From (2), m must be equal to 9, in order to make x positive.

Then $$x = 1,$$
$$y = 9.$$

Hence, the number is 149.

26. Some men earning each \$2.50 a day, and some women earning each \$1.75 a day, receive all together for their daily wages \$44.75. Determine the number of men and the number of women.

Let $x =$ number of men,

and $y =$ number of women.

Then $$\frac{5x}{2} + \frac{7y}{4} = \frac{179}{4},$$
$$10x + 7y = 179,$$
$$y = 25 - x + \frac{4 - 3x}{7}.$$

Transpose, $$y - 25 + x = \frac{4 - 3x}{7}.$$

Multiply by 5, $$5y - 125 + 5x = 2 - 2x + \frac{6 - x}{7}.$$

Let $$\frac{6 - x}{7} = m,$$
$$x = 6 - 7m.$$

Substitute $6 - 7m$ for x in the original equation,

$$60 - 70m + 7y = 179,$$
$$7y = 119 + 70m.$$
$$\therefore\ y = 17 + 10m.$$

If $m = 0,\quad x = 6,\quad y = 17.$

If $m = -1,\quad x = 13,\quad y = 7.$

27. A wishes to pay B a debt of £1 12*s*., but has only half-crowns in his pocket, while B has only four-penny pieces. How may they settle the matter most simply?

Let $x =$ number of half-crowns,

and $y =$ number of four-penny pieces.

Then half-crowns $= 30x$ pence,

and four-penny pieces $= 4y$ pence.

$$£1 + 12s. = 384 \text{ pence.}$$

But $£1 + 12s. = 30x - 4y.$

$$\therefore 30x - 4y = 384 \qquad (1)$$

$$4y = 30x - 384.$$

$$\therefore y = \frac{30x - 384}{4},$$

$$\text{or } y = 7x - 96 + \frac{x}{2}$$

Let $\frac{x}{2} = m.$

Then $x = 2m.$

Substitute value of x in (1),

$$60m - 4y = 384.$$

$$\therefore y = 15m - 96.$$

If $m = 7,\ x = 14,\ y = 9.$

Hence, A can give B 14 half-crowns, and receive from B 9 four-penny pieces.

29. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the whole price £100. If the oxen cost £5, the sheep £1, and the hens 1 *s*. each, how many of each did he buy?

Let $x =$ number of oxen,

and $y =$ number of sheep.

Then $100 - x - y =$ number of hens.

$$5x + y + \frac{100 - x - y}{20} = 100 \qquad (1)$$

$$100x + 20y + 100 - x - y = 2000,$$

$$99x + 19y = 1900 \qquad (2)$$

Transpose, $19y = 1900 - 99x.$

Divide by 19, $y = 100 - 5x - \frac{4x}{19}.$

Transpose, $100 - 5x - y = \frac{4x}{19}.$

Multiply by 5,

$$500 - 25x - 5y = x + \frac{x}{19}.$$

Let $\frac{x}{19} = m.$

Then $x = 19m.$

Substitute value of x in (2),

$$1881m + 19y = 1900.$$

Transpose, $19y = 1900 - 1881m,$

$$y = 100 - 99m.$$

If $m = 1,\ x = 19,\ y = 1,$

and $100 - x - y = 80.$

Hence, he buys 19 oxen, 1 sheep, and 80 hens.

30. A number of lengths 3 feet, 5 feet, and 8 feet are cut; how may 48 of them be taken so as to measure 175 feet all together?

Let $x =$ number 8 feet long,

$y =$ number 5 feet long,

and $48 - x - y =$ number 3 feet long.

$$8x + 5y + 3(48 - x - y) = 175 \qquad (1)$$

Simplify, $5x + 2y = 31.$

Transpose, $2y = 31 - 5x \qquad (2)$

$$\therefore y = 15 - 2x + \frac{1-x}{2}.$$

Let $\frac{1-x}{2} = m.$

Then $1 - x = 2m.$

$$\therefore x = 1 - 2m.$$

Substitute value of x in (2), $2y = 31 - 5 + 10m.$

$$y = 13 + 5m \qquad (3)$$

If $m = 0,\ -1,\ -2,$

$x = 1,\ 3,\ 5 =$ number of 8-ft. lengths,

$y = 13,\ 8,\ 3 =$ number of 5-ft. lengths,

$48 - x - y = 34,\ 37,\ 40 =$ number of 3-ft. lengths.

31. A field containing an integral number of acres less than 10 is divided into 8 lots of one size, and 7 of 4 times that size; and has also a road passing through it containing 1300 square yards. Find the size of the lots in square yards.

Let $x =$ number of acres.

$\therefore 4840x =$ number of square yards.

$y =$ number of square yards in 1 lot,

$8y =$ number of square yards in 8 lots,

$28y =$ number of square yards in 7 lots of second kind.

$$8y + 28y + 1300 = 4840x \quad (1)$$

$$9y + 325 = 1210x,$$

$$9y = 1210x - 325 \quad (2)$$

$$y = 134x - 36 + \frac{4x-1}{9},$$

$$y - 134x + 36 = \frac{4x-1}{9}.$$

Multiply by 7,

$$7y - 938x + 252 = 3x + \frac{x-7}{9}.$$

Let $\frac{x-7}{9} = m.$

Then $x - 7 = 9m.$

$\therefore x = 9m + 7.$

Substitute value of x in (2),

$$9y = 10890m + 8470 - 325,$$

$$9y = 10890m + 8145.$$

$$\therefore y = 1210m + 905.$$

If $m = 0,$

$x = 7 \quad =$ number of acres.

$y = 905 \quad =$ number of sq. yds. in 1st lot.

$4y = 3620 =$ number of sq. yds. in 2d lot.

32. Two wheels are to be made, the circumference of one of which is to be a multiple of the other. What circumferences may be taken so that when the first has gone round three times and the other five, the difference in the length of rope coiled on them may be 17 feet?

Let $x =$ circumference of the first wheel,
and $y =$ circumference of the second wheel.

$3x - 5y = 17$, difference of length of rope coiled on them when first wheel goes round three times, and second five times.

Let
$$x = my,$$
$$3x - 5y = 17,$$
$$3my - 5y = 17,$$
$$y(3m - 5) = 17.$$
$$\therefore y = \frac{17}{3m - 5}$$
If
$$m = 2,\ y = 17,\ x = 34.$$

33. In how many ways can a person pay a sum of £15 in half-crowns, shillings, and sixpences, so that the number of shillings and sixpences together shall be equal to the number of half-crowns?

Let $x =$ number of shillings,
and $y =$ number of sixpences.
Then $x + y =$ number of half-crowns.

y sixpences $= \frac{1}{2}y$ shillings.

$(x + y)$ half-crowns $= \frac{5}{2}(x + y)$ shillings.

Then $x + \frac{1}{2}y + \frac{5}{2}(x + y) =$ whole number of shillings.
But $300 =$ whole number of shillings.
$$\therefore x + \frac{1}{2}y + \frac{5}{2}(x + y) = 300.$$
$$2x + y + 5x + 5y = 600,$$
$$7x + 6y = 600.$$
$$x = 85 + \frac{5 - 6y}{7}.$$
Transpose, and multiply by 6,
$$6x - 510 = 4 - 5y + \frac{2 - y}{7}.$$
Let
$$\frac{2 - y}{7} = m.$$
$$\therefore y = 2 - 7m.$$
Substitute value of y,
$$7x + 12 - 42m = 600.$$
$$\therefore x = 6m + 84.$$
If $m = 0, -2, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11, -12, -13$, then x and y would each have 14 positive values.

Hence, there are 14 ways.

Exercise XCVII.

1.

$a^2 + 3b^2$ is $> 2b(a+b)$.

If $a^2 + 3b^2$ is $> 2ab + 2b^2$,

if (transposing), $a^2 + b^2$ is $> 2ab$.

But $a^2 + b^2$ is $> 2ab$. § 249.

$\therefore a^2 + 3b^2$ is $> 2b(a+b)$.

2.

$a^3b + ab^3$ is $> 2a^2b^2$.

If (dividing both sides by ab),

$a^2 + b^2$ is $> 2ab$.

But $a^2 + b^2$ is $> 2ab$. § 249.

$\therefore a^3b + ab^3$ is $> 2a^2b^2$.

3.

$(a^2+b^2)(a^4+b^4)$ is $> (a^3+b^3)^2$.

If (simplifying),

$a^6 + a^4b^2 + a^2b^4 + b^6$ is $> a^6 + 2a^3b^3 + b^6$,

if (transposing), $a^4b^2 + a^2b^4$ is $> 2a^3b^3$,

if (dividing by a^2b^2), $a^2 + b^2$ is $> 2ab$.

But $a^2 + b^2$ is $> 2ab$. § 249.

$\therefore (a^2+b^2)(a^4+b^4)$ is $> (a^3+b^3)^2$.

4.

$a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2$ is $> 6abc$.

$a(b^2+c^2) + b(a^2+c^2) + c(a^2+b^2)$ is $> 6abc$.

Since (b^2+c^2) is $> 2bc$, § 249.

$\therefore a(b^2+c^2)$ is $> 2abc$.

Since (a^2+c^2) is $> 2ac$,

$\therefore b(a^2+c^2)$ is $> 2abc$.

Since (a^2+b^2) is $> 2ab$,

$\therefore c(a^2+b^2)$ is $> 2abc$.

Therefore (by adding),

$a(b^2+c^2) + b(a^2+c^2) + c(a^2+b^2)$ is $> 6abc$.

5. **The sum of any fraction and its reciprocal is >2.**

Let $\frac{a}{b}=$ the fraction.

Then $\frac{b}{a}=$ the reciprocal,

and $\frac{a}{b}+\frac{b}{a}$ is >2,

if (multiplying by ab), a^2+b^2 is $>2ab$.

But a^2+b^2 is $>2ab$. § 249.

$\therefore \frac{a}{b}+\frac{b}{a}$ is >2.

6.

If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, xy is not less than $ac+bd$, or $ad+bc$.

Now, if xy equals or is $>ac+bd$,

then x^2y^2 equals or is $>(ac+bd)^2$,

and (by substituting the values of x^2 and y^2),

$(a^2+b^2)(c^2+d^2)$ equals or is $>(ac+bd)^2$,

or (simplifying),

$a^2c^2+a^2d^2+c^2b^2+b^2d^2$ equals or is $>a^2c^2+2abcd+b^2d^2$,

and $a^2d^2+b^2c^2$ equals or is $>2abcd$.

But $a^2d^2+b^2c^2$ equals or is $>2abcd$. § 249.

$\therefore xy$ equals or is $>ac+bd$.

7.

$ab+ac+bc$ is $<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$,

if (by expanding and combining),

$ab+ac+bc$ is $<3a^2+3b^2+3c^2-2ab-2ac-2bc$,

if $3ab+3ac+3bc$ is $<3a^2+3b^2+3c^2$,

if $ab+ac+bc$ is $<a^2+b^2+c^2$.

$$\begin{array}{llll} \text{But} & 2ab & \text{is} < & a^2+b^2 \\ & 2ac & \text{is} < & a^2 \qquad +c^2 \\ & 2bc & \text{is} < & \qquad b^2+c^2 \\ \hline \therefore & \multicolumn{3}{l}{2ab+2ac+2bc \text{ is } <2a^2+2b^2+2c^2} \end{array}$$

or $ab+ac+bc$ is $<a^2+b^2+c^2$.

$\therefore ab+ac+bc$ is $<(a+b-c)^2+(a+c-b)^2+(b+c-a)^2$.

8. **Which is the greater,**

$$(a^2+b^2)(c^2+d^2) \text{ or } (ac+bd)^2?$$

Simplify, $a^2c^2+b^2c^2+a^2d^2+b^2d^2$ is $> a^2c^2+2abcd+b^2d^2$.

If $\quad b^2c^2+a^2d^2$ is $> 2abcd$

But $\quad b^2c^2+a^2d^2$ is $> 2abcd$, § 249.

$\therefore (a^2+b^2)(c^2+d^2)$ is $> (ac+bd)^2$.

9. **Which is the greater,**

$$m^2+m \text{ or } m^3+1?$$

m^2+m is $>$ or $< m^3+1$,

as $\quad m(m+1)$ is $>$ or $< (m^2-m+1)(m+1)$,

as $\quad m >$ or $< m^2-m+1$,

as $\quad 2m$ is $>$ or $< m^2+1$.

But $\quad m^2+1$ is $> 2m$, § 249.

$\therefore m^3+1$ is $> m^2+m$.

10. **Which is the greater,**

$$a^4-b^4 \text{ or } 4a^3(a-b), \text{ when } a \text{ is} > b?$$

$4a^3(a-b)$ is $>$ or $< a^4-b^4$,

as (dividing by $a-b$), $\quad 4a^3$ is $>$ or $< a^3+a^2b+ab^2+b^3$,

as (subtracting a^3 from both sides),

$3a^3$ is $>$ or $< a^2b+ab^2+b^3$,

as (transposing a^2b), $3a^3-a^2b$ is $>$ or $< ab^2+b^3$,

as $\quad a^2(3a-b)$ is $>$ or $< b^2(a+b)$,

if the factor a^2 be taken out from the left side, and the factor b^2 from the right side, since a is $> b$, the left side will have been divided by a greater number than the right; so that, if the left is greater than the right, after both factors have been taken out, it must have been greater before.

If, therefore, $\quad 3a-b$ is $> a+b$,

if (by adding $b-a$ to both sides),

$2a$ is $> 2b$.

But $\quad 2a$ is $> 2b$.

$\therefore 4a^3(a-b)$ is $> a^4-b^4$.

11. Which is the greater,

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \text{ or } \sqrt{a}+\sqrt{b}?$$

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \text{ is} > \text{or} < \sqrt{a}+\sqrt{b},$$

as (squaring), $\frac{a^2}{b}+\sqrt{2ab}+\frac{b^2}{a}$ is $>$ or $< a+\sqrt{2ab}+b$,

as (transposing), $\frac{a^2}{b}+\frac{b^2}{a}$ is $>$ or $< a+b$,

as (multiplying by ab), a^3+b^3 is $>$ or $< a^2b+ab^2$,

as $(a+b)(a^2-ab+b^2)$ is $>$ or $< ab(a+b)$,

as (a^2+b^2) is $>$ or $< 2ab$.

But (a^2+b^2) is $> 2ab$, § 249.

$$\therefore \sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \text{ is} > \sqrt{a}+\sqrt{b}.$$

12. Which is the greater,

$$\frac{a+b}{2} \text{ or } \frac{2ab}{a+b}?$$

$$\frac{a+b}{2} \text{ is} > \text{or} < \frac{2ab}{a+b},$$

as $a^2+2ab+b^2$ is $>$ or $< 4ab$,

as a^2+b^2 is $>$ or $< 2ab$.

But a^2+b^2 is $> 2ab$, § 249.

$$\therefore \frac{a+b}{2} \text{ is} > \frac{2ab}{a+b}.$$

13. Which is the greater,

$$\frac{a}{b^2}+\frac{b}{a^2} \text{ or } \frac{1}{b}+\frac{1}{a}?$$

$$\frac{a}{b^2}+\frac{b}{a^2} \text{ is} > \text{or} < \frac{1}{b}+\frac{1}{a},$$

as a^3+b^3 is $>$ or $< a^2b+ab^2$,

as $(a+b)(a^2-ab+b^2)$ is $>$ or $< (a+b)ab$,

as a^2+b^2 is $>$ or $< 2ab$.

But a^2+b^2 is $> 2ab$, § 249.

$$\therefore \frac{a}{b^2}+\frac{b}{a^2} \text{ is} > \frac{1}{b}+\frac{1}{a}.$$

Exercise XCVIII.

1. $\sqrt{x^3} = x^{\frac{3}{2}}$.
$\sqrt[3]{x^2} = x^{\frac{2}{3}}$.
$(\sqrt{x})^5 = x^{\frac{5}{2}}$.
$\sqrt[3]{a^4} = a^{\frac{4}{3}}$.
$\sqrt[7]{a^6} = a^{\frac{6}{7}}$.
$(\sqrt[3]{a})^7 = a^{\frac{7}{3}}$.
$\sqrt[6]{a^3b^2} = a^{\frac{1}{2}}b^{\frac{1}{3}}$.

2. $\sqrt[3]{xy^2z^3} = x^{\frac{1}{3}}y^{\frac{2}{3}}z$.
$\sqrt[5]{x^3y^2z^4} = x^{\frac{3}{5}}y^{\frac{2}{5}}z^{\frac{4}{5}}$.
$\sqrt[7]{a^5b^6c^7} = a^{\frac{5}{7}}b^{\frac{6}{7}}c$.
$5\sqrt{a^2bc^3x^4} = 5ab^{\frac{1}{2}}c^{\frac{3}{2}}x^2$.

3. $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.
$a^{\frac{1}{2}}b^{\frac{1}{6}} = \sqrt[6]{a^3b}$.
$4x^{\frac{1}{6}}y^{-\frac{5}{6}} = 4\sqrt[6]{xy^{-5}}$.
$3x^{\frac{1}{3}}y^{-\frac{2}{3}} = 3\sqrt[3]{xy^{-2}}$.

4. $a^{-2} = \frac{1}{a^2}$.
$3x^{-1}y^{-3} = \frac{3}{xy^3}$.
$6x^{-3}y = \frac{6y}{x^3}$.
$x^4y^{-5} = \frac{x^4}{y^5}$.
$\frac{2a^{-1}x}{3^{-1}b^2y^{-3}} = \frac{6xy^3}{ab^2}$.

5. $\frac{3xy}{z^2} = 3xyz^{-2}$.
$\frac{z}{x^3y^4} = x^{-3}y^{-4}z$.
$\frac{a}{bc} = ab^{-1}c^{-1}$.
$\frac{c^2}{a^3b^{-2}} = a^{-3}b^2c^2$.
$\frac{x^{-\frac{1}{2}}}{y^{-\frac{2}{3}}} = x^{-\frac{1}{2}}y^{\frac{2}{3}}$.
$\frac{x^{-2}}{y^{\frac{1}{3}}} = x^{-2}y^{-\frac{1}{3}}$.

6. $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}}$.
$b^{\frac{1}{3}} \times b^{\frac{1}{6}} = b^{\frac{1}{2}}$.
$c^{\frac{2}{3}} \times c^{\frac{1}{12}} = c^{\frac{3}{4}}$.
$d^{\frac{3}{4}} \times d^{\frac{1}{12}} = d^{\frac{5}{6}}$.

7. $m^{\frac{1}{2}} \times m^{-\frac{1}{6}} = m^{\frac{1}{3}}$.
$n^{\frac{3}{4}} \times n^{-\frac{1}{12}} = n^{\frac{2}{3}}$.
$a^0 \times a^{\frac{1}{2}} = a^{\frac{1}{2}}$.
$a^0 \times a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}}$.

8. $a^{\frac{1}{2}} \times \sqrt{a} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$.
$c^{-\frac{1}{2}} \times \sqrt{c} = \frac{1}{c^{\frac{1}{2}}} \times c^{\frac{1}{2}} = 1$.
$y^{\frac{1}{4}} \times \sqrt[4]{y} = y^{\frac{1}{4}} \times y^{\frac{1}{4}} = y^{\frac{1}{2}}$.
$x^{\frac{5}{6}} \times \sqrt{x^{-1}} = x^{\frac{5}{6}} \times x^{-\frac{1}{2}} = x^{\frac{1}{3}}$.

9. $ab^{\frac{1}{2}}c \times a^{-\frac{1}{2}}bc^{\frac{1}{3}} = a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{4}{3}}.$

$a^{\frac{2}{3}}b^{\frac{1}{2}}c^{-\frac{1}{4}} \times a^{\frac{1}{3}}b^{-\frac{1}{2}}c^{\frac{1}{2}}d = ac^{\frac{1}{4}}d.$

10. $x^{\frac{1}{3}}y^{\frac{3}{4}}z^{\frac{1}{6}} \times x^{-\frac{2}{3}}y^{-\frac{1}{2}}z^{-\frac{1}{2}} = x^{-\frac{1}{3}}y^{\frac{1}{4}}z^{-\frac{1}{3}}.$

$x^{\frac{5}{6}}y^{\frac{1}{4}}z^{\frac{1}{2}} \times x^{-\frac{1}{6}}y^{-\frac{1}{2}}z^{-\frac{1}{2}} = x^{\frac{2}{3}}y^{-\frac{1}{4}}.$

11. $a^{\frac{1}{2}} \times a^{-\frac{1}{3}} \times a^{-\frac{1}{4}} \times a^{-\frac{1}{3}} = a^{-\frac{5}{12}}.$

$$\left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{3}} \times \left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$$

$$= \frac{a^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \times \frac{b^{\frac{1}{3}}x^{\frac{1}{3}}}{y^{\frac{2}{3}}} \times \frac{y^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= \frac{y^{\frac{1}{3}}}{b^{\frac{1}{6}}x^{\frac{1}{6}}}.$$

12. $a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{1}{6}}.$

$c^{\frac{5}{6}} \div c^{\frac{1}{2}} = c^{\frac{1}{3}}.$

$n^{\frac{7}{12}} \div n^{\frac{3}{4}} = n^{-\frac{1}{6}}.$

$a^{\frac{5}{6}} \div \sqrt[3]{a^2} = a^{\frac{5}{6}} \div a^{\frac{2}{3}} = a^{\frac{1}{6}}.$

13. $(a^6)^{\frac{1}{2}} \div (a^6)^{\frac{2}{3}} = (a^6)^{-\frac{1}{6}} = a^{-1}$

$= \frac{1}{a}.$

$(c^{-\frac{1}{2}})^{\frac{2}{3}} = c^{-\frac{1}{3}} = \frac{1}{c^{\frac{1}{3}}}.$

$(m^{-\frac{1}{2}})^4 = m^{-2} = \frac{1}{m^2}.$

$(n^{\frac{1}{3}})^{-3} = n^{-1} = \frac{1}{n}.$

$(x^{\frac{3}{4}})^{\frac{4}{3}} = x^{\frac{12}{12}} = x.$

14. $(p^{-\frac{3}{4}})^{-\frac{2}{3}} = p^{\frac{1}{2}}.$

$(q^{\frac{2}{3}})^{-\frac{1}{2}} = q^{-\frac{1}{3}}.$

$(x^{-\frac{1}{2}}y^{\frac{3}{4}})^{-\frac{4}{3}} = x^{\frac{2}{3}}y^{-1}.$

$(a^{\frac{2}{3}} \times a^{\frac{4}{7}})^{-\frac{14}{13}} = (a^{\frac{26}{21}})^{-\frac{14}{13}} = a^{-\frac{4}{3}}.$

15. $(4a^{-\frac{2}{3}})^{-\frac{3}{2}} = 4^{-\frac{3}{2}}a = \frac{a}{8}.$

$(27b^{-3})^{-\frac{2}{3}} = 27^{-\frac{2}{3}}b^2 = \frac{b^2}{9}.$

$(64c^{16})^{-\frac{5}{6}} = 64^{-\frac{5}{6}}c^{-\frac{40}{3}}$

$= \frac{1}{32c^{\frac{40}{3}}}.$

$(32c^{-10})^{\frac{2}{5}} = 32^{\frac{2}{5}}c^{-4} = \frac{4}{c^4}.$

16. $\left(\frac{16a^{-4}}{81b^3}\right)^{-\frac{3}{4}} = \frac{16^{-\frac{3}{4}}a^3}{81^{-\frac{3}{4}}b^{-\frac{9}{4}}}$

$= \frac{81^{\frac{3}{4}}a^3b^{\frac{9}{4}}}{16^{\frac{3}{4}}}$

$= \frac{27a^3b^{\frac{9}{4}}}{8}.$

$\left(\frac{9a^4}{16b^{-3}}\right)^{-\frac{3}{2}} = \frac{9^{-\frac{3}{2}}a^{-6}}{16^{-\frac{3}{2}}b^{\frac{9}{2}}}$

$= \frac{16^{\frac{3}{2}}}{9^{\frac{3}{2}}a^6b^{\frac{9}{2}}}$

$= \frac{64}{27a^6b^{\frac{9}{2}}}.$

$(3^{\frac{3}{8}}a^{-3})^{-\frac{2}{3}} = 3^{-\frac{1}{4}}a^2 = \frac{a^2}{3^{\frac{1}{4}}}.$

$\left(\frac{256}{625}\right)^{-\frac{3}{4}} = \frac{256^{-\frac{3}{4}}}{625^{-\frac{3}{4}}} = \frac{625^{\frac{3}{4}}}{256^{\frac{3}{4}}}$

$= \frac{125}{64}.$

Exercise XCIX.

1.

$$\begin{array}{l}
x^{2p} + x^{p}y^{p} + y^{2p} \\
\underline{x^{2p} - x^{p}y^{p} + y^{2p}} \\
x^{4p} + x^{3p}y^{p} + x^{2p}y^{2p} \\
\qquad - x^{3p}y^{p} - x^{2p}y^{2p} - x^{p}y^{3p} \\
\qquad\qquad\qquad x^{2p}y^{2p} + x^{p}y^{3p} + y^{4p} \\
\hline
x^{4p} \qquad\qquad + x^{2p}y^{2p} \qquad\qquad + y^{4p}
\end{array}$$

2.

$$\begin{array}{l}
x^{mn-n} - y^{n} \\
\underline{x^{n} + y^{mn-n}} \\
x^{mn} - x^{n}y^{n} \\
\qquad\qquad + x^{mn-n}y^{mn-n} \\
\qquad\qquad\qquad\qquad\qquad - y^{mn} \\
\hline
x^{mn} - x^{n}y^{n} + x^{mn-n}y^{mn-n} - y^{mn}
\end{array}$$

3.

$$\begin{array}{l}
x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1 \\
\underline{x^{\frac{1}{3}} - 1} \\
x - 2x^{\frac{2}{3}} + x^{\frac{1}{3}} \\
\quad - x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 1 \\
\hline
x - 3x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 1
\end{array}$$

4.

$$\begin{array}{l}
8a^{\frac{3}{7}} + 4a^{\frac{2}{7}}b^{\frac{1}{7}} + 5a^{\frac{1}{7}}b^{\frac{2}{7}} + 9b^{\frac{3}{7}} \\
\underline{2a^{\frac{4}{7}} - b^{\frac{4}{7}}} \\
16a + 8a^{\frac{6}{7}}b^{\frac{1}{7}} + 10a^{\frac{5}{7}}b^{\frac{2}{7}} + 18a^{\frac{4}{7}}b^{\frac{3}{7}} \\
\qquad - 8a^{\frac{3}{7}}b^{\frac{4}{7}} - 4a^{\frac{2}{7}}b^{\frac{5}{7}} - 5a^{\frac{1}{7}}b^{\frac{6}{7}} - 9b
\end{array}$$

5.

$$\begin{array}{l}
1 + ab^{-1} + a^{2}b^{-2} \\
\underline{1 - ab^{-1} + a^{2}b^{-2}} \\
1 + ab^{-1} + a^{2}b^{-2} \\
\quad - ab^{-1} - a^{2}b^{-2} - a^{3}b^{-3} \\
\qquad\qquad + a^{2}b^{-2} + a^{3}b^{-3} + a^{4}b^{-4} \\
\hline
1 \qquad\qquad + a^{2}b^{-2} \qquad\qquad + a^{4}b^{-4}
\end{array}$$

6.

$$\begin{array}{l}
a^{2}b^{-2} + 2 + a^{-2}b^{2} \\
\underline{a^{2}b^{-2} - 2 - a^{-2}b^{2}} \\
a^{4}b^{-4} + 2a^{2}b^{-2} + 1 \\
\qquad - 2a^{2}b^{-2} - 4 - 2a^{-2}b^{2} \\
\qquad\qquad\qquad - 1 - 2a^{-2}b^{2} - a^{-4}b^{4} \\
\hline
a^{4}b^{-4} \qquad\qquad - 4 - 4a^{-2}b^{2} - a^{-4}b^{4}
\end{array}$$

7.

$$\begin{array}{l}
4x^{-3} + 3x^{-2} + 2x^{-1} + 1 \\
\underline{\;x^{-2} - \;x^{-1} + 1} \\
4x^{-5} + 3x^{-4} + 2x^{-3} + \;x^{-2} \\
\qquad - 4x^{-4} - 3x^{-3} - 2x^{-2} - \;x^{-1} \\
\qquad\qquad + 4x^{-3} + 3x^{-2} + 2x^{-1} + 1 \\
\hline
4x^{-5} - \;x^{-4} + 3x^{-3} + 2x^{-2} + \;x^{-1} + 1
\end{array}$$

8.

$$
\begin{array}{l|l}
x^{4n} - y^{4n} & x^{n} - y^{n} \\ \cline{2-2}
x^{4n} - x^{3n}y^{n} & x^{3n} + x^{2n}y^{n} + x^{n}y^{2n} + y^{3n} \\ \cline{1-1}
\qquad x^{3n}y^{n} - y^{4n} & \\
\qquad x^{3n}y^{n} - x^{2n}y^{2n} & \\ \cline{1-1}
\qquad\qquad x^{2n}y^{2n} - y^{4n} & \\
\qquad\qquad x^{2n}y^{2n} - x^{n}y^{3n} & \\ \cline{1-1}
\qquad\qquad\qquad x^{n}y^{3n} - y^{4n} & \\
\qquad\qquad\qquad x^{n}y^{3n} - y^{4n} & \\ \cline{1-1}
\end{array}
$$

9.

$$
\begin{array}{l|l}
x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} & x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} \\ \cline{2-2}
x + x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{2}{3}}z^{\frac{1}{3}} & x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{1}{3}} + y^{\frac{2}{3}} - y^{\frac{1}{3}}z^{\frac{1}{3}} + z^{\frac{2}{3}} \\ \cline{1-1}
-x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{2}{3}}z^{\frac{1}{3}} - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + y + z & \\
-x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} & \\ \cline{1-1}
\qquad -x^{\frac{2}{3}}z^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + y + z & \\
\qquad -x^{\frac{2}{3}}z^{\frac{1}{3}} - x^{\frac{1}{3}}z^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} & \\ \cline{1-1}
\qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} + x^{\frac{1}{3}}z^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + y + z & \\
\qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} + y^{\frac{2}{3}}z^{\frac{1}{3}} + y & \\ \cline{1-1}
\qquad\qquad\qquad -x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + x^{\frac{1}{3}}z^{\frac{2}{3}} - y^{\frac{2}{3}}z^{\frac{1}{3}} + z & \\
\qquad\qquad\qquad -x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} - y^{\frac{2}{3}}z^{\frac{1}{3}} - y^{\frac{1}{3}}z^{\frac{2}{3}} & \\ \cline{1-1}
\qquad\qquad\qquad\qquad x^{\frac{1}{3}}z^{\frac{2}{3}} + y^{\frac{1}{3}}z^{\frac{2}{3}} + z & \\
\qquad\qquad\qquad\qquad x^{\frac{1}{3}}z^{\frac{2}{3}} + y^{\frac{1}{3}}z^{\frac{2}{3}} + z & \\ \cline{1-1}
\end{array}
$$

10.

$$
\begin{array}{l|l}
 & x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}} \\ \cline{2-2}
x + y & \\
x - x^{\frac{4}{5}}y^{\frac{1}{5}} + x^{\frac{3}{5}}y^{\frac{2}{5}} - x^{\frac{2}{5}}y^{\frac{3}{5}} + x^{\frac{1}{5}}y^{\frac{4}{5}} & x^{\frac{1}{5}} + y^{\frac{1}{5}} \\ \cline{1-1}
\qquad x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y & \\
\qquad x^{\frac{4}{5}}y^{\frac{1}{5}} - x^{\frac{3}{5}}y^{\frac{2}{5}} + x^{\frac{2}{5}}y^{\frac{3}{5}} - x^{\frac{1}{5}}y^{\frac{4}{5}} + y & \\ \cline{1-1}
\end{array}
$$

11.

$$
\begin{array}{l|l}
x^2y^{-2} + 2 + x^{-2}y^2 & xy^{-1} + x^{-1}y \\ \hline
x^2y^{-2} + 1 & xy^{-1} + x^{-1}y \\ \cline{1-1}
\qquad 1 + x^{-2}y^2 & \\
\qquad 1 + x^{-2}y^2 & \\ \cline{1-1}
\end{array}
$$

12.

$$
\begin{array}{l|l}
a^{-4} + a^{-2}b^{-2} + b^{-4} & a^{-2} - a^{-1}b^{-1} + b^{-2} \\ \hline
a^{-4} + a^{-2}b^{-2} - a^{-3}b^{-1} & a^{-2} + a^{-1}b^{-1} + b^{-2} \\ \cline{1-1}
\qquad a^{-3}b^{-1} + b^{-4} & \\
\qquad a^{-3}b^{-1} - a^{-2}b^{-2} + a^{-1}b^{-3} & \\ \cline{1-1}
\qquad\qquad a^{-2}b^{-2} - a^{-1}b^{-3} + b^{-4} & \\
\qquad\qquad a^{-2}b^{-2} - a^{-1}b^{-3} + b^{-4} & \\ \cline{1-1}
\end{array}
$$

13. $(4ab^{-1})^2 = 16a^2b^{-2}.$

$(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 = a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b.$

$(a + a^{-1})^2 = a^2 + 2 + a^{-2}.$

$(2a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{-\frac{1}{2}}b^{\frac{2}{3}})^2 = 4ab^{\frac{2}{3}} - 4b + a^{-1}b^{\frac{4}{3}}.$

14. $a^{\frac{1}{2}}b = 4^{\frac{1}{2}} \times 2 = 4.$

$5ab^{-1} = 5 \times 4 \times \frac{1}{2} = 10.$

$2(ab)^{\frac{1}{3}} = 2\sqrt[3]{8} = 4.$

$a^{-\frac{1}{2}}b^{-1}c^{\frac{2}{3}} = \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}.$

$12a^{-2}b^{-3} = 12 \times \frac{1}{16} \times \frac{1}{8} = \frac{3}{32}.$

15. $(a^{\frac{1}{2}} - b^{\frac{1}{3}})^3 = a^{\frac{3}{2}} - 3ab^{\frac{1}{3}} + 3a^{\frac{1}{2}}b^{\frac{2}{3}} - b.$

$$
\begin{aligned}
(2x^{-1} + x)^4 &= (2x^{-1})^4 + 4(2x^{-1})^3(x) + 6(2x^{-1})^2(x)^2 \\
&\quad + 4(2x^{-1})(x)^3 + x^4 \\
&= 16x^{-4} + 32x^{-2} + 24 + 8x^2 + x^4.
\end{aligned}
$$

$$
\begin{aligned}
(ab^{-1} - by^{-1})^6 &= a^6b^{-6} - 6(a^5b^{-5} \times by^{-1}) + 15(a^4b^{-4} \times b^2y^{-2}) \\
&\quad - 20(a^3b^{-3} \times b^3y^{-3}) + 15(a^2b^{-2} \times b^4y^{-4}) \\
&\quad - 6(ab^{-1} \times b^5y^{-5}) + b^6y^{-6} \\
&= a^6b^{-6} - 6a^5b^{-4}y^{-1} + 15a^4b^{-2}y^{-2} - 20a^3y^{-3} \\
&\quad + 15a^2b^2y^{-4} - 6ab^4y^{-5} + b^6y^{-6}.
\end{aligned}
$$

16.

$$\begin{array}{r|l}
 & 9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2 \;\underline{|\,3x^{-2} - 3x^{-1}y^{\frac{1}{2}} + y} \\
 & 9x^{-4} \\
6x^{-2} - 3x^{-1}y^{\frac{1}{2}} & -18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y \\
 & -18x^{-3}y^{\frac{1}{2}} + 9x^{-2}y \\
6x^{-2} - 6x^{-1}y^{\frac{1}{2}} + y & 6x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2 \\
 & 6x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2
\end{array}$$

17.

$$\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{|\,2x + 1 - 3x^{-1}} \\
 & 8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3} \\
 & 8x^3 \\
3(2x)^2 = 12x^2 & 12x^2 - 30x - 35 \\
(6x+1)(1) = \quad +6x+1 & \\
12x^2+6x+1 & 12x^2 + 6x + 1 \\
12x^2+12x+3 & -36x - 36 + 45x^{-1} + 27x^{-2} - 27x^{-3} \\
-18+9x^{-1}+9x^{-2} & -36x - 36 + 45x^{-1} + 27x^{-2} - 27x^{-3} \\
12x^2+12x-15+9x^{-1}+9x^{-2} &
\end{array}$$

18.

$$\sqrt[3]{12} = \sqrt[3]{3 \times 2 \times 2} = 3^{\frac{1}{3}} \times 2^{\frac{2}{3}}.$$
$$\sqrt[4]{72} = \sqrt[4]{3^2 \times 2^3} \quad = 3^{\frac{1}{2}} \times 2^{\frac{3}{4}}.$$
$$\sqrt[6]{96} = \sqrt[6]{3 \times 2^5} \quad = 3^{\frac{1}{6}} \times 2^{\frac{5}{6}}.$$
$$\sqrt[8]{64} = \sqrt[8]{2^6} \quad = 2^{\frac{3}{4}}$$
$$= 2^3 \times 3 = 24.$$

19.

$$[(x^{5ab})^3 \times (x^{5b})^{-2}]^{\frac{1}{3a-2}}$$
$$= [(x^{15ab}) \times (x^{-10b})]^{\frac{1}{3a-2}}$$
$$= (x^{15ab-10b})^{\frac{1}{3a-2}}$$
$$= x^{5b}.$$

20.

$$(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}$$
$$= (x^{18a-12})^{\frac{1}{3a-2}}$$
$$= x^6.$$

21.

$$3(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - 4(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + (a^{\frac{1}{2}} - 2b^{\frac{1}{2}})^2$$
$$= 3a + 6a^{\frac{1}{2}}b^{\frac{1}{2}} + 3b - 4a + 4b + a - 4a^{\frac{1}{2}}b^{\frac{1}{2}} + 4b$$
$$= 2a^{\frac{1}{2}}b^{\frac{1}{2}} + 11b.$$

22.

$$\{(a^m)^{m-\frac{1}{m}}\}^{\frac{1}{m+1}}$$
$$= (a^{m^2-1})^{\frac{1}{m+1}}$$
$$= a^{m-1}.$$

23.

$$\left(\frac{x^{p+q}}{x^q}\right)^p \div \left(\frac{x^q}{x^{q-p}}\right)^{p-q}$$
$$= (x^p)^p \div (x^p)^{p-q}$$
$$= x^{pq}.$$

24. $[\{(a^{-m})^{-n}\}^{p}]^{q} \div [\{(a^{m})^{n}\}^{-p}]^{-q}$
$= a^{mnpq} \div a^{mnpq}$
$= 1.$

25. $\dfrac{x^{2p(q-1)} - y^{2q(p-1)}}{x^{p(q-1)} + y^{q(p-1)}}.$

By factoring the numerator,

$$= \frac{(x^{p(q-1)} + y^{q(p-1)})(x^{p(q-1)} - y^{q(p-1)})}{x^{p(q-1)} + y^{q(p-1)}}$$
$$= x^{p(q-1)} - y^{q(p-1)}.$$

Exercise C.

1. $3\sqrt{5} = \sqrt{5 \times (3)^2} = \sqrt{45}.$
$3\sqrt{21} = \sqrt{21 \times (3)^2} = \sqrt{189}.$
$5\sqrt{32} = \sqrt{32 \times (5)^2} = \sqrt{800}.$
$a^2b\sqrt{bc} = \sqrt{bc \times (a^2b)^2} = \sqrt{a^4b^3c}.$
$x\sqrt[3]{x^2y^2} = \sqrt[3]{x^2y^2(x)^3} = \sqrt[3]{x^5y^2}.$

2. $3y^2\sqrt[4]{x^3y} = \sqrt[4]{81x^3y^9}.$
$2x\sqrt[5]{xy} = \sqrt[5]{32x^6y}.$
$a^3\sqrt[4]{a^3b^2} = \sqrt[4]{a^{15}b^2}.$
$3c^2\sqrt[3]{abc} = \sqrt[3]{27abc^7}.$
$5abc\sqrt{abc^{-1}} = \sqrt{25a^3b^3c}.$

3. $\frac{2}{7}\sqrt{\frac{91}{8}} = \sqrt{\frac{4}{49} \times \frac{91}{8}} = \sqrt{\frac{13}{14}}.$
$16\sqrt{\frac{275}{288}} = \sqrt{256 \times \frac{275}{288}} = \sqrt{244\frac{4}{9}}.$
$(x+y)\sqrt{\dfrac{xy}{x^2+2xy+y^2}} = \sqrt{x^2+2xy+y^2 \times \dfrac{xy}{x^2+2xy+y^2}} = \sqrt{xy}.$

4. $\sqrt{x^2y^4z} = xy^2\sqrt{z}.$
$\sqrt{8a^3b} = \sqrt{4a^2 \times 2ab} = 2a\sqrt{2ab}.$
$\sqrt[3]{54a^4x^2y^3} = \sqrt[3]{27a^3y^3 \times 2ax^2} = 3ay\sqrt[3]{2ax^2}.$
$\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}.$
$\sqrt{125a^4d^3} = \sqrt{25a^4d^2 \times 5d} = 5a^2d\sqrt{5d}.$

5. $\sqrt[3]{1000a} = \sqrt[3]{10 \times 10 \times 10 \times a} = 10\sqrt[3]{a}.$
$\sqrt[3]{160x^4y^7} = \sqrt[3]{20xy \times 8x^3y^6} = 2xy^2\sqrt[3]{20xy}.$
$\sqrt[3]{108m^9n^{10}} = \sqrt[3]{27m^9n^9 \times 4n} = 3m^3n^3\sqrt[3]{4n}.$
$\sqrt[3]{1372a^{15}b^{16}} = \sqrt[3]{343a^{15}b^{15} \times 4b} = 7a^5b^5\sqrt[3]{4b}.$

6. $\sqrt[3]{a^4 - 3a^3b + 3a^2b^2 - ab^3} = \sqrt[3]{a(a^3 - 3a^2b + 3ab^2 - b^3)} = (a-b)\sqrt[3]{a}.$
$\sqrt{50a^2 - 100ab + 50b^2} = \sqrt{2 \times 25(a^2 - 2ab + b^2)} = 5(a-b)\sqrt{2}.$

7. $2\sqrt[4]{80a^5b^2c^6} = 2\sqrt[4]{16a^4c^4\times 5ab^2c^2} = 4ac\sqrt[4]{5ab^2c^2}.$

$7\sqrt{396x} = 7\sqrt{36\times 11x} = 42\sqrt{11x}.$

$9\sqrt[3]{81x^2y^3z} = 9\sqrt[3]{27y^3\times 3x^2z} = 27y\sqrt[3]{3x^2z}.$

$5\sqrt{726} = 5\sqrt{121\times 6} = 55\sqrt{6}.$

8. $\sqrt{\frac{5}{9}} = \sqrt{5\times\frac{1}{9}} = \frac{1}{3}\sqrt{5}.$

$\sqrt{1\frac{11}{16}} = \sqrt{\frac{27}{16}} = \sqrt{3\times\frac{9}{16}} = \frac{3}{4}\sqrt{3}.$

$\sqrt{3\frac{1}{8}} = \sqrt{\frac{25}{8}} = \sqrt{\frac{25\times 2}{16}} = \frac{5}{4}\sqrt{2}.$

$\frac{3}{5}\sqrt{90\frac{5}{8}} = \frac{3}{5}\sqrt{\frac{725}{8}} = \frac{3}{5}\sqrt{\frac{2\times 5^2\times 29}{2^4}} = \frac{3}{4}\sqrt{58}.$

$2\sqrt[3]{\frac{1}{2}} = \sqrt[3]{8\times\frac{1}{2}} = \sqrt[3]{4}.$

9. $\sqrt[3]{\frac{2xy^2}{z}} = \sqrt[3]{\frac{2xy^2z^2}{z^3}} = \frac{1}{z}\sqrt[3]{2xy^2z^2}.$

$\sqrt[3]{\frac{4}{25}} = \sqrt[3]{\frac{20}{125}} = \frac{1}{5}\sqrt[3]{20}.$

$\frac{a}{b}\sqrt[4]{\frac{b}{2a^3}} = \frac{a}{b}\sqrt[4]{\frac{8ab}{16a^4}} = \frac{1}{2b}\sqrt[4]{8ab}.$

$\sqrt{\frac{3a^2bx}{4cy^3}} = \sqrt{\frac{3a^2bcxy}{4c^2y^4}} = \frac{a}{2cy^2}\sqrt{3bcxy}.$

10. $\frac{12}{\sqrt{5}} = \frac{12\sqrt{5}}{5} = 2\frac{2}{5}\sqrt{5}.$

$\frac{2}{\sqrt{1701}} = \sqrt{\frac{4}{3^5\times 7}} = \sqrt{\frac{4\times 21}{3^6\times 7^2}} = \frac{2}{189}\sqrt{21}.$

$\left(\frac{x^3y^2}{z^2}\right)\left(\frac{z^5}{x^5y^5}\right)^{\frac{1}{2}} = \left(\frac{x^6y^4z^5}{x^5y^5z^4}\right)^{\frac{1}{2}} = \left(\frac{xz}{y}\right)^{\frac{1}{2}}.$

$\left(\frac{a^3b^2}{c^4}\right)\left(\frac{c^9b^3}{a}\right)^{\frac{1}{3}} = \left(\frac{a^9c^9b^9}{ac^{12}}\right)^{\frac{1}{3}} = \left(\frac{a^8b^9}{c^3}\right)^{\frac{1}{3}} = \frac{a^2b^3}{c}\sqrt[3]{a^2}.$

11. $(ax)\times(b^2x)^{\frac{1}{2}} = (a^2b^2x^3)^{\frac{1}{2}} = abx\sqrt{x}.$

$(2a^2b^4)\times(b^2x^3)^{\frac{1}{3}} = 2(a^6b^{14}x^3)^{\frac{1}{3}} = 2a^2b^4x\sqrt[3]{b^2}.$

$5(3a^3b^4y)\times(a^5b^{-4}y^3)^{\frac{1}{4}} = 15a^4b^3y\sqrt[4]{ay^3}.$

12. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.

$$\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}.$$
$$\sqrt{45} = \sqrt{5 \times 9} = 3\sqrt{5}.$$
$$\sqrt{\tfrac{4}{5}} = \sqrt{\tfrac{20}{25}} = \sqrt{\tfrac{4}{25} \times 5} = \tfrac{2}{5}\sqrt{5}.$$

Therefore, they are similar surds because they have the same surd factor.

13. Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\frac{a^6}{b}}$ are similar surds.

$$2\sqrt[3]{a^3b^2} = 2a\sqrt[3]{b^2}.$$
$$\sqrt[3]{8b^5} = 2b\sqrt[3]{b^2}.$$
$$\tfrac{1}{2}\sqrt[3]{\frac{a^6}{b}} = \tfrac{1}{2}\sqrt[3]{\frac{a^6b^2}{b^3}} = \frac{a^2}{2b}\sqrt[3]{b^2}.$$

Since they all have the same surd factor, they are similar surds.

14. If $\sqrt{2} = 1.414213$, find the values of

$$\sqrt{50};\ \tfrac{5}{2}\sqrt{288};\ \frac{1}{\sqrt{2}};\ \frac{3}{\sqrt{450}}.$$

$$\begin{aligned}\sqrt{50} &= 5\sqrt{2}\\ &= 5 \times 1.414213\\ &= 7.071065.\end{aligned}$$

$$\begin{aligned}\tfrac{5}{2}\sqrt{288} &= \tfrac{5}{2}\sqrt{144 \times 2}\\ &= 30\sqrt{2}\\ &= 30 \times 1.414213\\ &= 42.426390.\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{2}} &= \sqrt{\tfrac{1}{2}}\\ &= \tfrac{1}{2}\sqrt{2}\\ &= \frac{1.414213}{2}\\ &= 0.707107.\end{aligned}$$

$$\begin{aligned}\frac{3}{\sqrt{450}} &= \sqrt{\tfrac{9}{450}}\\ &= \sqrt{\tfrac{1}{50}}\\ &= \sqrt{\tfrac{1}{100} \times 2}\\ &= \tfrac{1}{10}\sqrt{2}\\ &= \tfrac{1}{10} \times 1.414213\\ &= 0.1414213.\end{aligned}$$

EXERCISE CI.

1. Which is the greater,

$$3\sqrt{7} \text{ or } 2\sqrt{15}?$$

$3\sqrt{7} = \sqrt{9 \times 7} = \sqrt{63}.$

$2\sqrt{15} = \sqrt{4 \times 15} = \sqrt{60}.$

Since 63 is > 60,

$\therefore 3\sqrt{7} > 2\sqrt{15}.$

2. Arrange in order of magnitude

$$9\sqrt{3},\ 6\sqrt{7},\ 5\sqrt{10}.$$

$9\sqrt{3} = \sqrt{243}.$

$6\sqrt{7} = \sqrt{252}.$

$5\sqrt{10} = \sqrt{250}.$

Since $\sqrt{252} > \sqrt{250} > \sqrt{243}$,

$\therefore 6\sqrt{7} > 5\sqrt{10} > 9\sqrt{3}.$

3. Arrange in order of magnitude

$$4\sqrt[3]{4},\ 3\sqrt[3]{5},\ 5\sqrt[3]{3}.$$

$4\sqrt[3]{4} = \sqrt[3]{256}.$

$3\sqrt[3]{5} = \sqrt[3]{135}.$

$5\sqrt[3]{3} = \sqrt[3]{375}.$

Since $\sqrt[3]{135} < \sqrt[3]{256} < \sqrt[3]{375}$,

$\therefore 3\sqrt[3]{5} < 4\sqrt[3]{4} < 5\sqrt[3]{3}.$

4. $3\sqrt{2} \times 4\sqrt{6} = 12\sqrt{12}$

$= 24\sqrt{3}.$

$\frac{2}{7}\sqrt{10} \times \frac{7}{10}\sqrt{15} = \frac{1}{5}\sqrt{150}$

$= \sqrt{6}.$

5. $5\sqrt{\frac{2}{7}} \times \frac{3}{7}\sqrt{162}.$

$5\sqrt{\frac{2}{7}} = \frac{5}{7}\sqrt{14};$

$\frac{3}{7}\sqrt{162} = \frac{27}{7}\sqrt{2}.$

$\frac{5}{7}\sqrt{14} \times \frac{27}{7}\sqrt{2} = \frac{135}{49}\sqrt{28}$

$= \frac{270}{49}\sqrt{7}.$

$\frac{1}{2}\sqrt[3]{4} \times 2\sqrt[3]{2} = \sqrt[3]{8} = 2.$

6. $2\sqrt{5} \div 3\sqrt{15} = \frac{2}{3}\sqrt{\frac{1}{3}}$

$= \frac{2}{9}\sqrt{3}.$

$\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}} = \frac{2}{3}\sqrt{60}$

$= 1\frac{1}{3}\sqrt{15}.$

7. $\frac{2}{5}\sqrt{3} \times \frac{4}{9}\sqrt{5} \div \frac{6}{7}\sqrt{2}$

$= \frac{8}{45}\sqrt{15} \div \frac{6}{7}\sqrt{2}$

$= \frac{28}{135}\sqrt{\frac{15}{2}}$

$= \frac{28}{135}\sqrt{\frac{30}{4}}$

$= \frac{14}{135}\sqrt{30}.$

8. $\dfrac{2\sqrt{10}}{3\sqrt{27}} \times \dfrac{7\sqrt{48}}{5\sqrt{14}} \div \dfrac{4\sqrt{15}}{15\sqrt{21}}$

$= (\frac{2}{3} \times \frac{7}{5} \times \frac{15}{4})\sqrt{\frac{10}{27} \times \frac{48}{14} \times \frac{21}{15}}$

$= \frac{7}{2}\sqrt{\frac{16}{9}} = \frac{28}{6} = 4\frac{2}{3}.$

9. $2\sqrt[3]{4} \times 5\sqrt[3]{32} \div \sqrt[3]{108}.$

$2\sqrt[3]{4} \times 5\sqrt[3]{32} = 10\sqrt[3]{128}.$

$10\sqrt[3]{128} \div \sqrt[3]{108} = 10\sqrt[3]{\frac{128}{108}}$

$= 10\sqrt[3]{\frac{32}{27}}$

$= 10\sqrt[3]{\frac{8}{27} \times 4}$

$= 6\frac{2}{3}\sqrt[3]{4}.$

EXERCISE CII.

3

1. $2\sqrt[3]{3} = 2(3)^{\frac{1}{3}} = 2(3)^{\frac{2}{6}} = \sqrt[6]{576}$.
$3\sqrt{2} = 3(2)^{\frac{1}{2}} = 3(2)^{\frac{3}{6}} = \sqrt[6]{5832}$.
$\frac{5}{2}\sqrt[4]{4} = \frac{5}{2}(2)^{\frac{1}{2}} = \frac{5}{2}(2)^{\frac{3}{6}} = \sqrt[6]{1953\frac{1}{8}}$.
$\therefore$ the order of magnitude is $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{4}$, $2\sqrt[3]{3}$.

2. $\sqrt{\frac{3}{5}} = (\frac{3}{5})^{\frac{1}{2}} = (\frac{3}{5})^{\frac{3}{6}} = \sqrt[6]{\frac{27}{125}}$.
$\sqrt[3]{\frac{14}{15}} = (\frac{14}{15})^{\frac{1}{3}} = (\frac{14}{15})^{\frac{2}{6}} = \sqrt[6]{\frac{196}{225}}$.
$\sqrt[6]{\frac{27}{125}} = \sqrt[6]{\frac{243}{1125}}$.
$\sqrt[6]{\frac{196}{225}} = \sqrt[6]{\frac{980}{1125}}$.
$\therefore\ \sqrt[3]{\frac{14}{15}} > \sqrt{\frac{3}{5}}$.

3. $2\sqrt[3]{22} = \sqrt[3]{176} = \sqrt[6]{176^2} = \sqrt[6]{30976}$.
$3\sqrt[3]{7} = \sqrt[3]{189} = \sqrt[6]{189^2} = \sqrt[6]{35721}$.
$4\sqrt{2} = \sqrt{32} = \sqrt[6]{32^3} = \sqrt[6]{32768}$.
$\therefore$ the order of magnitude is $3\sqrt[3]{7}$, $4\sqrt{2}$, $2\sqrt[3]{22}$.

4. $3\sqrt{19} = \sqrt{171} = 171^{\frac{1}{2}} = 171^{\frac{3}{6}} = \sqrt[6]{171^3} = \sqrt[6]{5000211}$.
$5\sqrt[3]{2} = \sqrt[3]{250} = 250^{\frac{1}{3}} = 250^{\frac{2}{6}} = \sqrt[6]{250^2} = \sqrt[6]{62500}$.
$3\sqrt[3]{3} = \sqrt[3]{81} = 81^{\frac{1}{3}} = 81^{\frac{2}{6}} = \sqrt[6]{81^2} = \sqrt[6]{6561}$.
$\therefore$ the order of magnitude is $3\sqrt{19}$, $5\sqrt[3]{2}$, $3\sqrt[3]{3}$.

5. $2\sqrt{ax}\times\sqrt[3]{3a^2b}\times\sqrt{2bx}$; $\sqrt[4]{a^3xy^3}\times\sqrt[5]{a^2xy}$.
$2\sqrt{ax} = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$.
$\sqrt[3]{3a^2b} = 3^{\frac{1}{3}}a^{\frac{2}{3}}b^{\frac{1}{3}}$.
$\sqrt{2bx} = 2^{\frac{1}{2}}b^{\frac{1}{2}}x^{\frac{1}{2}}$.
$\sqrt[6]{2^6\times 3^2\times 2^3}\times a^{\frac{7}{6}}\times b^{\frac{5}{6}}\times x = 2ax\sqrt[6]{72ab^5}$.

$\sqrt[4]{a^3xy^3} = (a^3xy^3)^{\frac{1}{4}} = (a^3xy^3)^{\frac{5}{20}} = \sqrt[20]{a^{15}x^5y^{15}}$.
$\sqrt[5]{a^2xy} = (a^2xy)^{\frac{1}{5}} = (a^2xy)^{\frac{4}{20}} = \sqrt[20]{a^8x^4y^4}$.
$\sqrt[20]{a^{15}x^5y^{15}}\times\sqrt[20]{a^8x^4y^4} = \sqrt[20]{a^{23}\times x^9\times y^{19}} = a\sqrt[20]{a^3x^9y^{19}}$.

6. $3(4ab^2)^{\frac{1}{3}} \div (2a^3b)^{\frac{1}{2}}.$

$3(4ab^2)^{\frac{1}{3}} = 3(4ab^2)^{\frac{2}{6}} = 3\sqrt[6]{16a^2b^4}.$

$(2a^3b)^{\frac{1}{2}} = (2a^3b)^{\frac{3}{6}} = \sqrt[6]{8a^9b^3} = a\sqrt[6]{8a^3b^3}.$

$3\sqrt[6]{16a^2b^4} \div a\sqrt[6]{8a^3b^3} = \frac{3}{a}\sqrt[6]{2a^{-1}b} = \frac{3}{a^2}\sqrt[6]{2a^5b}.$

$(2a^3b^2)^{\frac{1}{4}} \times (a^5b^3)^{\frac{1}{3}} \div (a^3b^5)^{\frac{1}{2}}.$

$(2^{\frac{1}{4}}a^{\frac{3}{4}}b^{\frac{1}{2}}) \times (a^{\frac{5}{3}}b) = 2^{\frac{1}{4}}a^{\frac{29}{12}}b^{\frac{3}{2}}.$

$2^{\frac{1}{4}}a^{\frac{29}{12}}b^{\frac{3}{2}} \div a^{\frac{3}{2}}b^{\frac{5}{2}} = 2^{\frac{1}{4}}a^{\frac{11}{12}}b^{-1} = \frac{1}{b}\sqrt[12]{8a^{11}}.$

7. $(2ab)^{\frac{1}{2}} \times (3ab^2)^{\frac{1}{3}} \div (5ab^3)^{\frac{1}{6}}$

$= (2ab)^{\frac{3}{6}} \times (3ab^2)^{\frac{2}{6}} \div (5ab^3)^{\frac{1}{6}}$

$= \sqrt[6]{(2ab)^3} \times \sqrt[6]{(3ab^2)^2} \div \sqrt[6]{5ab^3}$

$= \sqrt[6]{2^3a^3b^3} \times \sqrt[6]{3^2a^2b^4} \div \sqrt[6]{5ab^3}$

$= \sqrt[6]{2^3 \times 3^2 \times a^5 \times b^7} \div \sqrt[6]{5ab^3}$

$= \sqrt[6]{\frac{2^3 \times 3^2a^5b^7}{5ab^3}}$

$= \sqrt[6]{\frac{2^3 \times 3^2 \times 5^5a^4b^4}{5^6}}$

$= \frac{1}{5}\sqrt[6]{225000\,a^4b^4}.$

$4\sqrt{12} \div 2\sqrt{3}$

$= 2\sqrt{4} = 2 \times 2 = 4.$

8. $\left(\frac{ay}{x}\right)^{\frac{1}{2}} \times \left(\frac{bx}{y^2}\right)^{\frac{1}{3}} \div \left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$

$= \frac{a^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \times \frac{b^{\frac{1}{3}}x^{\frac{1}{3}}}{y^{\frac{2}{3}}} \times \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{y^{\frac{1}{2}}}$

$= \frac{ab^{\frac{5}{6}}}{x^{\frac{1}{6}}y^{\frac{2}{3}}}$

$= \frac{ab^{\frac{5}{6}}x^{\frac{5}{6}}y^{\frac{1}{3}}}{xy}$ 3

$= \frac{a}{xy}\sqrt[6]{b^5x^5y^2}.$

9. $(7\sqrt{2} - 5\sqrt{6} - 3\sqrt{8} + 4\sqrt{20}) \times 3\sqrt{2}.$

$$\begin{array}{l} 7\sqrt{2} - 5\sqrt{6} - 3\sqrt{8} + 4\sqrt{20} \\ 3\sqrt{2} \\ \hline 42 - 30\sqrt{3} - 36 + 24\sqrt{10} \end{array}$$

$= 6 - 30\sqrt{3} + 24\sqrt{10}.$

10. $\sqrt{(\frac{16}{25})^7} \times \sqrt{(\frac{25}{64})^6}$

$= (\frac{4}{5})^7 \times (\frac{5}{8})^6$

$= \left(\frac{2^2}{5}\right)^7 \times \left(\frac{5}{2^3}\right)^6$

$= \frac{1}{80}.$

$\sqrt[3]{(4ab^2)^x} \times \sqrt[3]{(2a^2b)^x}$

$= \sqrt[3]{(8a^3b^3)^x}$

$= (2ab)^x.$

11. $a^{\frac{9}{7}}b^{\frac{3}{7}} \times a^{\frac{12}{7}}b^{\frac{48}{7}}$

$= a^3b^7\sqrt[7]{b^2}.$

$\frac{a^{\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{3}{4}}d^{-\frac{5}{6}}}{a^{\frac{7}{8}}b^{-\frac{9}{10}}c^{-\frac{10}{11}}d^{\frac{11}{12}}}$

$= a^{-\frac{3}{8}}b^{\frac{7}{30}}c^{\frac{73}{44}}d^{-\frac{7}{4}}.$

Exercise CIII.

1. $$\begin{aligned} \sqrt{27} &= \sqrt{9\times 3} = 3\sqrt{3} \\ 2\sqrt{48} &= 2\sqrt{16\times 3} = 8\sqrt{3} \\ 3\sqrt{108} &= 3\sqrt{36\times 3} = 18\sqrt{3} \\ \hline \text{Sum} & \quad\quad = 29\sqrt{3} \end{aligned}$$

$$\begin{aligned} 3\sqrt{1000} &= 3\sqrt{100\times 10} = 30\sqrt{10} \\ 4\sqrt{50} &= 4\sqrt{25\times 2} = 20\sqrt{2} \\ 12\sqrt{288} &= 12\sqrt{144\times 2} = 144\sqrt{2} \\ \hline \text{Sum} &= 164\sqrt{2} + 30\sqrt{10} \end{aligned}$$

2. $$\begin{aligned} \sqrt[3]{128} &= \sqrt[3]{64\times 2} = 4\sqrt[3]{2} \\ \sqrt[3]{686} &= \sqrt[3]{343\times 2} = 7\sqrt[3]{2} \\ \sqrt[3]{16} &= \sqrt[3]{8\times 2} = 2\sqrt[3]{2} \\ \hline \text{Sum} & \quad\quad = 13\sqrt[3]{2} \end{aligned}$$

$$\begin{aligned} 7\sqrt[3]{54} &= 7\sqrt[3]{27\times 2} = 21\sqrt[3]{2} \\ 3\sqrt[3]{16} &= 3\sqrt[3]{8\times 2} = 6\sqrt[3]{2} \\ \sqrt[3]{432} &= \sqrt[3]{216\times 2} = 6\sqrt[3]{2} \\ \hline \text{Sum} & \quad\quad = 33\sqrt[3]{2} \end{aligned}$$

3. $$\begin{aligned} 12\sqrt{72} &= 12(2^3\times 3^2)^{\frac{1}{2}} = 72\sqrt{2} \\ 3\sqrt{128} &= 3(2^2\times 4^2\times 2)^{\frac{1}{2}} = 24\sqrt{2} \\ \hline \text{Difference} & \quad\quad = 48\sqrt{2} \end{aligned}$$

$$\begin{aligned} 7\sqrt[3]{81} &= 7(3^4)^{\frac{1}{3}} = 21\sqrt[3]{3} \\ 3\sqrt[3]{1029} &= 3(3\times 7^3)^{\frac{1}{3}} = 21\sqrt[3]{3} \\ \hline \text{Difference} & \quad\quad = 0 \end{aligned}$$

4. $$\begin{aligned} & 2\sqrt{3} + 3\sqrt{1\tfrac{1}{3}} - \sqrt{5\tfrac{1}{3}} \\ &= 2\sqrt{3} + 3\sqrt{\tfrac{4}{3}} - \sqrt{\tfrac{16}{3}} \\ &= 2\sqrt{3} + \tfrac{6}{3}\sqrt{3} - \tfrac{4}{3}\sqrt{3} \\ &= \tfrac{8}{3}\sqrt{3}. \end{aligned}$$

$$\begin{aligned} & 2\sqrt{\tfrac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\tfrac{3}{5}} \\ &= \tfrac{2}{3}\sqrt{15} + 2\sqrt{15} - \sqrt{15} - \tfrac{1}{5}\sqrt{15} \\ &= \tfrac{22}{15}\sqrt{15}. \end{aligned}$$

5. $$\sqrt{\frac{a^4c}{b^3}} = \frac{a^2}{b^2}\sqrt{bc}.$$

$$\sqrt{\frac{a^2c^3}{bd^2}} = \frac{ac}{bd}\sqrt{bc}.$$

$$\sqrt{\frac{a^2cd^2}{bm^2}} = \frac{ad}{bm}\sqrt{bc}.$$

$$\therefore \sqrt{\frac{a^4c}{b^3}} - \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{bm^2}} = \left(\frac{a^2}{b^2} - \frac{ac}{bd} - \frac{ad}{bm}\right)\sqrt{bc}.$$

$$\begin{aligned} & 3\sqrt{\tfrac{2}{5}} + 2\sqrt{\tfrac{1}{10}} - 4\sqrt{\tfrac{1}{40}} \\ &= \tfrac{3}{5}\sqrt{10} + \tfrac{1}{5}\sqrt{10} - \tfrac{1}{5}\sqrt{10} \\ &= \tfrac{3}{5}\sqrt{10}. \end{aligned}$$

6. $\sqrt{4a^3b} + \sqrt{25ab^3} - (a-5b)\sqrt{ab}$
$= 2a\sqrt{ab} + 5b\sqrt{ab} - (a-5b)\sqrt{ab}$
$= (a+10b)\sqrt{ab}.$

7. $c\sqrt[5]{a^6b^7c^3} - a\sqrt[5]{ab^7c^8} + b\sqrt[5]{a^6b^2c^8}$
$= abc\sqrt[5]{ab^2c^3} - abc\sqrt[5]{ab^2c^3} + abc\sqrt[5]{ab^2c^3}$
$= abc\sqrt[5]{ab^2c^3}.$

8. $2\sqrt[3]{40} = 4\sqrt[3]{5}.$
$3\sqrt[3]{108} = 3\sqrt[3]{3^3 \times 2^2} = 9\sqrt[3]{4}.$
$\sqrt[3]{500} = \sqrt[3]{5^3 \times 2^2} = 5\sqrt[3]{4}.$
$-\sqrt[3]{320} = -\sqrt[3]{2^6 \times 5} = -4\sqrt[3]{5}.$
$-2\sqrt[3]{1372} = -2\sqrt[3]{2^2 \times 7^3} = -14\sqrt[3]{4}.$
And $4\sqrt[3]{5} + 9\sqrt[3]{4} + 5\sqrt[3]{4} - 4\sqrt[3]{5} - 14\sqrt[3]{4} = 0.$

9. $(2\sqrt[6]{3a^4b})^3 = [2(3a^4b)^{\frac{1}{6}}]^3$
$= 2^3(3a^4b)^{\frac{1}{2}} = 8a^2\sqrt{3b}.$

$(3\sqrt[6]{3})^2 = [3(3)^{\frac{1}{6}}]^2$
$= 3^2(3)^{\frac{1}{3}} = 9\sqrt[3]{3}.$

10. $\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)^{\frac{1}{3}} = \left(\frac{a}{3}\right)^{\frac{1}{3}}\left(\frac{a}{3}\right)^{\frac{1}{6}}$
$= \left(\frac{a}{3}\right)^{\frac{1}{2}} = \left(\frac{3a}{9}\right)^{\frac{1}{2}} = \frac{1}{3}\sqrt{3a}.$
$(\sqrt{27})^{\frac{1}{2}} = (27^{\frac{1}{2}})^{\frac{1}{2}} = 27^{\frac{1}{4}} = \sqrt[4]{27}.$

11. $(\sqrt[3]{81})^{\frac{1}{4}} = (\sqrt[3]{3^4})^{\frac{1}{4}} = \sqrt[3]{3}.$
$(\sqrt[4]{512})^{\frac{1}{3}} = (\sqrt[4]{2^9})^{\frac{1}{3}} = 2^{\frac{3}{4}} = \sqrt[4]{8}.$
$(\sqrt[3]{256})^{-\frac{1}{4}} = 256^{-\frac{1}{12}} = \left(\frac{1}{2^8}\right)^{\frac{1}{12}} = \left(\frac{2^4}{2^{12}}\right)^{\frac{1}{12}} = \frac{1}{2}\sqrt[3]{2}.$
$\sqrt[12]{16} = (2^4)^{\frac{1}{12}} = \sqrt[3]{2}.$
$\sqrt[12]{27} = (3^3)^{\frac{1}{12}} = \sqrt[4]{3}.$

12. $\sqrt[10]{4} = (2^2)^{\frac{1}{10}} = \sqrt[5]{2}.$
$\sqrt[10]{36} = (6^2)^{\frac{1}{10}} = \sqrt[5]{6}.$
$\sqrt[10]{32} = (2^5)^{\frac{1}{10}} = \sqrt{2}.$
$\sqrt[10]{243} = (3^5)^{\frac{1}{10}} = \sqrt{3}.$
$\sqrt[9]{125} = (5^3)^{\frac{1}{9}} = \sqrt[3]{5}.$
$\sqrt[4]{49} = (7^2)^{\frac{1}{4}} = \sqrt{7}.$

13. $\sqrt[9]{8x^6} = (2^3x^6)^{\frac{1}{9}} = \sqrt[3]{2x^2}.$
$\sqrt[6]{9a^2b^4} = (3^2a^2b^4)^{\frac{1}{6}} = \sqrt[3]{3ab^2}.$
$\sqrt[8]{16a^{12}} = \sqrt[8]{2^4a^{12}} = 2^{\frac{1}{2}}a^{\frac{3}{2}}$
$= a\sqrt{2a}.$
$\sqrt[5]{32a^{10}} = (2^5a^{10})^{\frac{1}{5}} = 2a^2.$

14. $(\sqrt[9]{8})^4 = (\sqrt[9]{2^3})^4 = 2^{\frac{4}{3}} = 2\sqrt[3]{2}.$

$(\sqrt[9]{27})^4 = (\sqrt[9]{3^3})^4 = 3^{\frac{4}{3}} = 3\sqrt[3]{3}.$

$(\sqrt[8]{64})^3 = (\sqrt[8]{2^6})^3 = 2^{\frac{9}{4}} = 4\sqrt[4]{2}.$

$(\sqrt[3]{4})^2 = (\sqrt[3]{2^2})^2 = 2^{\frac{4}{3}} = 2\sqrt[3]{2}.$

15. $(a\sqrt[3]{a})^{-3} = (a^{\frac{4}{3}})^{-3} = a^{-4}.$

$(x\sqrt[3]{x})^{-\frac{1}{2}} = (x^{\frac{4}{3}})^{-\frac{1}{2}} = x^{-\frac{2}{3}}.$

$(p^2\sqrt{p})^{\frac{1}{5}} = (\sqrt{p^5})^{\frac{1}{5}} = (p^{\frac{5}{2}})^{\frac{1}{5}} = p^{\frac{1}{2}}.$

$(a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{4}} = (\sqrt[4]{a^{-15}})^{-\frac{1}{4}} = (a^{-\frac{15}{4}})^{-\frac{1}{4}} = a^{\frac{15}{16}}.$

16. $(\sqrt{a}+\sqrt{b})^5 = (a^{\frac{1}{2}}+b^{\frac{1}{2}})^5$

$= a^{\frac{5}{2}} + 5a^2b^{\frac{1}{2}} + 10a^{\frac{3}{2}}b + 10ab^{\frac{3}{2}} + 5a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$

$= a^2\sqrt{a} + 5a^2\sqrt{b} + 10ab\sqrt{a} + 10ab\sqrt{b} + 5b^2\sqrt{a} + b^2\sqrt{b}.$

$(\sqrt[3]{m^2}+\sqrt{x^3})^3 = (m^{\frac{2}{3}}+x^{\frac{3}{2}})^3$

$= (m^{\frac{2}{3}})^3 + 3(m^{\frac{2}{3}})^2x^{\frac{3}{2}} + 3m^{\frac{2}{3}}(x^{\frac{3}{2}})^2 + (x^{\frac{3}{2}})^3$

$= m^2 + 3mx\sqrt[6]{m^2x^3} + 3x^3\sqrt[3]{m^2} + x^4\sqrt{x}.$

$(\sqrt{a}-2\sqrt{b})^5 = (a^{\frac{1}{2}}-2b^{\frac{1}{2}})^5$

$= (a^{\frac{1}{2}})^5 - 5(a^{\frac{1}{2}})^4(2b^{\frac{1}{2}}) + 10(a^{\frac{1}{2}})^3(2b^{\frac{1}{2}})^2 - 10(a^{\frac{1}{2}})^2(2b^{\frac{1}{2}})^3$

$+ 5(a^{\frac{1}{2}})(2b^{\frac{1}{2}})^4 - (2b^{\frac{1}{2}})^5$

$= a^2\sqrt{a} - 10a^2\sqrt{b} + 40ab\sqrt{a} - 80ab\sqrt{b} + 80b^2\sqrt{a} - 32b^2\sqrt{b}.$

17. $(2a^2-\frac{1}{2}\sqrt{a})^6$

$= (2a^2)^6 - 6(2a^2)^5(\frac{1}{2}\sqrt{a}) + 15(2a^2)^4(\frac{1}{2}\sqrt{a})^2 - 20(2a^2)^3(\frac{1}{2}\sqrt{a})^3$

$+ 15(2a^2)^2(\frac{1}{2}\sqrt{a})^4 - 6(2a^2)(\frac{1}{2}\sqrt{a})^5 + (\frac{1}{2}\sqrt{a})^6$

$= 64a^{12} - 96a^{10}\sqrt{a} + 60a^9 - 20a^7\sqrt{a} + 3\frac{3}{4}a^6 - \frac{3}{8}a^4\sqrt{a} + \frac{a^3}{64}.$

$(2\sqrt[5]{x^4}-\frac{1}{2}y^2)^4$

$= (2x^{\frac{4}{5}})^4 - 4(2x^{\frac{4}{5}})^3\left(\frac{y^2}{2}\right) + 6(2x^{\frac{4}{5}})^2\left(\frac{y^2}{2}\right)^2 - 4(2x^{\frac{4}{5}})\left(\frac{y^2}{2}\right)^3 + \left(\frac{y^2}{2}\right)^4$

$= 16x^3\sqrt[5]{x} - 16x^2y^2\sqrt[5]{x^2} + 6xy^4\sqrt[5]{x^3} - y^6\sqrt[5]{x^4} + \frac{y^8}{16}.$

$$\left(\frac{2x^2}{y}-\sqrt[3]{y^2}\right)^6=(2x^2y^{-1}-y^{\frac{2}{3}})^6$$

$$=(2x^2y^{-1})^6-6(2x^2y^{-1})^5(y^{\frac{2}{3}})+15(2x^2y^{-1})^4(y^{\frac{2}{3}})^2-20(2x^2y^{-1})^3(y^{\frac{2}{3}})^3$$
$$+15(2x^2y^{-1})^2(y^{\frac{2}{3}})^4-6(2x^2y^{-1})(y^{\frac{2}{3}})^5+(y^{\frac{2}{3}})^6$$

$$=64x^{12}y^{-6}-192x^{10}y^{-4}\sqrt[3]{y^{-1}}+240x^8y^{-2}\sqrt[3]{y^{-2}}-160x^6y^{-1}$$
$$+60x^4\sqrt[3]{y^2}-12x^2y^2\sqrt[3]{y}+y^4.$$

18. $$\left(\sqrt{ab}-\frac{c}{2\sqrt{b}}\right)^5$$

$$=(\sqrt{ab})^5-5(\sqrt{ab})^4\left(\frac{c}{2\sqrt{b}}\right)+10(\sqrt{ab})^3\left(\frac{c}{2\sqrt{b}}\right)^2$$
$$-10(\sqrt{ab})^2\left(\frac{c}{2\sqrt{b}}\right)^3+5(\sqrt{ab})\left(\frac{c}{2\sqrt{b}}\right)^4-\left(\frac{c}{2\sqrt{b}}\right)^5$$

$$=a^{\frac{5}{2}}b^{\frac{5}{2}}-\tfrac{5}{2}a^2b^{\frac{3}{2}}c+\tfrac{5}{2}a^{\frac{3}{2}}b^{\frac{1}{2}}c^2-\tfrac{5}{4}ab^{-\frac{1}{2}}c^3+\tfrac{5}{16}a^{\frac{1}{2}}b^{-\frac{3}{2}}c^4-\tfrac{1}{32}b^{-\frac{5}{2}}c^5.$$

$$\left(\frac{a^2}{2c}-\frac{\sqrt{c}}{3}\right)^5$$

$$=\left(\frac{a^2}{2c}\right)^5-5\left(\frac{a^2}{2c}\right)^4\left(\frac{\sqrt{c}}{3}\right)+10\left(\frac{a^2}{2c}\right)^3\left(\frac{\sqrt{c}}{3}\right)^2-10\left(\frac{a^2}{2c}\right)^2\left(\frac{\sqrt{c}}{3}\right)^3$$
$$+5\left(\frac{a^2}{2c}\right)\left(\frac{\sqrt{c}}{3}\right)^4-\left(\frac{\sqrt{c}}{3}\right)^5$$

$$=\frac{a^{10}c^{-5}}{32}-\tfrac{5}{48}a^8c^{-\frac{7}{2}}+\tfrac{5}{36}a^6c^{-2}-\tfrac{5}{54}a^4c^{-\frac{1}{2}}+\tfrac{5}{162}a^2c-\frac{c^{\frac{5}{2}}}{243}.$$

$$\left(a^2b-\frac{\sqrt{b}}{2a}\right)^4=\left(a^2b-\frac{b^{\frac{1}{2}}}{2a}\right)^4$$

$$=(a^2b)^4-4(a^2b)^3\left(\frac{b^{\frac{1}{2}}}{2a}\right)+6(a^2b)^2\left(\frac{b^{\frac{1}{2}}}{2a}\right)^2-4(a^2b)\left(\frac{b^{\frac{1}{2}}}{2a}\right)^3+\left(\frac{b^{\frac{1}{2}}}{2a}\right)^4$$

$$=a^8b^4-2a^5b^{\frac{7}{2}}+\tfrac{3}{2}a^2b^3-\frac{a^{-1}b^{\frac{5}{2}}}{2}+\frac{a^{-4}b^2}{16}.$$

19. $$\left(\frac{a}{b}\sqrt{\frac{c}{d}}-\sqrt{\frac{d^2}{c^2}}\right)^3$$

$$=\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^3-3\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)^2\left(\sqrt{\frac{d^2}{c^2}}\right)+3\left(\frac{a}{b}\sqrt{\frac{c}{d}}\right)\left(\sqrt{\frac{d^2}{c^2}}\right)^2-\left(\sqrt{\frac{d^2}{c^2}}\right)^3$$

$$=\frac{a^3c}{b^3d}\sqrt{\frac{c}{d}}-3\frac{a^2}{b^2}+\frac{3ad^2}{bc^2}\sqrt{\frac{c}{d}}-\frac{d^3}{c^3}$$

$$=a^3b^{-3}c^{\frac{3}{2}}d^{-\frac{3}{2}}-3a^2b^{-2}+3ab^{-1}c^{-\frac{3}{2}}d^{\frac{3}{2}}-c^{-3}d^3.$$

$$(n^{\frac{n}{2}} - a^{-\frac{n}{2}})^4$$
$$= (a^{\frac{n}{2}})^4 - 4(a^{\frac{n}{2}})^3(a^{-\frac{n}{2}}) + 6(a^{\frac{n}{2}})^2(a^{-\frac{n}{2}})^2 - 4(a^{\frac{n}{2}})(a^{-\frac{n}{2}})^3 + (a^{-\frac{n}{2}})^4$$
$$= a^{2n} - 4a^n + 6 - 4a^{-n} + a^{-2n}.$$

$$\left(\frac{2a}{b^2} - \tfrac{1}{3}b\sqrt{a}\right)^4$$
$$= \left(\frac{2a}{b^2}\right)^4 - 4\left(\frac{2a}{b^2}\right)^3\left(\tfrac{1}{3}b\sqrt{a}\right) + 6\left(\frac{2a}{b^2}\right)^2\left(\tfrac{1}{3}b\sqrt{a}\right)^2$$
$$- 4\left(\frac{2a}{b^2}\right)\left(\tfrac{1}{3}b\sqrt{a}\right)^3 + \left(\tfrac{1}{3}b\sqrt{a}\right)^4$$
$$= \frac{16a^4}{b^8} - \frac{32a^{\frac{7}{2}}}{3b^5} + \frac{8a^3}{3b^2} - \frac{8a^{\frac{5}{2}}}{27} + \frac{a^2b^4}{81}$$
or $$16a^4b^{-8} - \tfrac{32}{3}a^{\frac{7}{2}}b^{-5} + \tfrac{8}{3}a^3b^{-2} - \tfrac{8}{27}a^{\frac{5}{2}}b + \frac{a^2b^4}{81}.$$

20. $$\left(\sqrt{\frac{a}{bc}} - \frac{\sqrt{c}}{3ab}\right)^3$$
$$= \left(\frac{a^{\frac{1}{2}}}{(bc)^{\frac{1}{2}}}\right)^3 - 3\left(\frac{a^{\frac{1}{2}}}{(bc)^{\frac{1}{2}}}\right)^2\left(\frac{c^{\frac{1}{2}}}{3ab}\right) + 3\left(\frac{a^{\frac{1}{2}}}{(bc)^{\frac{1}{2}}}\right)\left(\frac{c^{\frac{1}{2}}}{3ab}\right)^2 - \left(\frac{c^{\frac{1}{2}}}{3ab}\right)^3$$
$$= \frac{a^{\frac{3}{2}}}{(bc)^{\frac{3}{2}}} - 3\left(\frac{a}{bc} \times \frac{c^{\frac{1}{2}}}{3ab}\right) + 3\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}c^{\frac{1}{2}}}\right)\left(\frac{c}{9a^2b^2}\right) - \frac{c^{\frac{3}{2}}}{27a^3b^3}$$
$$= \frac{a^{\frac{3}{2}}}{(bc)^{\frac{3}{2}}} - \frac{3ac^{\frac{1}{2}}}{3ab^2c} + \frac{3a^{\frac{1}{2}}c}{9a^2b^{\frac{5}{2}}c^{\frac{1}{2}}} - \frac{c^{\frac{3}{2}}}{27a^3b^3}$$
$$= \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}c^{\frac{3}{2}}} - \frac{1}{b^2c^{\frac{1}{2}}} + \frac{c^{\frac{1}{2}}}{3a^{\frac{3}{2}}b^{\frac{5}{2}}} - \frac{c^{\frac{3}{2}}}{27a^3b^3}.$$

$$\left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}} - 3\sqrt{b}\right)^3$$
$$= \left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}}\right)^3 - 3\left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}}\right)^2(3\sqrt{b}) + 3\left(\frac{\sqrt{a}}{2\sqrt[3]{b^2}}\right)(3\sqrt{b})^2 - (3\sqrt{b})^3$$
$$= \frac{a^{\frac{3}{2}}}{8b^2} - \frac{9a}{4b^{\frac{5}{6}}} + \frac{27a^{\frac{1}{2}}b^{\frac{1}{3}}}{2} - 27b^{\frac{3}{2}}.$$

$$\left(\frac{a\sqrt{a}}{\sqrt[6]{b^5}}-\frac{\sqrt[4]{b}}{2a}\right)^3$$

$$=\left(\frac{a^{\frac{3}{2}}}{b^{\frac{5}{6}}}-\frac{b^{\frac{1}{4}}}{2a}\right)^3$$

$$=\left(\frac{a^{\frac{3}{2}}}{b^{\frac{5}{6}}}\right)^3-3\left(\frac{a^{\frac{3}{2}}}{b^{\frac{5}{6}}}\right)^2\left(\frac{b^{\frac{1}{4}}}{2a}\right)+3\left(\frac{a^{\frac{3}{2}}}{b^{\frac{5}{6}}}\right)\left(\frac{b^{\frac{1}{4}}}{2a}\right)^2-\left(\frac{b^{\frac{1}{4}}}{2a}\right)^3$$

$$=\frac{a^{\frac{9}{2}}}{b^{\frac{5}{2}}}-\frac{3a^2}{2b^{\frac{17}{12}}}+\frac{3}{4a^{\frac{1}{2}}b^{\frac{1}{3}}}-\frac{b^{\frac{3}{4}}}{8a^3}.$$

21.

$$\begin{array}{r|l}
 & x^{4m}+6x^{3m}y^n+11x^{2m}y^{2n}+6x^my^{3n}+y^{4n}\ \underline{|\,x^{2m}+3x^my^n+y^{2n}} \\
 & x^{4m} \\ \hline
2x^{2m}+3x^my^n & 6x^{3m}y^n+11x^{2m}y^{2n} \\
 & 6x^{3m}y^n+\ \ 9x^{2m}y^{2n} \\ \hline
2x^{2m}+6x^my^n+y^{2n} & 2x^{2m}y^{2n}+6x^my^{3n}+y^{4n} \\
 & 2x^{2m}y^{2n}+6x^my^{3n}+y^{4n} \\ \hline
\end{array}$$

22.

$$\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{|\,1+2x^{-\frac{1}{3}}-3x^{-\frac{2}{3}}+4x^{-1}} \\
 & 1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-\ 4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \\
 & 1 \\ \hline
2+2x^{-\frac{1}{3}} & 4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}} \\
 & 4x^{-\frac{1}{3}}+4x^{-\frac{2}{3}} \\ \hline
2+4x^{-\frac{1}{3}}-3x^{-\frac{2}{3}} & -6x^{-\frac{2}{3}}-\ 4x^{-1}+25x^{-\frac{4}{3}} \\
 & -6x^{-\frac{2}{3}}-12x^{-1}+\ 9x^{-\frac{4}{3}} \\ \hline
2+4x^{-\frac{1}{3}}-6x^{-\frac{2}{3}}+4x^{-1} & 8x^{-1}+16x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \\
 & 8x^{-1}+16x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2} \\ \hline
\end{array}$$

Exercise CIV.

1.

(1) $\dfrac{3}{\sqrt{7}+\sqrt{5}}$

$=\dfrac{3(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$

$=\dfrac{3\sqrt{7}-3\sqrt{5}}{2}$

$=\frac{3}{2}(\sqrt{7}-\sqrt{5}).$

(2) $\dfrac{7}{2\sqrt{5}-\sqrt{6}}$

$=\dfrac{7}{2\sqrt{5}-\sqrt{6}}\times\dfrac{2\sqrt{5}+\sqrt{6}}{2\sqrt{5}+\sqrt{6}}$

$=\dfrac{7(2\sqrt{5}+\sqrt{6})}{14}$

$=\frac{1}{2}(2\sqrt{5}+\sqrt{6}).$

(3) $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$

$=\dfrac{(4-\sqrt{2})(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$

$=\dfrac{4-5\sqrt{2}+2}{-1}$

$=5\sqrt{2}-6.$

(4) $\dfrac{6}{5-2\sqrt{6}}$

$=\dfrac{6}{5-2\sqrt{6}}\times\dfrac{5+2\sqrt{6}}{5+2\sqrt{6}}$

$=\dfrac{30+12\sqrt{6}}{1}$

$=30+12\sqrt{6}.$

2.

(1) $\dfrac{a}{\sqrt{b}-\sqrt{c}}$

$=\dfrac{a(\sqrt{b}+\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}$

$=\dfrac{a(\sqrt{b}+\sqrt{c})}{b-c}.$

(2) $\dfrac{a+b}{a-\sqrt{b}}$

$=\dfrac{a+b}{a-\sqrt{b}}\times\dfrac{a+\sqrt{b}}{a+\sqrt{b}}$

$=\dfrac{(a+b)(a+\sqrt{b})}{a^2-b}.$

(3) $\dfrac{2x-\sqrt{xy}}{\sqrt{xy}-2y}\times\dfrac{\sqrt{xy}+2y}{\sqrt{xy}+2y}$

$=\dfrac{2(x-y)\sqrt{xy}+3xy}{xy-4y^2}.$

3.

(1) $\dfrac{2}{\sqrt{3}}=\dfrac{2(\sqrt{3})}{(\sqrt{3})(\sqrt{3})}$

$=\dfrac{2\sqrt{3}}{3}=\dfrac{3.463}{3}$

$=1.154.....$

(2) $\dfrac{1}{\sqrt{5}-\sqrt{2}}$

$=\dfrac{1(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$

$=\dfrac{\sqrt{5}+\sqrt{2}}{3}$

$=\dfrac{3.649.....}{3}$

$=1.216.....$

(3) $\dfrac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}$

$=\dfrac{7\sqrt{5}(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}$

$=\dfrac{7(\sqrt{35}-\sqrt{15})}{4}$

$=\dfrac{7(5.9160.....-3.8729.....)}{4}$

$=3.576.....$

(4) $\dfrac{7+2\sqrt{10}}{7-2\sqrt{10}}$

$=\dfrac{(7+2\sqrt{10})(7+2\sqrt{10})}{(7-2\sqrt{10})(7+2\sqrt{10})}$

$=\dfrac{49+28\sqrt{10}+40}{49-40}$

$=\dfrac{89+28\sqrt{10}}{9}$

$=\dfrac{89+28\times 3.162.....}{9}$

$=19.726.....$

Exercise CV.

1. $4+\sqrt{-3}=4+3^{\frac{1}{2}}\sqrt{-1}.$
$4-\sqrt{-3}=4-3^{\frac{1}{2}}\sqrt{-1}.$
$(4+3^{\frac{1}{2}}\sqrt{-1})(4-3^{\frac{1}{2}}\sqrt{-1})=16-3(-1)=19.$
$\sqrt{3}-2\sqrt{-2}=\sqrt{3}-2\times 2^{\frac{1}{2}}\sqrt{-1}.$
$\sqrt{3}+2\sqrt{-2}=\sqrt{3}+2\times 2^{\frac{1}{2}}\sqrt{-1}.$
$(\sqrt{3}-2\times 2^{\frac{1}{2}}\sqrt{-1})(\sqrt{3}+2\times 2^{\frac{1}{2}}\sqrt{-1})=3-8(-1)=11.$

2. $\sqrt{54}\times\sqrt{-2}=3\sqrt{6}\times\sqrt{-2}=3\sqrt{-12}=6\sqrt{-3}.$
$a\sqrt{-b}\times x\sqrt{-y}=(ab^{\frac{1}{2}}\sqrt{-1})\times(xy^{\frac{1}{2}}\sqrt{-1})=ab^{\frac{1}{2}}xy^{\frac{1}{2}}(-1)$
$=-ax\sqrt{by}.$

3. $\sqrt{-a}+\sqrt{-b}=a^{\frac{1}{2}}\sqrt{-1}+b^{\frac{1}{2}}\sqrt{-1}.$
$\sqrt{-a}-\sqrt{-b}=a^{\frac{1}{2}}\sqrt{-1}-b^{\frac{1}{2}}\sqrt{-1}.$
$(a^{\frac{1}{2}}\sqrt{-1}+b^{\frac{1}{2}}\sqrt{-1})(a^{\frac{1}{2}}\sqrt{-1}-b^{\frac{1}{2}}\sqrt{-1})=a(-1)-b(-1)=b-a.$
$a\sqrt{-a^2b^4}=a^2b^2\sqrt{-1}.$
$\sqrt{-a^4b^5}=a^2b^{\frac{5}{2}}\sqrt{-1}.$
$a^2b^2\sqrt{-1}\times a^2b^{\frac{5}{2}}\sqrt{-1}=a^4b^{\frac{9}{2}}(-1)=-a^4b^4\sqrt{b}.$

4. $(\sqrt{-10})(\sqrt{-2})=(10^{\frac{1}{2}}\sqrt{-1})(2^{\frac{1}{2}}\sqrt{-1})=20^{\frac{1}{2}}(-1)=-2\sqrt{5}.$
$(2\sqrt{3}-6\sqrt{-5})(4\sqrt{3}-\sqrt{-5})$
$=(2\times 3^{\frac{1}{2}}-6\times 5^{\frac{1}{2}}\sqrt{-1})(4\times 3^{\frac{1}{2}}-5^{\frac{1}{2}}\sqrt{-1})$
$=24-(24\times 3^{\frac{1}{2}}\times 5^{\frac{1}{2}}\sqrt{-1})-[2\times 3^{\frac{1}{2}}\times 5^{\frac{1}{2}}\sqrt{-1}+30(-1)]$
$=-6-26(3^{\frac{1}{2}}\times 5^{\frac{1}{2}}\sqrt{-1})=-6-26\sqrt{-15}=-2(3+13\sqrt{-15}).$

Exercise CIV.

1.

(1) $\dfrac{3}{\sqrt{7}+\sqrt{5}}$

$=\dfrac{3(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})}$

$=\dfrac{3\sqrt{7}-3\sqrt{5}}{2}$

$=\frac{3}{2}(\sqrt{7}-\sqrt{5}).$

(2) $\dfrac{7}{2\sqrt{5}-\sqrt{6}}$

$=\dfrac{7}{2\sqrt{5}-\sqrt{6}}\times\dfrac{2\sqrt{5}+\sqrt{6}}{2\sqrt{5}+\sqrt{6}}$

$=\dfrac{7(2\sqrt{5}+\sqrt{6})}{14}$

$=\frac{1}{2}(2\sqrt{5}+\sqrt{6}).$

(3) $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$

$=\dfrac{(4-\sqrt{2})(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$

$=\dfrac{4-5\sqrt{2}+2}{-1}$

$=5\sqrt{2}-6.$

(4) $\dfrac{6}{5-2\sqrt{6}}$

$=\dfrac{6}{5-2\sqrt{6}}\times\dfrac{5+2\sqrt{6}}{5+2\sqrt{6}}$

$=\dfrac{30+12\sqrt{6}}{1}$

$=30+12\sqrt{6}.$

2.

(1) $\dfrac{a}{\sqrt{b}-\sqrt{c}}$

$=\dfrac{a(\sqrt{b}+\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}$

$=\dfrac{a(\sqrt{b}+\sqrt{c})}{b-c}.$

(2) $\dfrac{a+b}{a-\sqrt{b}}$

$=\dfrac{a+b}{a-\sqrt{b}}\times\dfrac{a+\sqrt{b}}{a+\sqrt{b}}$

$=\dfrac{(a+b)(a+\sqrt{b})}{a^2-b}$

(3) $\dfrac{2x-\sqrt{xy}}{\sqrt{xy}-2y}\times\dfrac{\sqrt{x}\text{[illegible]}}{\sqrt{}\text{[illegible]}}$

$=\dfrac{2(x-y)\sqrt{xy}}{xy-4\text{[illegible]}}$

3.

(1) $\dfrac{2}{\sqrt{3}}=\dfrac{2(\sqrt{}\text{[illegible]}}{(\sqrt{3})(\text{[illegible]}}$

$=\dfrac{2\sqrt{3}}{3}=\dfrac{3.4\text{[illegible]}}{\text{[illegible]}}$

$=1.154.....$

(2) $\dfrac{1}{\sqrt{5}-\sqrt{2}}$

$=\dfrac{1(\sqrt{}\text{[illegible]}}{(\sqrt{5}-\sqrt{}\text{[illegible]}}$

$=\dfrac{\sqrt{5}+\sqrt{}\text{[illegible]}}{3}$

5.

20.

5.

$\overline{13 - 2\sqrt{30}}$.

$\overline{13 + 2\sqrt{30}}$.

$\sqrt{169 - 120}$.

7.

13.

$= 10,$

$y = 3.$

$y = \sqrt{10} - \sqrt{3}.$

10.

$x - \sqrt{y} = \sqrt{11 - 6\sqrt{2}}.$

$\sqrt{x} + \sqrt{y} = \sqrt{11 + 6\sqrt{2}}.$

ltiplying,

$$x - y = \sqrt{121 - 72}.$$

$$\therefore x - y = 7.$$

$$x + y = 11.$$

$$\therefore x = 9,$$

and $y = 2.$

$$\therefore \sqrt{x} - \sqrt{y} = 3 - \sqrt{2}.$$

11.

Let $\sqrt{x}-\sqrt{y}=\sqrt{14-4\sqrt{6}}.$

Then $\sqrt{x}+\sqrt{y}=\sqrt{14+4\sqrt{6}}.$

By multiplying,

$$x-y=\sqrt{196-96}.$$

$$\therefore x-y=10.$$

But $x+y=14.$

$$\therefore x=12,$$

and $y=2.$

$$\therefore \sqrt{x}-\sqrt{y}=2\sqrt{3}-\sqrt{2}.$$

12.

Let $\sqrt{x}-\sqrt{y}=\sqrt{38-12\sqrt{10}}.$

Then $\sqrt{x}+\sqrt{y}=\sqrt{38+12\sqrt{10}}.$

By multiplying,

$$x-y=\sqrt{1444-1440}.$$

$$\therefore x-y=2.$$

But $x+y=38.$

$$\therefore x=20,$$

and $y=18.$

$$\therefore \sqrt{x}-\sqrt{y}=2\sqrt{5}-3\sqrt{2}.$$

13.

Let $\sqrt{x}-\sqrt{y}=\sqrt{103-12\sqrt{11}}.$

Then $\sqrt{x}+\sqrt{y}=\sqrt{103+12\sqrt{11}}.$

By multiplying,

$$x-y=\sqrt{10609-1584}.$$

$$\therefore x-y=95.$$

But $x+y=103.$

$$\therefore x=99,$$

and $y=4.$

$$\therefore \sqrt{x}-\sqrt{y}=3\sqrt{11}-2.$$

14.

Let $\sqrt{x}-\sqrt{y}=\sqrt{57-12\sqrt{15}}.$

Then $\sqrt{x}+\sqrt{y}=\sqrt{57+12\sqrt{15}}.$

By multiplying,

$$x-y=\sqrt{3249-2160}.$$

$$\therefore x-y=33.$$

But $x+y=57.$

$$\therefore x=45,$$

and $y=12.$

$$\therefore \sqrt{x}-\sqrt{y}=3\sqrt{5}-2\sqrt{3}.$$

15.

Let $\sqrt{x}-\sqrt{y}=\sqrt{\tfrac{7}{2}-\sqrt{10}}.$

Then $\sqrt{x}+\sqrt{y}=\sqrt{\tfrac{7}{2}+\sqrt{10}}.$

By multiplying,

$$x-y=\sqrt{\tfrac{49}{4}-10}.$$

$$\therefore x-y=\tfrac{3}{2}.$$

But $x+y=\tfrac{7}{2}.$

$$\therefore x=\tfrac{5}{2},$$

and $y=1.$

$$\therefore \sqrt{x}-\sqrt{y}=\tfrac{1}{2}\sqrt{10}-1.$$

16.

Let $\sqrt{x}+\sqrt{y}=\sqrt{2a+2\sqrt{a^2-b^2}}.$

Then $\sqrt{x}-\sqrt{y}=\sqrt{2a-2\sqrt{a^2-b^2}}.$

By multiplying,

$$x-y=\sqrt{4a^2-4a^2+4b^2}.$$

$$\therefore x-y=2b.$$

But $x+y=2a.$

$$\therefore x=a+b,$$

and $y=a-b.$

$$\therefore \sqrt{x}+\sqrt{y}=\sqrt{a+b}+\sqrt{a-b}.$$

17.

Let $\sqrt{x} - \sqrt{y} = \sqrt{a^2 - 2b\sqrt{a^2 - b^2}}.$

Then $\sqrt{x} + \sqrt{y} = \sqrt{a^2 + 2b\sqrt{a^2 - b^2}}.$

By multiplying,

$$x - y = \sqrt{a^4 - 4a^2b^2 + 4b^4}.$$

$$\therefore x - y = a^2 - 2b^2.$$

But $x + y = a^2.$

$$\therefore x = a^2 - b^2,$$

$$\text{and } y = b^2.$$

$$\therefore \sqrt{x} - \sqrt{y} = \sqrt{a^2 - b^2} - b.$$

18.

Let $\sqrt{x} - \sqrt{y} = \sqrt{87 - 12\sqrt{42}}.$

Then $x - y = \sqrt{7569 - 6048}.$

$$\therefore x - y = 39.$$

But $x + y = 87.$

$$\therefore x = 63,$$

$$\text{and } y = 24.$$

$$\therefore \sqrt{x} - \sqrt{y} = \sqrt{63} - \sqrt{24}$$

$$= 3\sqrt{7} - 2\sqrt{6}.$$

19.

Let $\sqrt{x} - \sqrt{y} = \sqrt{(a + b)^2 - 4(a - b)\sqrt{ab}}.$

Then $\sqrt{x} + \sqrt{y} = \sqrt{(a + b)^2 + 4(a - b)\sqrt{ab}}.$

$$\therefore x - y = \sqrt{a^4 - 12a^3b + 38a^2b^2 - 12ab^3 + b^4}$$

$$= a^2 - 6ab + b^2.$$

But $x + y = a^2 + 2ab + b^2.$

$$\therefore x = (a - b)^2,$$

$$\text{and } y = 4ab.$$

$$\therefore \sqrt{x} - \sqrt{y} = a - b - 2\sqrt{ab}.$$

Exercise CVII.

1. $\sqrt{x-5}=2.$

Squaring, $x-5=4,$

$$x=9.$$

2. $2\sqrt{3x+4}-x=4,$

$$2\sqrt{3x+4}=x+4.$$

Squaring,

$$12x+16=x^2+8x+16,$$

$$x^2-4x=0,$$

$$x(x-4)=0.$$

$$\therefore x=4 \text{ or } 0.$$

3. $3-\sqrt{x^2-1}=2x,$

$$3-2x=\sqrt{x^2-1}.$$

Squaring,

$$9-12x+4x^2=x^2-1,$$

$$3x^2-12x=-10,$$

$$9x^2-36x=-30,$$

$$9x^2-(\ )+36=6,$$

$$3x-6=\pm\sqrt{6},$$

$$3x=6\pm\sqrt{6}.$$

$$\therefore x=2\pm\tfrac{1}{3}\sqrt{6}.$$

4. $\sqrt{3x-2}=2(x-4).$

Squaring,

$$3x-2=4x^2-32x+64$$

$$4x^2-35x=-66,$$

$$64x^2-(\ )+(35)^2=169,$$

$$8x-35=\pm 13.$$

$$\therefore x=6 \text{ or } 2\tfrac{3}{4}.$$

5. $4x-12\sqrt{x}=16.$

Divide by 4,

$$x-3\sqrt{x}=4,$$

$$3\sqrt{x}=x-4.$$

Squaring, $9x=x^2-8x+16,$

$$x^2-17x=-16,$$

$$4x^2-68x+289=225,$$

$$2x-17=\pm 15.$$

$$\therefore x=16 \text{ or } 1.$$

6.

$$\sqrt{x+4}+\sqrt{2x-1}=6.$$

Squaring,

$$x+4+2\sqrt{2x^2+7x-4}+2x-1=36,$$

$$2\sqrt{2x^2+7x-4}=33-3x.$$

Squaring,

$$8x^2+28x-16=1089-198x+9x^2,$$

$$x^2-226x=-1105,$$

$$x^2-(\ )+(113)^2=(113)^2-1105,$$

$$x-113=\pm 108.$$

$$\therefore x=221 \text{ or } 5.$$

7.

$$\sqrt{13x-1}-\sqrt{2x-1}=5.$$

Squaring,

$$13x-1-2\sqrt{26x^2-15x+1}+2x-1=25,$$

$$15x-27=2\sqrt{26x^2-15x+1}.$$

Squaring, $$225x^2-810x+729=104x^2-60x+4,$$

$$121x^2-750x=-725,$$

$$58564x^2-(\ )+(750)^2=211600,$$

$$242x-750=\pm 460,$$

$$242x=1210 \text{ or } 290.$$

$$\therefore x=5 \text{ or } 1\tfrac{24}{121}.$$

8.

$$\sqrt{4+x}+\sqrt{x}=3.$$

Squaring, $$4+x+2\sqrt{4x+x^2}+x=9,$$

$$2\sqrt{4x+x^2}=5-2x.$$

Squaring, $$16x+4x^2=25-20x+4x^2,$$

$$36x=25.$$

$$\therefore x=\tfrac{25}{36}.$$

9.

$$\sqrt{25+x}+\sqrt{25-x}=8.$$

Squaring, $$25+x+2\sqrt{625-x^2}+25-x=64,$$

$$2\sqrt{625-x^2}=14,$$

$$2500-4x^2=196,$$

$$4x^2=2304,$$

$$2x=\pm 48.$$

$$\therefore x=\pm 24.$$

10.

$$x^2=21+\sqrt{x^2-9}.$$

$$x^2-21=\sqrt{x^2-9},$$

$$x^4-42x^2+441=x^2-9,$$

$$x^4-43x^2=-450,$$

$$4x^4-172x^2=-1800,$$

$$4x^4-(\ )+(43)^2=49,$$

$$2x^2-43=\pm 7,$$

$$x^2=25 \text{ or } 18.$$

$$\therefore x=\pm 5 \text{ or } \pm 3\sqrt{2}.$$

11.

$$2x-\sqrt[3]{8x^3+26}+2=0.$$
$$\sqrt[3]{8x^3+26}=2x+2,$$
$$8x^3+26=8x^3+24x^2+24x+8,$$
$$24x^2+24x=18,$$
$$4x^2+4x=3,$$
$$4x^2+(\ )+1=4,$$
$$2x+1=\pm 2.$$
$$\therefore x=\tfrac{1}{2} \text{ or } -\tfrac{3}{2}.$$

12.

$$\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$$
$$x+1+2\sqrt{x^2+17x+16}+x+16=x+25,$$
$$2\sqrt{x^2+17x+16}=8-x,$$
$$4x^2+68x+64=64-16x+x^2,$$
$$3x^2+84x=0,$$
$$x^2+28x=0,$$
$$x(x+28)=0.$$
$$\therefore x=0 \text{ or } -28.$$

13.

$$\sqrt{2x+1}-\sqrt{x+4}=\tfrac{1}{3}\sqrt{x-3}.$$
$$2x+1-2\sqrt{2x^2+9x+4}+x+4=\frac{x-3}{9},$$
$$27x+45-18\sqrt{2x^2+9x+4}=x-3,$$
$$18\sqrt{2x^2+9x+4}=26x+48,$$
$$9\sqrt{2x^2+9x+4}=13x+24,$$
$$162x^2+729x+324=169x^2+624x+576,$$
$$-7x^2+105x=252,$$
$$x^2-15x=-36,$$
$$4x^2-(\ )+225=81,$$
$$2x-15=\pm 9.$$
$$\therefore x=12 \text{ or } 3.$$

14.

$$\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$$
$$x+3+2\sqrt{x^2+11x+24}+x+8=25x,$$
$$2\sqrt{x^2+11x+24}=23x-11,$$
$$4x^2+44x+96=529x^2-506x+121,$$
$$525x^2-550x=-25,$$
$$21x^2-22x=-1,$$
$$1764x^2-(\ )+484=400,$$
$$42x-22=\pm 20.$$
$$\therefore x=1 \text{ or } \tfrac{1}{21}.$$

15.

$$\sqrt{3+x}+\sqrt{x}=\frac{6}{\sqrt{3+x}}.$$

Clear of fractions, $3+x+\sqrt{3x+x^2}=6,$

$$\sqrt{3x+x^2}=3-x,$$

$$3x+x^2=9-6x+x^2,$$

$$9x=9.$$

$$\therefore x=1.$$

16.

$$\sqrt{x^2-1}+6=\frac{16}{\sqrt{x^2-1}}.$$

$$x^2-1+6\sqrt{x^2-1}=16,$$

$$6\sqrt{x^2-1}=17-x^2,$$

$$36x^2-36=289-34x^2+x^4,$$

$$x^4-70x^2=-325,$$

$$x^4-(\ )+(35)^2=900,$$

$$x^2-35=\pm 30,$$

$$x^2=65 \text{ or } 5.$$

$$\therefore x=\pm\sqrt{65} \text{ or } \pm\sqrt{5}.$$

17.

$$\frac{1}{\sqrt{x+1}}+\frac{1}{\sqrt{x-1}}=\frac{1}{\sqrt{x^2-1}}.$$

$$\sqrt{x-1}+\sqrt{x+1}=1,$$

$$\sqrt{x-1}=1-\sqrt{x+1},$$

$$x-1=1-2\sqrt{x+1}+x+1,$$

$$2\sqrt{x+1}=3,$$

$$4x+4=9.$$

$$\therefore x=1\tfrac{1}{4}.$$

18.

$$\frac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\frac{x}{2a}.$$

$$2a\sqrt{x+2a}-2a\sqrt{x-2a}=x\sqrt{x-2a}+x\sqrt{x+2a},$$

$$(2a-x)\sqrt{x+2a}=(2a+x)\sqrt{x-2a},$$

$$(4a^2-4ax+x^2)(x+2a)=(4a^2+4ax+x^2)(x-2a),$$

$$x^3-2ax^2-4a^2x+8a^3=x^3+2ax^2-4a^2x-8a^3,$$

$$4ax^2=16a^3.$$

$$\therefore x=\pm 2a.$$

19.

$$\frac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$$

$$3x+\sqrt{4x-x^2}=6x-2\sqrt{4x-x^2},$$
$$3\sqrt{4x-x^2}=3x,$$
$$\sqrt{4x-x^2}=x,$$
$$4x-x^2=x^2,$$
$$2x^2-4x=0,$$
$$2x(x-2)=0.$$
$$\therefore\ x=0 \text{ or } 2.$$

20.

$$\frac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7.$$

$$\sqrt{7x^2+4}+2\sqrt{3x-1}=7\sqrt{7x^2+4}-14\sqrt{3x-1},$$
$$16\sqrt{3x-1}=6\sqrt{7x^2+4},$$
$$8\sqrt{3x-1}=3\sqrt{7x^2+4},$$
$$192x-64=63x^2+36,$$
$$63x^2-192x=-100,$$
$$4(63x)^2-(\ )+(192)^2=11664,$$
$$126x-192=\pm 108.$$
$$\therefore\ x=2\tfrac{8}{21} \text{ or } \tfrac{2}{3}.$$

21.

$$\sqrt{(x-a)^2+2ab+b^2}=x-a+b.$$
$$x^2-2ax+a^2+2ab+b^2=x^2+a^2+b^2-2ax-2ab+2bx,$$
$$2bx=4ab,$$
$$2x=4a.$$
$$\therefore\ x=2a.$$

22.

$$\sqrt{(x+a)^2+2ab+b^2}=b-a-x.$$
$$x^2+2ax+a^2+2ab+b^2=b^2+a^2+x^2-2ab+2ax-2bx,$$
$$2bx=-4ab.$$
$$\therefore\ x=-2a.$$

23.

$$\sqrt{\frac{x}{4}+3}+\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}.$$

$$\sqrt{3x+36}+\sqrt{3x-36}=\sqrt{8x}.$$

$$3x+36+2\sqrt{9x^2-1296}+3x-36=8x,$$

$$2\sqrt{9x^2-1296}=2x,$$

$$\sqrt{9x^2-1296}=x,$$

$$9x^2-1296=x^2,$$

$$x^2=162.$$

$$\therefore x=\pm 9\sqrt{2}.$$

24.

$$4x^{\frac12}-3(x^{\frac12}+1)(x^{\frac12}-2)=x^{\frac12}(10-3x^{\frac12}).$$

$$4x^{\frac12}-3x+3x^{\frac12}+6=10x^{\frac12}-3x,$$

$$3x^{\frac12}=6,$$

$$9x=36.$$

$$\therefore x=4.$$

25.

$$(x^{\frac23}-2)(x^{\frac43}-4)=x^{\frac23}(x^{\frac23}-1)^2-12.$$

$$x^{\frac63}-2x^{\frac43}-4x^{\frac23}+8=x^{\frac63}-2x^{\frac43}+x^{\frac23}-12,$$

$$5x^{\frac23}=20,$$

$$x^{\frac23}=4.$$

Raise to third power, $\quad x^2=64.$

$$\therefore x=\pm 8.$$

26. $x^3-4x^{\frac32}=96.$

$$x^3-4x^{\frac32}+4=100,$$

$$x^{\frac32}-2=\pm 10,$$

$$x^{\frac32}=12 \text{ or } -8,$$

$$x^{\frac12}=\sqrt[3]{12} \text{ or } -2.$$

$$\therefore x=(\sqrt[3]{144}) \text{ or } 4$$

$$=2\sqrt[3]{18} \text{ or } 4.$$

27. $x+x^{-1}=2.9.$

$$x+\frac{1}{x}=2.9,$$

$$x^2+1=2.9x,$$

$$10x^2-29x=-10,$$

$$400x^2-(\ )+(29)^2=441,$$

$$20x-29=\pm 21.$$

$$\therefore x=\tfrac{2}{5} \text{ or } 2\tfrac{1}{2}.$$

28.

$$x^{\frac{1}{2}} + 2a^2x^{-\frac{1}{2}} = 3a.$$

Multiply by $x^{\frac{1}{2}}$, $\quad x + 2a^2 = 3ax^{\frac{1}{2}}$,

$$x - 3ax^{\frac{1}{2}} = -2a^2,$$
$$4x - (\;) + 9a^2 = a^2,$$
$$\sqrt{4x} - 3a = \pm a,$$
$$\sqrt{4x} = 3a \pm a,$$
$$4x = 16a^2 \text{ or } 4a^2.$$
$$\therefore x = 4a^2 \text{ or } a^2$$

29.

$$81\sqrt[3]{x} + \frac{81}{\sqrt[3]{x}} = 52x.$$
$$81x^{\frac{2}{3}} + 81 = 52x^{\frac{4}{3}}.$$

Transpose, $\quad 52x^{\frac{4}{3}} - 81x^{\frac{2}{3}} = 81$,

$$(104)^2x^{\frac{4}{3}} - (\;) + (81)^2 = 23409,$$
$$104x^{\frac{2}{3}} - 81 = \pm 153,$$
$$104x^{\frac{2}{3}} = 234 \text{ or } -72,$$
$$x^{\frac{2}{3}} = \tfrac{9}{4} \text{ or } -\tfrac{9}{13},$$
$$x^2 = \tfrac{729}{64} \text{ or } -\tfrac{729}{2197}.$$
$$\therefore x = \pm\tfrac{27}{8} \text{ or } \pm\tfrac{27}{169}\sqrt{-13}.$$

Exercise CVIII.

1.

$$x^2 - 3x - 6\sqrt{x^2 - 3x - 3} + 2 = 0.$$

Add -3 to both sides,

$$(x^2 - 3x - 3) - 6(x^2 - 3x - 3)^{\frac{1}{2}} = -5,$$
$$(x^2 - 3x - 3) - (\;) + 9 = 4,$$
$$(x^2 - 3x - 3)^{\frac{1}{2}} - 3 = \pm 2,$$
$$(x^2 - 3x - 3)^{\frac{1}{2}} = 5 \text{ or } 1,$$
$$x^2 - 3x - 3 = 25 \text{ or } 1,$$
$$x^2 - 3x = 28 \text{ or } 4,$$
$$4x^2 - (\;) + 9 = 121 \text{ or } 25,$$
$$2x - 3 = \pm 11 \text{ or } \pm 5,$$
$$2x = 14, -8, 8, -2.$$
$$\therefore x = 7, -4, 4, -1.$$

2.

$$x^2 + 3x - \frac{3}{x} + \frac{1}{x^2} = \frac{7}{36}.$$

$$\left(x^2 + \frac{1}{x^2}\right) + \left(3x - \frac{3}{x}\right) = \frac{7}{36}.$$

Subtract 2 from $\left(x^2 + \frac{1}{x^2}\right)$, and from $\frac{7}{36}$.

Since $$\left(x^2 - 2 + \frac{1}{x^2}\right) = \left(x - \frac{1}{x}\right)^2,$$

$$\left(x - \frac{1}{x}\right)^2 + 3\left(x - \frac{1}{x}\right) = -\frac{65}{36},$$

$$4\left(x - \frac{1}{x}\right)^2 + (\ ) + 9 = \frac{16}{9},$$

$$2\left(x - \frac{1}{x}\right) + 3 = \pm\frac{4}{3},$$

$$2x - \frac{2}{x} = -\frac{5}{3} \text{ or } -\frac{13}{3},$$

$$6x^2 - 6 = -5x \text{ or } -13x,$$

$$6x^2 + 5x = 6,$$

$$144x^2 + (\ ) + 25 = 169,$$

$$12x + 5 = \pm 13.$$

$$\therefore x = \tfrac{2}{3} \text{ or } -1\tfrac{1}{2}.$$

From $$6x^2 - 6 = -13x,$$

$$6x^2 + 13x = 6,$$

$$144x^2 + (\ ) + 169 = 313,$$

$$12x + 13 = \pm\sqrt{313}.$$

$$\therefore x = \tfrac{1}{12}(-13 \pm \sqrt{313}).$$

3.

$$(2x^2 - 3x)^2 - 2(2x^2 - 3x) = 15.$$

$$(2x^2 - 3x)^2 - (\ ) + 1 = 16,$$

$$(2x^2 - 3x) - 1 = \pm 4,$$

$$2x^2 - 3x = 5 \text{ or } -3,$$

$$16x^2 - (\ ) + 9 = 49 \text{ or } -15,$$

$$4x - 3 = \pm 7 \text{ or } \pm\sqrt{-15},$$

$$4x = 10,\ -4,\ 3 \pm \sqrt{-15}.$$

$$\therefore x = 2\tfrac{1}{2},\ -1,\ \tfrac{1}{4}(3 \pm \sqrt{-15}).$$

4.

$$(ax-b)^2+4a(ax-b)=\frac{9a^2}{4}$$

$$4(ax-b)^2+16a(ax-b)=9a^2,$$
$$4(ax-b)^2+(\ )+16a^2=25a^2,$$
$$2(ax-b)+4a=\pm 5a,$$
$$2(ax-b)=a \text{ or } -9a,$$
$$2ax=a+2b \text{ or } 2b-9a.$$
$$\therefore x=\frac{a+2b}{2a} \text{ or } \frac{2b-9a}{2a}$$

5.

$$3(2x^2-x)-(2x^2-x)^{\frac{1}{2}}=2.$$

$$36(2x^2-x)-(\ )+1=25,$$
$$6(2x^2-x)^{\frac{1}{2}}-1=\pm 5,$$
$$6(2x^2-x)^{\frac{1}{2}}=6 \text{ or } -4,$$
$$(2x^2-x)^{\frac{1}{2}}=1 \text{ or } -\tfrac{2}{3},$$
$$2x^2-x=1 \text{ or } \tfrac{4}{9},$$
$$16x^2-8x=8 \text{ or } \tfrac{32}{9},$$
$$16x^2-(\ )+1=9 \text{ or } \tfrac{41}{9},$$
$$4x-1=\pm 3 \text{ or } \pm\tfrac{1}{3}\sqrt{41}.$$
$$\therefore x=1,\ -\tfrac{1}{2},\ \tfrac{1}{4}(1\pm\tfrac{1}{3}\sqrt{41}).$$

6.

$$15x-3x^2+4(x^2-5x+5)^{\frac{1}{2}}=16.$$

Change signs and add 15 to both sides,

$$(3x^2-15x+15)-4(x^2-5x+5)^{\frac{1}{2}}=-1,$$
$$3(x^2-5x+5)-4(x^2-5x+5)^{\frac{1}{2}}=-1,$$
$$36(x^2-5x+5)-(\ )+16=4,$$
$$6(x^2-5x+5)^{\frac{1}{2}}=6 \text{ or } 2,$$
$$(x^2-5x+5)^{\frac{1}{2}}=1 \text{ or } \tfrac{1}{3},$$
$$x^2-5x+5=1 \text{ or } \tfrac{1}{9},$$
$$x^2-5x=-4 \text{ or } -4\tfrac{8}{9},$$
$$4x^2-(\ )+25=9 \text{ or } \tfrac{49}{9},$$
$$2x-5=\pm 3 \text{ or } \pm\tfrac{7}{3}.$$
$$\therefore x=4,\ 1,\ 3\tfrac{2}{3},\ 1\tfrac{1}{3}.$$

7.

$$x^2 + x^{-2} + x + x^{-1} = 4.$$

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 4,$$

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6,$$

$$4\left(x + \frac{1}{x}\right)^2 + 4\left(x + \frac{1}{x}\right) = 24,$$

$$4\left(x + \frac{1}{x}\right)^2 + (\ ) + 1 = 25.$$

Extract the root, $$2\left(x + \frac{1}{x}\right) = 4 \text{ or } -6,$$

$$x + \frac{1}{x} = 2 \text{ or } -3,$$

$$x^2 + 1 = 2x \text{ or } -3x.$$

For first value, $$x^2 + 1 = 2x,$$

$$x^2 - 2x = -1,$$

$$x^2 - 2x + 1 = 0,$$

$$x - 1 = 0.$$

$$\therefore x = 1.$$

For second value, $$x^2 + 1 = -3x,$$

$$x^2 + 3x = -1,$$

$$4x^2 + (\ ) + 9 = 5,$$

$$2x + 3 = \pm\sqrt{5}.$$

$$\therefore x = \tfrac{1}{2}(-3 \pm \sqrt{5}).$$

8.

$$x^2 + \sqrt{x^2 - 7} = 19.$$

Subtract 7 from each side,

$$(x^2 - 7) + (x^2 - 7)^{\frac{1}{2}} = 12,$$

$$4(x^2 - 7) + (\ ) + 1 = 49,$$

$$2(x^2 - 7)^{\frac{1}{2}} + 1 = \pm 7,$$

$$2(x^2 - 7)^{\frac{1}{2}} = 6 \text{ or } -8,$$

$$(x^2 - 7)^{\frac{1}{2}} = 3 \text{ or } -4,$$

$$x^2 - 7 = 9 \text{ or } 16,$$

$$x^2 = 16 \text{ or } 23.$$

$$\therefore x = \pm 4 \text{ or } \pm\sqrt{23}.$$

9.

$$x^2 + x + \tfrac{1}{6}(x^2 + x)^{\frac{1}{2}} = \tfrac{7}{6}.$$
$$6(x^2 + x) + (x^2 + x)^{\frac{1}{2}} = 7,$$
$$144(x^2 + x) + (\quad) + 1 = 169,$$
$$12(x^2 + x)^{\frac{1}{2}} + 1 = \pm 13,$$
$$(x^2 + x)^{\frac{1}{2}} = 1 \text{ or } -\tfrac{7}{6},$$
$$(x^2 + x) = 1 \text{ or } \tfrac{49}{36},$$
$$4x^2 + 4x + 1 = 5 \text{ or } \tfrac{232}{36},$$
$$2x + 1 = \pm\sqrt{5} \text{ or } \pm\tfrac{1}{3}\sqrt{58}.$$
$$\therefore x = \tfrac{1}{2}(-1 \pm \sqrt{5}) \text{ or } \tfrac{1}{2}(-1 \pm \tfrac{1}{3}\sqrt{58}).$$

10.

$$(x + 1)^{\frac{1}{2}} + (x - 1)^{\frac{1}{2}} = 5.$$

Squaring, $$x + 1 + 2(x^2 - 1)^{\frac{1}{2}} + x - 1 = 25,$$
$$2(x^2 - 1)^{\frac{1}{2}} = 25 - 2x,$$
$$4x^2 - 4 = 625 - 100x + 4x^2,$$
$$100x = 629.$$
$$\therefore x = 6\tfrac{29}{100}.$$

11.

$$x - 1 = 2 + 2x^{-\frac{1}{2}}.$$
$$x - 1 = 2 + \frac{2}{x^{\frac{1}{2}}},$$
$$x - \frac{2}{x^{\frac{1}{2}}} = 3,$$
$$x^{\frac{3}{2}} - 2 = 3x^{\frac{1}{2}},$$
$$x^{\frac{3}{2}} - 3x^{\frac{1}{2}} = 2.$$

Squaring, $$x^3 - 6x^2 + 9x = 4,$$
$$x^3 - 6x^2 + 9x - 4 = 0,$$
$$(x - 1)(x^2 - 5x + 4) = 0.$$
$$\therefore (x - 1) \text{ or } (x^2 - 5x + 4) = 0.$$

If $$(x - 1) = 0,$$
$$x = 1.$$

If $$x^2 - 5x + 4 = 0,$$
$$x^2 - 5x = -4,$$
$$4x^2 - 20x = -16,$$
$$4x^2 - 20x + 25 = 9,$$
$$2x - 5 = \pm 3,$$
$$2x = 8 \text{ or } 2.$$
$$\therefore x = 4 \text{ or } 1.$$

12.

$$\sqrt{3x+5}-\sqrt{3x-5}=4.$$
$$\sqrt{3x+5}=\sqrt{3x-5}+4,$$
$$(3x+5)=(3x-5)+8\sqrt{3x-5}+16,$$
$$6=8\sqrt{3x-5},$$
$$36=192x-320,$$
$$-192x=-356.$$
$$\therefore x=1\tfrac{41}{48}.$$

13.

$$(x^4+1)-x(x^2+1)=-2x^2.$$

Transpose, $\quad x^4+2x^2+1-x(x^2+1)=0,$

$$(x^2+1)^2-x(x^2+1)=0.$$

Multiply by 4, and complete the square,

$$(x^2+1)^2-(\ )+x^2=x^2.$$

Extract the root, $\quad 2(x^2+1)-x=\pm x,$

$$2x^2+2=2x \text{ or } 0.$$

For first value, $\quad 2x^2-2x=-2.$

Multiply by 2, and complete the square,

$$4x^2-(\ )+1=-3.$$

Extract the root, $\quad 2x-1=\pm\sqrt{-3}.$

$$\therefore x=\tfrac{1}{2}(1\pm\sqrt{-3}).$$

For second value, $\quad 2x^2=-2,$

$$x^2=-1.$$
$$\therefore x=\pm\sqrt{-1}.$$

14.

$$2x^2-2\sqrt{2x^2-5x}=5(x+3).$$
$$2x^2-5x-2\sqrt{2x^2-5x}=15,$$
$$(2x^2-5x)-2(2x^2-5x)^{\frac{1}{2}}+1=16,$$
$$(2x^2-5x)^{\frac{1}{2}}-1=\pm 4,$$
$$(2x^2-5x)^{\frac{1}{2}}=5 \text{ or } -3,$$
$$2x^2-5x=25 \text{ or } 9,$$
$$16x^2-(\ )+25=225 \text{ or } 97,$$
$$4x-5=\pm 15 \text{ or } \pm\sqrt{97},$$
$$4x=20,\ -10,\ (5+\sqrt{97}),\ (5-\sqrt{97}).$$
$$\therefore x=5,\ -2\tfrac{1}{2},\ \tfrac{1}{4}(5+\sqrt{97}),\ \tfrac{1}{4}(5-\sqrt{97}).$$

15.

$$x + 2 - 4x\sqrt{x+2} = 12x^2.$$

Complete the square,

$$(x+2) - 4x\sqrt{x+2} + 4x^2 = 16x^2,$$
$$\sqrt{x+2} - 2x = \pm 4x,$$
$$\sqrt{x+2} = 6x \text{ or } -2x,$$
$$x + 2 = 36x^2 \text{ or } 4x^2,$$
$$36x^2 - x = 2,$$
$$5184x^2 - 144x = 288,$$
$$5184x^2 - (\ ) + 1 = 289,$$
$$72x - 1 = \pm 17,$$
$$72x = 18 \text{ or } -16.$$
$$\therefore x = \tfrac{1}{4} \text{ or } -\tfrac{2}{9}.$$

Also,

$$x + 2 = 4x^2,$$
$$4x^2 - x = 2,$$
$$64x^2 - 16x = 32,$$
$$64x^2 - (\ ) + 1 = 33,$$
$$8x - 1 = \pm\sqrt{33}.$$
$$\therefore x = \tfrac{1}{8}(1 \pm \sqrt{33}).$$

16.

$$\sqrt{2x+a} + \sqrt{2x-a} = b.$$
$$2x + a + 2\sqrt{4x^2 - a^2} + 2x - a = b^2,$$
$$2\sqrt{4x^2 - a^2} = b^2 - 4x,$$
$$16x^2 - 4a^2 = b^4 - 8b^2x + 16x^2,$$
$$8b^2x = 4a^2 + b^4.$$
$$\therefore x = \frac{4a^2 + b^4}{8b^2}.$$

17.

$$\sqrt{9x^2 + 21x + 1} - \sqrt{9x^2 + 6x + 1} = 3x.$$
$$9x^2 + 21x + 1 - 2\sqrt{81x^4 + 243x^3 + 144x^2 + 27x + 1} + 9x^2 + 6x + 1 = 9x^2$$
$$2\sqrt{81x^4 + 243x^3 + 144x^2 + 27x + 1} = 9x^2 + 27x + 2,$$
$$324x^4 + 972x^3 + 576x^2 + 108x + 4 = 81x^4 + 729x^2 + 4 + 486x^3 + 36x^2 + 108x,$$
$$243x^4 + 486x^3 - 189x^2 = 0,$$
$$27x^2(9x^2 + 18x - 7) = 0.$$
$$\therefore x = 0.$$

Or,

$$9x^2 + 18x = 7,$$
$$9x^2 + (\ ) + 9 = 16,$$
$$3x + 3 = \pm 4,$$
$$3x = 1 \text{ or } -7.$$
$$\therefore x = \tfrac{1}{3} \text{ or } -2\tfrac{1}{3}.$$

18.

$$x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + x^{-\frac{4}{3}} + 4x^{-\frac{2}{3}} = -\tfrac{7}{4}.$$

$$\left(x^{\frac{4}{3}} + \frac{1}{x^{\frac{4}{3}}}\right) - \left(4x^{\frac{2}{3}} - \frac{4}{x^{\frac{2}{3}}}\right) = -\tfrac{7}{4}.$$

Since $$x^{\frac{4}{3}} - 2 + \frac{1}{x^{\frac{4}{3}}} = \left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right)^2,$$

$$\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right)^2 - 4\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right) = -\tfrac{15}{4},$$

$$\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right)^2 - (\quad) + 4 = \tfrac{1}{4},$$

$$\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right) - 2 = \pm\tfrac{1}{2},$$

$$\left(x^{\frac{2}{3}} - \frac{1}{x^{\frac{2}{3}}}\right) = 1\tfrac{1}{2} \text{ or } 2\tfrac{1}{2},$$

$$2x^{\frac{4}{3}} - 2 = 3x^{\frac{2}{3}} \text{ or } 5x^{\frac{2}{3}},$$

$$2x^{\frac{4}{3}} - 3x^{\frac{2}{3}} = 2,$$

$$16x^{\frac{4}{3}} - (\ ) + 9 = 25,$$

$$4x^{\frac{2}{3}} - 3 = \pm 5,$$

$$4x^{\frac{2}{3}} = 8 \text{ or } -2,$$

$$x^{\frac{2}{3}} = 2 \text{ or } -\tfrac{1}{2},$$

$$x^2 = 8 \text{ or } -\tfrac{1}{8}.$$

$$\therefore x = \pm 2\sqrt{2} \text{ or } \pm\tfrac{1}{4}\sqrt{-2}.$$

Also, $$2x^{\frac{4}{3}} - 2 = 5x^{\frac{2}{3}}$$

$$2x^{\frac{4}{3}} - 5x^{\frac{2}{3}} = 2,$$

$$16x^{\frac{4}{3}} - (\ ) + 25 = 41,$$

$$4x^{\frac{2}{3}} - 5 = \pm\sqrt{41},$$

$$x^{\frac{2}{3}} = \tfrac{1}{4}(5 \pm \sqrt{41}).$$

$$\therefore x = [\tfrac{1}{4}(5 \pm \sqrt{41})]^{\frac{3}{2}}.$$

19.

$$(2x+3y)^2-2(2x+3y)=8 \qquad (1)$$

$$x^2-y^2=21 \qquad (2)$$

Add 1 to both sides of (1),

$$(2x+3y)^2-2(2x+3y)+1=9.$$

Extract the root. $\quad (2x+3y)-1=\pm 3,$

$$2x+3y=4 \text{ or } -2 \qquad (3)$$

$$\therefore x=\frac{4-3y}{2} \text{ or } -\frac{2+3y}{2},$$

$$x^2=\frac{16-24y+9y^2}{4} \text{ or } \frac{4+12y+9y^2}{4}.$$

Substitute value of x^2 in (2),

$$\frac{16-24y+9y^2}{4}-y^2=21,$$

$$5y^2-24y=68,$$

$$100y^2-(\;)+576=1936,$$

$$10y-24=\pm 44,$$

$$10y=68 \text{ or } -20.$$

$$\therefore y=6\tfrac{4}{5} \text{ or } -2.$$

Substitute second value of x^2 in (2),

$$\frac{4+12y+9y^2}{4}-y^2=21,$$

$$5y^2+12y=80,$$

$$100y^2+(\;)+144=1744,$$

$$10y+12=\pm 4\sqrt{109},$$

$$10y=-12\pm 4\sqrt{109}.$$

$$\therefore y=-1\tfrac{1}{5}\pm\tfrac{2}{5}\sqrt{109}$$

$$=\tfrac{2}{5}(-3\pm\sqrt{109}).$$

Substitute values of y in (3), $\quad x=-8\tfrac{1}{5},\ 5,\ \tfrac{1}{5}(4\mp 3\sqrt{109}).$

20.

$$x + y + \sqrt{x+y} = a,$$
$$x - y + \sqrt{x-y} = b.$$
$$4(x+y) + (\;) + 1 = 4a + 1,$$
$$4(x-y) + (\;) + 1 = 4b + 1,$$
$$2(x+y)^{\frac{1}{2}} + 1 = \pm\sqrt{4a+1},$$
$$2(x-y)^{\frac{1}{2}} + 1 = \pm\sqrt{4b+1},$$
$$2(x+y)^{\frac{1}{2}} = -1 \pm \sqrt{4a+1},$$
$$2(x-y)^{\frac{1}{2}} = -1 \pm \sqrt{4b+1},$$
$$4(x+y) = 4a + 1 \mp 2\sqrt{4a+1} + 1,$$
$$4(x-y) = 4b + 1 \mp 2\sqrt{4b+1} + 1,$$
$$8x = 4a + 4b + 4 \mp 2\sqrt{4a+1} \mp 2\sqrt{4b+1}.$$
$$\therefore x = \tfrac{1}{2}(a+b+1) \mp \tfrac{1}{4}(\sqrt{4a+1} \pm \sqrt{4b+1}).$$
$$8y = 4a - 4b \mp 2\sqrt{4a+1} \pm 2\sqrt{4b+1}.$$
$$\therefore y = \tfrac{1}{2}(a-b) \mp \tfrac{1}{4}(\sqrt{4a+1} \mp \sqrt{4b+1}).$$

21.

$$x^4 - x^2y^2 + y^4 = 13 \qquad (1)$$
$$x^2 - xy + y^2 = 3 \qquad (2)$$

Square (2), $x^4 + 3x^2y^2 + y^4 - 2x^3y - 2xy^3 = 9 \qquad (3)$

Subtract (3) from (1), $2x^3y + 2xy^3 - 4x^2y^2 = 4.$

Divide by $2xy$, $x^2 - 2xy + y^2 = \dfrac{2}{xy}. \qquad (4)$

Subtract (4) from (2), $xy = 3 - \dfrac{2}{xy}.$

$$\therefore x^2y^2 - 3xy = -2,$$
$$4x^2y^2 - (\;) + 9 = 1,$$
$$2xy - 3 = \pm 1,$$
$$xy = 2 \text{ or } 1 \qquad (5)$$

Subtract (5) from (2), $x^2 - 2xy + y^2 = 1 \text{ or } 2,$

$$x - y = \pm 1 \text{ or } \pm\sqrt{2} \qquad (6)$$

Multiply (5) by 3, and add to (2),

$$x^2 + 2xy + y^2 = 9 \text{ or } 6,$$
$$x + y = \pm 3 \text{ or } \pm\sqrt{6} \qquad (7)$$

Add (6) and (7), $2x = \pm 4,\ \pm 2, \text{ or } \pm\sqrt{2} \pm \sqrt{6}.$

$$\therefore x = \pm 2,\ \pm 1, \text{ or } \tfrac{1}{2}(\pm\sqrt{2} \pm \sqrt{6}).$$

Subtract (6) from (7), $2y = \pm 2,\ \pm 4, \text{ or } \mp\sqrt{2} \pm \sqrt{6}.$

$$\therefore y = \pm 1,\ \pm 2, \text{ or } \tfrac{1}{2}(\mp\sqrt{2} \pm \sqrt{6}).$$

22.

$$x^2 + y^2 + x + y = 48 \qquad (1)$$

$$2xy = 24 \qquad (2)$$

Add (1) and (2), $x^2 + 2xy + y^2 + x + y = 72,$

$$(x+y)^2 + (x+y) = 72.$$

Complete the square, $4(x+y)^2 + (\ ) + 1 = 289.$

Extract the root, $2(x+y) + 1 = \pm 17,$

$$x + y = 8 \text{ or } -9 \qquad (3)$$

From (2), $x = \frac{12}{y}.$

Substitute value of x in (3), $\frac{12}{y} + y = 8 \text{ or } -9,$

$$12 + y^2 = 8y \text{ or } -9y,$$

$$y^2 - 8y = -12,$$

$$y^2 - (\ ) + 16 = 4,$$

$$y - 4 = \pm 2.$$

$$\therefore y = 6 \text{ or } 2.$$

Also, $12 + y^2 = -9y,$

$$y^2 + 9y = -12.$$

Complete the square, $4y^2 + (\ ) + 81 = 33,$

$$2y + 9 = \pm\sqrt{33},$$

$$2y = -9 \pm \sqrt{33}.$$

$$\therefore y = \tfrac{1}{2}(-9 \pm \sqrt{33}).$$

Substitute values of y in (3), $x = 2,\ 6,\ \tfrac{1}{2}(-9 \mp \sqrt{33}).$

23.

$$x^2 + xy + y^2 = a^2 \qquad (1)$$

$$x + \sqrt{xy} + y = b \qquad (2)$$

Divide (1) by 2, $x - \sqrt{xy} + y = \frac{a^2}{b} \qquad (3)$

Subtract (3) from (2), $2\sqrt{xy} = \frac{b^2 - a^2}{b}.$

Divide by 2, $\sqrt{xy} = \frac{b^2 - a^2}{2b}.$

Squaring, $xy = \frac{b^4 - 2a^2b^2 + a^4}{4b^2} \qquad (4)$

Add (1) and (4), $x^2+2xy+y^2=\dfrac{a^4+2a^2b^2+b^4}{4b^2}$.

Extract the root, $x+y=\pm\dfrac{a^2+b^2}{2b}$ (5)

From (4), $-3xy=-\dfrac{3(b^4-2a^2b^2+a^4)}{4b^2}$ (6)

Add (1) and (6), $x^2-2xy+y^2=\dfrac{10a^2b^2-3a^4-3b^4}{4b^2}$.

Extract the root, $x-y=\pm\dfrac{1}{2b}\sqrt{10a^2b^2-3a^4-3b^4}$ (7)

(5) is $x+y=\pm\dfrac{a^2+b^2}{2b}$.

Add, $2x=\pm\dfrac{a^2+b^2}{2b}\pm\dfrac{1}{2b}\sqrt{10a^2b^2-3a^4-3b^4}$.

Subtract (7) from (5), $2y=\pm\dfrac{a^2+b^2}{2b}\mp\dfrac{1}{2b}\sqrt{10a^2b^2-3a^4-3b^4}$.

$$\therefore x=\frac{1}{4b}\left[\pm(a^2+b^2)\pm\sqrt{10a^2b^2-3a^4-3b^4}\right].$$

$$\therefore y=\frac{1}{4b}\left[\pm(a^2+b^2)\mp\sqrt{10a^2b^2-3a^4-3b^4}\right].$$

24.

$$(x-y)^2-3(x-y)=10 \qquad (1)$$

$$x^2y^2-3xy=54 \qquad (2)$$

Complete the square of (1), $4(x-y)^2-(\ )+9=49$,

$$2(x-y)+3=\pm 7.$$

$$\therefore x-y=5 \text{ or } -2 \qquad (3)$$

Complete the square of (2), $4x^2y^2-(\ )+9=225$,

$$2xy-3=\pm 15.$$

$$\therefore xy=9 \text{ or } -6 \qquad (4)$$

From (3), $y=x-5$ or $x+2$.

Hence from (4), $x^2-5x=9$ or -6.

Therefore $x^2-5x+\frac{25}{4}=\frac{61}{4}$ or $\frac{1}{4}$,

$$x-\tfrac{5}{2}=\pm\tfrac{1}{2}\sqrt{61}, \text{ or } \pm\tfrac{1}{2},$$

$$x=3,\ 2, \text{ or } \tfrac{1}{2}(5\pm\sqrt{61}).$$

Putting $x+2$ for y in (4), we get

$$x=-1\pm\sqrt{10}, \text{ or } -1\pm\sqrt{-5}.$$

Whence $y=-2,\ -3,\ \frac{1}{2}(-5\pm\sqrt{61})$,

$$1\pm\sqrt{10},\ 1\pm\sqrt{-5}.$$

25.

$$\sqrt{x}-\sqrt{y}=x^{\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}}) \qquad (1)$$

$$(x+y)^2=2(x-y)^2 \qquad (2)$$

From (1), $$\sqrt{x}-\sqrt{y}=x+x^{\frac{1}{2}}y^{\frac{1}{2}} \qquad (3)$$

Square (3), $$x-2\sqrt{xy}+y=x^2+2x\sqrt{xy}+xy,$$

$$x-x^2-xy+y=2x\sqrt{xy}+2\sqrt{xy} \qquad (4)$$

From (2), $$x^2+2xy+y^2=2x^2-4xy+2y^2,$$

$$x^2+y^2=6xy \qquad (5)$$

Subtract $2xy$ from both sides of (5),

$$x^2-2xy+y^2=4xy.$$

Extract the root, $$x-y=\pm 2\sqrt{xy}.$$

Substitute $x-y$ for $2\sqrt{xy}$ in (4),

$$x-x^2-xy+y=x^2-xy+x-y,$$

$$\text{or } x^2-y=0.$$

$$\therefore y=x^2.$$

Substitute x^2 for y in (5), $$x^2+x^4=6x^3 \qquad (6)$$

$$x^2=0 \text{ or } 1+x^2=6x.$$

If $$x=0,\ y=0.$$

From $$1+x^2=6x,$$

$$x^2-6x=-1,$$

$$x^2-6x+9=8,$$

$$x-3=\pm\sqrt{8},$$

$$x=(3\pm 2\sqrt{2}).$$

Since $$y=x^2,$$

$$y=(3\pm 2\sqrt{2})^2.$$

26.

$$\left(\frac{3x}{x+y}\right)^{\frac{1}{2}}+\left(\frac{x+y}{3x}\right)^{\frac{1}{2}}=2 \qquad (1)$$

$$xy-(x+y)=54 \qquad (2)$$

Square (1), $$\frac{3x}{x+y}+2+\frac{x+y}{3x}=4.$$

Simplify, $$9x^2+x^2+2xy+y^2=6x^2+6xy,$$

$$4x^2-4xy+y^2=0.$$

Extract the root, $$2x-y=0 \qquad (3)$$

$$\therefore y=2x.$$

Substitute value of y in (2), $$2x^2-3x=54,$$

$$16x^2-(\ )+9=441,$$

$$4x-3=\pm 21,$$

$$4x=24 \text{ or } -18.$$

$$\therefore x=6 \text{ or } -4\tfrac{1}{2}.$$

$$\therefore y=12 \text{ or } -9.$$

27.

$$x + y + \sqrt{xy} = 28 \qquad (1)$$

$$x^2 + y^2 + xy = 336 \qquad (2)$$

Divide (2) by (1), $\quad x - \sqrt{xy} + y = 12 \qquad (3)$

Subtract (3) from (1), $\quad 2\sqrt{xy} = 16,$

$$\sqrt{xy} = 8,$$

$$xy = 64 \qquad (4)$$

Add (4) and (2), $\quad x^2 + 2xy + y^2 = 400.$

Extract the root, $\quad x + y = \pm 20 \qquad (5)$

Multiply (4) by 3, and subtract from (2),

$$x^2 - 2xy + y^2 = 144.$$

Extract the root, $\quad x - y = \pm 12 \qquad (6)$

Add (5) and (6), $\quad 2x = \pm 32 \text{ or } \pm 8.$

$$\therefore x = \pm 16 \text{ or } \pm 4.$$

Subtract (6) from (5), $\quad 2y = \pm 8 \text{ or } \pm 32.$

$$\therefore y = \pm 4 \text{ or } \pm 16.$$

28.

$$\frac{x^2}{a} - 3ax = \sqrt{4x^3 + 9ax^2} + \frac{27a^2}{4}.$$

$$4x^2 - 12a^2x = 4ax\sqrt{4x + 9a} + 27a^3,$$

$$12a^2x + 27a^3 - 4ax\sqrt{4x + 9a} = 4x^2,$$

$$3a^2(4x + 9a) - 4ax\sqrt{4x + 9a} = 4x^2,$$

$$36a^2(4x+9a) - 48ax\sqrt{4x+9a} + 16x^2 = 64x^2.$$

Extract the root, $\quad 6a(4x + 9a)^{\frac{1}{2}} - 4x = \pm 8x.$

Divide by 2, $\quad 6a(4x + 9a)^{\frac{1}{2}} = 12x \text{ or } -4x.$

Square, $\quad 3a(4x + 9a)^{\frac{1}{2}} = 6x \text{ or } -2x$

Transpose, $\quad 36a^2x + 81a^3 = 36x^2 \text{ or } 4x^2$

$$36x^2 - 36a^2x = 81a^3,$$

$$\text{and } 4x^2 - 36a^2x = 81a^3.$$

$$36x^2 - 36a^2x + 9a^4 = 9a^4 + 81a^3,$$

$$6x - 3a^2 = \pm 3a\sqrt{a^2 + 9a},$$

$$6x = 3a^2 \pm 3a\sqrt{a^2 + 9a}.$$

$$\therefore x = \frac{a}{2}(a \pm \sqrt{a^2 + 9a}).$$

From $\quad 4x^2 - 36a^2x = 81a^3,$

$$4x^2 - (\ ) + 81a^4 = 81a^2(a^2 + a),$$

$$2x - 9a^2 = \pm 9a(a^2 + a)^{\frac{1}{2}}.$$

$$\therefore x = \frac{9a}{2}(a \pm \sqrt{a^2 + a}).$$

29.

$$(x+1+x^{-1})(x-1+x^{-1}) = 5\tfrac{1}{4}.$$
$$\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right) = 5\tfrac{1}{4},$$
$$\left(\frac{x^2+x+1}{x}\right)\left(\frac{x^2-x+1}{x}\right) = 5\tfrac{1}{4},$$
$$\frac{x^4+x^2+1}{x^2} = \tfrac{21}{4},$$
$$4x^4+4x^2+4 = 21x^2,$$
$$4x^4-17x^2 = -4,$$
$$64x^4 - (\ ) + 289 = 225.$$

Extract the root, $8x^2-17 = \pm 15,$
$$8x^2 = 32 \text{ or } 2,$$
$$x^2 = 4 \text{ or } \tfrac{1}{4}.$$
$$\therefore x = \pm 2 \text{ or } \pm \tfrac{1}{2}.$$

30.

$$2(x^{\frac{1}{2}}-1)^{-1} - 2(x^{\frac{1}{2}}-4)^{-1} = 3(x^{\frac{1}{2}}-2)^{-1}.$$
$$\frac{2}{x^{\frac{1}{2}}-1} - \frac{2}{x^{\frac{1}{2}}-4} = \frac{3}{x^{\frac{1}{2}}-2}.$$
$$2x-12x^{\frac{1}{2}}+16-2x+6x^{\frac{1}{2}}-4 = 3x-15x^{\frac{1}{2}}+12,$$
$$-3x+9x^{\frac{1}{2}} = 0,$$
$$3x^{\frac{1}{2}}(x^{\frac{1}{2}}+3) = 0,$$
$$x^{\frac{1}{2}} = -3 \text{ or } 0.$$
$$\therefore x = 9 \text{ or } 0.$$

Exercise CIX.

1. $\log 6 = \log(3\times 2)$
$= \log 3 + \log 2.$
$\log 3 = 0.4771$
$\log 2 = 0.3010$
$\therefore \log 6 = 0.7781$

2. $\log 15 = \log(3\times 5)$
$= \log 3 + \log 5.$
$\log 3 = 0.4771$
$\log 5 = 0.6990$
$\therefore \log 15 = 1.1761$

3. $\log 21 = \log(7\times 3)$
$= \log 7 + \log 3.$
$\log 7 = 0.8451$
$\log 3 = 0.4771$
$\therefore \log 21 = 1.3222$

4. $\log 14 = \log(7\times 2)$
$= \log 7 + \log 2.$
$\log 7 = 0.8451$
$\log 2 = 0.3010$
$\therefore \log 14 = 1.1461$

5. $\log 35 = \log(7 \times 5)$

$\qquad = \log 7 + \log 5.$

$$\begin{aligned} \log\ 7 &= 0.8451 \\ \log\ 5 &= \underline{0.6990} \\ \therefore \log 35 &= 1.5441 \end{aligned}$$

6. $\log\ 9 = \log(3 \times 3)$

$\qquad = \log 3 + \log 3.$

$$\begin{aligned} \log\ 3 &= 0.4771 \\ \log\ 3 &= \underline{0.4771} \\ \therefore \log\ 9 &= 0.9542 \end{aligned}$$

7. $\log\ 8 = \log(2 \times 2 \times 2)$

$\qquad = \log 2 + \log 2 + \log 2.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 2 &= \underline{0.3010} \\ \therefore \log\ 8 &= 0.9030 \end{aligned}$$

8. $\log 49 = \log(7 \times 7)$

$\qquad = \log 7 + \log 7.$

$$\begin{aligned} \log\ 7 &= 0.8451 \\ \log\ 7 &= \underline{0.8451} \\ \therefore \log 49 &= 1.6902 \end{aligned}$$

9. $\log 25 = \log(5 \times 5)$

$\qquad = \log 5 + \log 5.$

$$\begin{aligned} \log\ 5 &= 0.6990 \\ \log\ 5 &= \underline{0.6990} \\ \therefore \log 25 &= 1.3980 \end{aligned}$$

10. $\log 30 = \log(2 \times 3 \times 5)$

$\qquad = \log 2 + \log 3 + \log 5.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 3 &= 0.4771 \\ \log\ 5 &= \underline{0.6990} \\ \therefore \log 30 &= 1.4771 \end{aligned}$$

11. $\log 42 = \log(7 \times 2 \times 3)$

$\qquad = \log 7 + \log 2 + \log 3.$

$$\begin{aligned} \log\ 7 &= 0.8451 \\ \log\ 3 &= 0.4771 \\ \log\ 2 &= \underline{0.3010} \\ \therefore \log 42 &= 1.6232 \end{aligned}$$

12. $\log 420 = \log(2^2 \times 3 \times 5 \times 7)$

$\qquad = \log 2^2 + \log 3$

$\qquad\qquad + \log 5 + \log 7.$

$$\begin{aligned} \log\ 2^2 &= 0.6020 \\ \log\ 3 &= 0.4771 \\ \log\ 5 &= 0.6990 \\ \log\ 7 &= \underline{0.8451} \\ \therefore \log 420 &= 2.6232 \end{aligned}$$

13. $\log 12 = \log(2 \times 2 \times 3)$

$\qquad = \log 2 + \log 2 + \log 3.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 3 &= \underline{0.4771} \\ \therefore \log 12 &= 1.0791 \end{aligned}$$

14. $\log 60 = \log(2 \times 3 \times 2 \times 5)$

$\qquad = \log 2^2 + \log 3 + \log 5.$

$$\begin{aligned} \log\ 2 &= 0.3010 \\ \log\ 2 &= 0.3010 \\ \log\ 3 &= 0.4771 \\ \log\ 5 &= \underline{0.6990} \\ \therefore \log 60 &= 1.7781 \end{aligned}$$

15. $\log 75 = \log(5 \times 5 \times 3)$

$\qquad = \log 5 + \log 5 + \log 3.$

$$\begin{aligned} \log\ 5 &= 0.6990 \\ \log\ 5 &= 0.6990 \\ \log\ 3 &= \underline{0.4771} \\ \therefore \log 75 &= 1.8751 \end{aligned}$$

16. $\log 7.5 = \log(3 \times 5 \times 5 \times 0.1)$

$\qquad = \log 3 + \log 5$

$\qquad\qquad + \log 5 + \log 0.1.$

$$\begin{aligned} \log\ 3 &= 0.4771 \\ \log\ 5 &= 0.6990 \\ \log\ 5 &= 0.6990 \\ \log\ .1 &= \underline{9.0000 - 10} \\ \therefore \log 7.5 &= 0.8751 \end{aligned}$$

17.
$$\begin{aligned}
\log 0.021 &= \log(7\times 3\times 0.001)\\
&= \log 7+\log 3+\log 0.001.\\
\log 7 &= 0.8451\\
\log 3 &= 0.4771\\
\log 0.001 &= 7.0000-10\\
\therefore \log 0.021 &= 8.3222-10
\end{aligned}$$

18.
$$\begin{aligned}
\log 0.35 &= \log(7\times 5\times 0.01)\\
&= \log 7+\log 5+\log 0.01.\\
\log 7 &= 0.8451\\
\log 5 &= 0.6990\\
\log 0.01 &= 8.0000-10\\
\therefore \log 0.35 &= 9.5441-10
\end{aligned}$$

19.
$$\begin{aligned}
\log 0.0035 &= \log(7\times 5\times 0.0001)\\
&= \log 7+\log 5+\log 0.0001.\\
\log 7 &= 0.8451\\
\log 5 &= 0.6990\\
\log 0.0001 &= 6.0000-10\\
\therefore \log 0.0035 &= 7.5441-10
\end{aligned}$$

20.
$$\begin{aligned}
\log 0.004 &= \log(2\times 2\times 0.001)\\
&= \log 2+\log 2+\log 0.001.\\
\log 2 &= 0.3010\\
\log 2 &= 0.3010\\
\log 0.001 &= 7.0000-10\\
\therefore \log 0.004 &= 7.6020-10
\end{aligned}$$

21.
$$\begin{aligned}
\log 0.05 &= \log(5\times 0.01)\\
&= \log 5+\log 0.01.\\
\log 5 &= 0.6990\\
\log 0.01 &= 8.0000-10\\
\therefore \log 0.05 &= 8.6990-10
\end{aligned}$$

22.
$$\begin{aligned}
\log 12.5 &= \log(5^3\times 0.1)\\
&= \log 5+\log 5+\log 5+\log 0.1.\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 0.1 &= 9.0000-10\\
\therefore \log 12.5 &= 1.0970
\end{aligned}$$

23.
$$\begin{aligned}
\log 1.25 &= \log(5^3\times 0.01)\\
&= \log 5+\log 5+\log 5+\log 0.01.\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 0.01 &= 8.0000-10\\
\therefore \log 1.25 &= 0.0970
\end{aligned}$$

24.
$$\begin{aligned}
\log 37.5 &= \log(5^3\times 3\times 0.1)\\
&= \log 5+\log 5+\log 5+\log 3+\log 0.1.\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 5 &= 0.6990\\
\log 3 &= 0.4771\\
\log 0.1 &= 9.0000-10\\
\therefore \log 37.5 &= 1.5741
\end{aligned}$$

25.
$$\begin{aligned}
\log 2.1 &= \log(3\times 7\times 0.1)\\
&= \log 3+\log 7+\log 0.1.\\
\log 3 &= 0.4771\\
\log 7 &= 0.8451\\
\log 0.1 &= 9.0000-10\\
\therefore \log 2.1 &= 0.3222
\end{aligned}$$

26.
$$\begin{aligned}
\log 16 &= \log(2^4)\\
&= \log 2+\log 2+\log 2+\log 2.\\
\log 2 &= 0.3010\\
\log 2 &= 0.3010\\
\log 2 &= 0.3010\\
\log 2 &= 0.3010\\
\therefore \log 16 &= 1.2040
\end{aligned}$$

27. $\log 0.056 = \log(2\times2\times2\times7\times0.001)$
$= \log 2 + \log 2 + \log 2 + \log 7 + \log 0.001.$

$\log\ 2 = 0.3010$
$\log\ 2 = 0.3010$
$\log\ 2 = 0.3010$
$\log\ 7 = 0.8451$
$\log 0.001 = 7.0000 - 10$

$\therefore \log 0.056 = 8.7481 - 10$

28. $\log 0.63 = \log(3\times3\times7\times0.01)$
$= \log 3 + \log 3 + \log 7 + \log 0.01.$

$\log\ 3 = 0.4771$
$\log\ 3 = 0.4771$
$\log\ 7 = 0.8451$
$\log 0.01 = 8.0000 - 10$

$\therefore \log 0.63 = 9.7993 - 10$

29. $\log 1.75 = \log(5\times5\times7\times0.01)$
$= \log 5 + \log 5 + \log 7 + \log 0.01.$

$\log\ 5 = 0.6990$
$\log\ 5 = 0.6990$
$\log\ 7 = 0.8451$
$\log 0.01 = 8.0000 - 10$

$\therefore \log 1.75 = 0.2431$

30. $\log 105 = \log(5\times3\times7)$
$= \log 5 \times \log 3 \times \log 7.$

$\log\ 5 = 0.6990$
$\log\ 3 = 0.4771$
$\log\ 7 = 0.8451$

$\therefore \log 105 = 2.0212$

31. $\log 0.0105 = \log(3\times7\times5\times0.0001)$
$= \log 3 + \log 7 + \log 5 + \log 0.0001.$

$\log\ 3 = 0.4771$
$\log\ 7 = 0.8451$
$\log\ 5 = 0.6990$
$\log 0.0001 = 6.0000 - 10$

$\therefore \log 0.0105 = 8.0212 - 10$

32. $\log 1.05 = \log(7\times3\times5\times0.01)$
$= \log 7 + \log 3 + \log 5 + \log 0.01.$

$\log\ 7 = 0.8451$
$\log\ 3 = 0.4771$
$\log\ 5 = 0.6990$
$\log 0.01 = 8.0000 - 10$

$\therefore \log 1.05 = 0.0212$

Exercise CX.

1. $\log 2^3 = 3 \times \log 2$
$= 3 \times 0.3010$
$= 0.9030.$

2. $\log 5^2 = 2 \times \log 5$
$= 2 \times 0.6990$
$= 1.3980.$

3. $\log 7^4 = 4 \times \log 7$
$= 4 \times 0.8451$
$= 3.3804.$

4. $\log 3^8 = 8 \times \log 3$
$= 8 \times 0.4771$
$= 3.8168.$

5. $\log 7^3 = 3 \times \log 7$
$= 3 \times 0.8451$
$= 2.5353.$

6. $\log 5^5 = 5 \times \log 5.$
$= 5 \times 0.6990$
$= 3.4950.$

7. $\log 2^{\frac{1}{3}} = \frac{1}{3}$ of $\log 2$
$= \frac{1}{3}$ of 0.3010
$= 0.1003.$

8. $\log 5^{\frac{1}{2}} = \frac{1}{2}$ of $\log 5$
$= \frac{1}{2}$ of 0.6990
$= 0.3495.$

9. $\log 3^{\frac{1}{8}} = \frac{1}{8}$ of $\log 3$
$= \frac{1}{8}$ of 0.4771
$= 0.0596.$

10. $\log 7^{\frac{1}{5}} = \frac{1}{5}$ of $\log 7$
$= \frac{1}{5}$ of 0.8451
$= 0.1690.$

11. $\log 5^{\frac{1}{5}} = \frac{1}{5}$ of $\log 5$
$= \frac{1}{5}$ of 0.6990
$= 0.1398.$

12. $\log 7^{\frac{1}{11}} = \frac{1}{11}$ of $\log 7$
$= \frac{1}{11}$ of 0.8451
$= 0.0768.$

13. $\log 2^{\frac{3}{4}} = \frac{3}{4}$ of $\log 2$
$= \frac{3}{4}$ of 0.3010
$= 0.2258.$

14. $\log 5^{\frac{2}{3}} = \frac{2}{3}$ of $\log 5$
$= \frac{2}{3}$ of 0.6990
$= 0.4660.$

15. $\log 3^{\frac{3}{7}} = \frac{3}{7}$ of $\log 3$
$= \frac{3}{7}$ of 0.4771
$= 0.2045.$

16. $\log 7^{\frac{2}{7}} = \frac{2}{7}$ of $\log 7$
$= \frac{2}{7}$ of 0.8451
$= 0.2415.$

17. $\log 5^{\frac{5}{3}} = \frac{5}{3}$ of $\log 5$
$= \frac{5}{3}$ of 0.6990
$= 1.1650.$

18. $\log 3^{\frac{9}{11}} = \frac{9}{11}$ of $\log 3$
$= \frac{9}{11}$ of 0.4771
$= 0.3904.$

19. $\log 7^{\frac{7}{2}} = \frac{7}{2}$ of $\log 7$
$= \frac{7}{2}$ of 0.8451
$= 2.9579.$

20. $\log 3^{\frac{4}{3}} = \frac{4}{3}$ of $\log 3$
$= \frac{4}{3}$ of 0.4771
$= 0.6361.$

21. $\log 5^{\frac{7}{2}} = \frac{7}{2}$ of $\log 5$
$= \frac{7}{2}$ of 0.6990
$= 2.4465.$

22. $\log 2^{\frac{11}{7}} = \frac{11}{7}$ of $\log 2$
$= \frac{11}{7}$ of 0.3010
$= 0.4730.$

23. $\log 5^{\frac{3}{4}} = \frac{3}{4}$ of $\log 5$
$= \frac{3}{4}$ of 0.6990
$= 0.5243.$

24. $\log 7^{\frac{11}{7}} = \frac{11}{7}$ of $\log 7$
$= \frac{11}{7}$ of 0.8451
$= 1.3280.$

25. $\log 21^{\frac{7}{8}} = \frac{7}{8}$ of $\log(7 \times 3).$
$\log 7 = 0.8451$
$\log 3 = 0.4771$
$\log 21 = 1.3222$
$\frac{7}{8}$ of $1.3222 = 1.1569.$

EXERCISE CXI.

1. $\log \frac{2}{5} = \log 2 + \text{colog}\, 5.$
$\log 2 = 0.3010$
$\text{colog}\, 5 = 9.3010 - 10$
$\therefore \log \frac{2}{5} = 9.6020 - 10$

2. $\log \frac{2}{7} = \log 2 + \text{colog}\, 7.$
$\log 2 = 0.3010$
$\text{colog}\, 7 = 9.1549 - 10$
$\therefore \log \frac{2}{7} = 9.4559 - 10$

3. $\log \frac{3}{5} = \log 3 + \text{colog}\, 5.$
$\log 3 = 0.4771$
$\text{colog}\, 5 = 9.3010 - 10$
$\therefore \log \frac{3}{5} = 9.7781 - 10$

4. $\log \frac{3}{7} = \log 3 + \text{colog}\, 7.$
$\log 3 = 0.4771$
$\text{colog}\, 7 = 9.1549 - 10$
$\therefore \log \frac{3}{7} = 9.6320 - 10$

5. $\log \frac{5}{7} = \log 5 + \text{colog}\, 7.$
$\log 5 = 0.6990$
$\text{colog}\, 7 = 9.1549 - 10$
$\therefore \log \frac{5}{7} = 9.8539 - 10$

6. $\log \frac{7}{5} = \log 7 + \text{colog}\, 5.$
$\log 7 = 0.8451$
$\text{colog}\, 5 = 9.3010 - 10$
$\therefore \log \frac{7}{5} = 0.1461$

7. $\log \frac{5}{3} = \log 5 + \text{colog}\, 3.$
$\log 5 = 0.6990$
$\text{colog}\, 3 = 9.5229 - 10$
$\therefore \log \frac{5}{3} = 0.2219$

8. $\log \frac{5}{2} = \log 5 + \text{colog}\, 2.$
$\log 5 = 0.6990$
$\text{colog}\, 2 = 9.6990 - 10$
$\therefore \log \frac{5}{2} = 0.3980$

9. $\log \frac{7}{3} = \log 7 + \text{colog}\, 3.$
$\log 7 = 0.8451$
$\text{colog}\, 3 = 9.5229 - 10$
$\therefore \log \frac{7}{3} = 0.3680$

10. $\log \frac{7}{2} = \log 7 + \text{colog}\, 2.$
$\log 7 = 0.8451$
$\text{colog}\, 2 = 9.6990 - 10$
$\therefore \log \frac{7}{2} = 0.5441$

11. $\log \frac{3}{2} = \log 3 + \text{colog}\, 2.$
$\log 3 = 0.4771$
$\text{colog}\, 2 = 9.6990 - 10$
$\therefore \log \frac{3}{2} = 0.1761$

12. $\log \frac{7}{0.5} = \log 7 + \text{colog}\, 0.5.$
$\log\ 7 = 0.8451$
$\text{colog}\, 0.5 = 0.3010$
$\therefore \log \frac{7}{0.5} = 1.1461$

13. $\log \frac{0.05}{3} = \log 0.05 + \text{colog}\, 3.$
$\log 0.05 = 8.6990 - 10$
$\text{colog}\ 3 = 9.5229 - 10$
$\therefore \log \frac{0.05}{3} = 8.2219 - 10$

14. $\log\frac{0.005}{2} = \log 0.005 + \text{colog}\, 2.$

$$\begin{aligned} \log 0.005 &= 7.6990 - 10 \\ \text{colog}\quad 2 &= 9.6990 - 10 \\ \therefore \log\frac{0.005}{2} &= 7.3980 - 10 \end{aligned}$$

15. $\log\frac{0.07}{5} = \log 0.07 + \text{colog}\, 5.$

$$\begin{aligned} \log\ 0.07 &= 8.8451 - 10 \\ \text{colog}\quad 5 &= 9.3010 - 10 \\ \therefore \log\frac{0.07}{5} &= 8.1461 - 10 \end{aligned}$$

16. $\log\frac{5}{0.07} = \log 5 + \text{colog}\, 0.07.$

$$\begin{aligned} \log\quad 5 &= 0.6990 \\ \text{colog}\ 0.07 &= 1.1549 \\ \therefore \log\frac{5}{0.07} &= 1.8539 \end{aligned}$$

17. $\log\frac{3}{0.007} = \log 3 + \text{colog}\, 0.007.$

$$\begin{aligned} \log\quad 3 &= 0.4771 \\ \text{colog}\, 0.007 &= 2.1549 \\ \therefore \log\frac{3}{0.007} &= 2.6320 \end{aligned}$$

18. $\log\frac{0.003}{7} = \log 0.003 + \text{colog}\, 7.$

$$\begin{aligned} \log 0.003 &= 7.4771 - 10 \\ \text{colog}\quad 7 &= 9.1549 - 10 \\ \therefore \log\frac{0.003}{7} &= 6.6320 - 10 \end{aligned}$$

19. $\log\frac{0.05}{0.003} = \log 0.05 + \text{colog}\, 0.003.$

$$\begin{aligned} \log\ 0.05 &= 8.6990 - 10 \\ \text{colog}\, 0.003 &= 2.5229 \\ \therefore \log\frac{0.05}{0.003} &= 1.2219 \end{aligned}$$

20. $\log\frac{0.007}{0.02} = \log 0.007 + \text{colog}\, 0.02.$

$$\begin{aligned} \log 0.007 &= 7.8451 - 10 \\ \text{colog}\ 0.02 &= 1.6990 \\ \therefore \log\frac{0.007}{0.02} &= 9.5441 - 10 \end{aligned}$$

21. $\log\frac{0.02}{0.007} = \log 0.02 + \text{colog}\, 0.007.$

$$\begin{aligned} \log\ 0.02 &= 8.3010 - 10 \\ \text{colog}\, 0.007 &= 2.1549 \\ \therefore \log\frac{0.02}{0.007} &= 0.4559 \end{aligned}$$

22. $\log\frac{0.005}{0.07} = \log 0.005 + \text{colog}\, 0.07.$

$$\begin{aligned} \log 0.005 &= 7.6990 - 10 \\ \text{colog}\ 0.07 &= 1.1549 \\ \therefore \log\frac{0.005}{0.07} &= 8.8539 - 10 \end{aligned}$$

23. $\log\frac{0.03}{7} = \log 0.03 + \text{colog}\, 7.$

$$\begin{aligned} \log\ 0.03 &= 8.4771 - 10 \\ \text{colog}\quad 7 &= 9.1549 - 10 \\ \therefore \log\frac{0.03}{7} &= 7.6320 - 10 \end{aligned}$$

24. $\log\frac{0.0007}{0.2} = \log 0.0007 + \text{colog}\, 0.2$

$$\begin{aligned} \log 0.0007 &= 6.8451 - 10 \\ \text{colog}\quad 0.2 &= 0.6990 \\ \therefore \log\frac{0.0007}{0.2} &= 7.5441 - 10 \end{aligned}$$

25. $\log\frac{0.02^2}{3^3} = \log 0.02^2 + \text{colog}\, 3^3.$

$$\begin{aligned} \log 0.02^2 &= 6.6020 - 10 \\ \text{colog}\quad 3^3 &= 8.5687 - 10 \\ \therefore \log\frac{0.02^2}{3^3} &= 5.1707 - 10 \end{aligned}$$

26. $\log \frac{3^3}{0.02^2} = \log 3^3 + \operatorname{colog} 0.02^2.$

$$\begin{aligned} \log \quad 3^3 &= 1.4313 \\ \operatorname{colog} 0.02^2 &= 3.3980 \\ \therefore \log \frac{3^3}{0.02^2} &= 4.8293 \end{aligned}$$

27. $\log \frac{7^3}{0.02^2} = \log 7^3 + \operatorname{colog} 0.02^2.$

$$\begin{aligned} \log \quad 7^3 &= 2.5353 \\ \operatorname{colog} 0.02^2 &= 3.3980 \\ \therefore \log \frac{7^3}{0.02^2} &= 5.9333 \end{aligned}$$

28. $\log \frac{0.07^3}{0.003^3} = \log 0.07^3 + \operatorname{colog} 0.003^3.$

$$\begin{aligned} \log \quad 0.07^3 &= 6.5353 - 10 \\ \operatorname{colog} 0.003^3 &= 7.5687 \\ \therefore \log \frac{0.07^3}{0.003^3} &= 4.1040 \end{aligned}$$

29. $\log \frac{0.005^2}{7^3} = \log 0.005^2 + \operatorname{colog} 7^3.$

$$\begin{aligned} \log 0.005^2 &= 5.3980 - 10 \\ \operatorname{colog} \quad 7^3 &= 7.4647 - 10 \\ \therefore \log \frac{0.005^2}{7^3} &= 2.8627 - 10 \end{aligned}$$

30. $\log \frac{7^3}{0.005^2} = \log 7^3 + \operatorname{colog} 0.005^2.$

$$\begin{aligned} \log \quad 7^3 &= 2.5353 \\ \operatorname{colog} 0.005^2 &= 4.6020 \\ \therefore \log \frac{7^3}{0.005^2} &= 7.1373 \end{aligned}$$

Exercise CXII.

1.

$\log \quad 60 = 1.7782.$

2.

$\log \quad 101 = 2.0043.$

3.

$\log \quad 999 = 2.9996.$

4.

$$\begin{aligned} \log 9901 &= 3.9956 + \tfrac{1}{10} \text{ of } 0.0005 \\ &= 3.9957. \end{aligned}$$

5.

$$\begin{aligned} \log 5406 &= 3.7324 + \tfrac{6}{10} \text{ of } 0.0008 \\ &= 3.7329. \end{aligned}$$

6.

$\log \quad 3780 = 3.5775.$

7.

$$\begin{aligned} \log 54327 &= 4.7348 + \tfrac{27}{100} \text{ of } 0.0008 \\ &= 4.7350. \end{aligned}$$

8.

$$\begin{aligned} \log 90801 &= 4.9581 + \tfrac{1}{100} \text{ of } 0.0005 \\ &= 4.9581. \end{aligned}$$

9.

$$\begin{aligned} \log 10001 &= 4.0000 + \tfrac{1}{100} \text{ of } 0.0043 \\ &= 4.0000. \end{aligned}$$

10.

$$\begin{aligned} \log 10010 &= 4.0000 + \tfrac{10}{100} \text{ of } 0.0043 \\ &= 4.0004. \end{aligned}$$

11.

log $70633 = 4.8488 + \frac{33}{100}$ of 0.0006
$= 4.8490.$

12.

log $12028 = 4.0792 + \frac{28}{100}$ of 0.0036
$= 4.0802.$

13.

log $0.00987 = 7.9943 - 10.$

14.

log $0.87701 = 9.9430 - 10.$

15.

log $1.0001 = 0.0000 + \frac{1}{100}$ of 0.0043
$= 0.0000.$

16.

log $877.08 = 2.9430 + \frac{8}{100}$ of 0.0005
$= 2.9430.$

17.

log $73.896 = 1.8681 + \frac{96}{100}$ of 0.0005
$= 1.8686.$

18.

log $7.0699 = 0.8488 + \frac{99}{100}$ of 0.0006
$= 0.8494.$

19.

log $0.0897 = 8.9528 - 10.$

20.

log $99.778 = 1.9987 + \frac{78}{100}$ of 0.0004
$= 1.9990.$

21.

Antilogarithm of 4.2488.
Number corresponding to 0.2488 is $1770 + \frac{1}{3}$ of $10 = 1773$.
$\therefore$ number required is 17730.

22.

Antilogarithm of 3.6330.
Number corresponding to 0.6330 is $4290 + \frac{5}{10}$ of $10 = 4295$.
$\therefore$ number required is 4295.

23.

Antilogarithm of 2.5310.
Number corresponding to 0.5310 is $3390 + \frac{8}{13}$ of $10 = 3396$.
$\therefore$ number required is 339.6.

24.

Antilogarithm of 1.9484.
Number corresponding to 0.9484 is 8880.
$\therefore$ number required is 88.8.

25.

Antilogarithm of 4.7317.
Number corresponding to 0.7317 is $5390 + \frac{1}{8}$ of $10 = 5391$.
$\therefore$ number required is 53910.

26.

Antilogarithm of 1.9730.
Number corresponding to 0.9730 is $9390 + \frac{3}{4}$ of $10 = 9398$.
$\therefore$ number required is 93.98.

27.

Antilogarithm of $9.8800 - 10$.
Number corresponding to 0.8800 is $7580 + \frac{3}{5}$ of $10 = 7586$.
$\therefore$ number required is 0.7586.

28.

Antilogarithm of 0.2787.
Number corresponding to 0.2787 is $1890 + \frac{22}{23}$ of $10 = 1900$.
$\therefore$ number required is 1.9.

29.

Antilogarithm of 9.0410—10.

Number corresponding to 0.0410 is $1090 + \frac{36}{40}$ of $10 = 1099$.

$\therefore$ number required is 0.1099.

30.

Antilogarithm of 9.8420—10.

Number corresponding to 0.8420 is 6950.

$\therefore$ number required is 0.6950.

31.

Antilogarithm of 7.0216—10.

Number corresponding to 0.0216 is $1050 + \frac{4}{41}$ of $10 = 1051$.

$\therefore$ number required is 0.001051.

32.

Antilogarithm of 8.6580—10.

Number corresponding to 0.6580 is 4550.

$\therefore$ number required is 0.0455.

33.

$948.76 \times 0.043875.$

$$\begin{aligned} \log\ 948.76 &= 2.9772 \\ \log 0.043875 &= \underline{8.6423 - 10} \\ &\quad 1.6195 \\ &= \log 41.64. \end{aligned}$$

34.

$3.4097 \times 0.0087634.$

$$\begin{aligned} \log\ 3.4097 &= 0.5328 \\ \log 0.0087634 &= \underline{7.9427 - 10} \\ &\quad 8.4755 - 10 \\ &= \log 0.02989. \end{aligned}$$

35.

$830.75 \times 0.0003769.$

$$\begin{aligned} \log\ 830.75 &= 2.9195 \\ \log 0.0003769 &= \underline{6.5762 - 10} \\ &\quad 9.4957 - 10 \\ &= \log 0.3131. \end{aligned}$$

36.

$8.4395 \times 0.98274.$

$$\begin{aligned} \log\ 8.4395 &= 0.9263 \\ \log 0.98274 &= \underline{9.9925 - 10} \\ &\quad 0.9188 \\ &= \log 8.294. \end{aligned}$$

37.

$7564 \times (-0.003764).$

$$\begin{aligned} \log\ 7564 &= 3.8787 \\ \log(-0.003764) &= \underline{7.5756^{n} - 10} \\ &\quad 1.4543^{n} \\ &= \log -28.47. \end{aligned}$$

38.

$3.7648 \times (-0.083497).$

$$\begin{aligned} \log\ 3.7648 &= 0.5757 \\ \log(-0.083497) &= \underline{8.9217^{n} - 10} \\ &\quad 9.4974^{n} - 10 \\ &= \log -0.3144. \end{aligned}$$

39.

$-5.840359 \times (-0.00178).$

$$\begin{aligned} \log(-5.840359) &= 0.7664^{n} \\ \log\ (-0.00178) &= \underline{7.2504^{n} - 10} \\ &\quad 8.0168 - 10 \\ &= \log 0.0104 \end{aligned}$$

40.

$-8945.07 \times 73.846.$

$$\begin{aligned} \log(-8945.07) &= 3.9515^{n} \\ \log\ 73.846 &= \underline{1.8683} \\ &\quad 5.8198^{n} \\ &= \log -660600. \end{aligned}$$

41.

$$\frac{70654}{54013} = \log 70654 + \operatorname{colog} 54013.$$

$$\begin{aligned} \log \quad 70654 &= 4.8491 \\ \operatorname{colog} 54013 &= \underline{5.2675 - 10} \\ &\ 0.1166 \\ &= \log 1.308. \end{aligned}$$

42.

$$\frac{58706}{93078} = \log 58706 + \operatorname{colog} 93078.$$

$$\begin{aligned} \log \quad 58706 &= 4.7686 \\ \operatorname{colog} 93078 &= \underline{5.0312 - 10} \\ &\ 9.7998 - 10 \\ &= \log 0.6307. \end{aligned}$$

43.

$$\frac{8.32165}{0.07891} = \log 8.32165 + \operatorname{colog} 0.07891.$$

$$\begin{aligned} \log \quad 8.32165 &= 0.9202 \\ \operatorname{colog} 0.07891 &= \underline{1.1028} \\ &\ 2.0230 \\ &= \log 105.4. \end{aligned}$$

44.

$$\frac{65039}{90761} = \log 65039 + \operatorname{colog} 90761.$$

$$\begin{aligned} \log \quad 65039 &= 4.8132 \\ \operatorname{colog} \quad 90761 &= \underline{5.0421 - 10} \\ &\ 9.8553 - 10 \\ &= \log 0.7167. \end{aligned}$$

45.

$$\frac{7.652}{-0.06875} = \log 7.652 + \operatorname{colog}(-0.06875).$$

$$\begin{aligned} \log \quad 7.652 &= \ 0.8838 \\ \operatorname{colog}(-0.06875) &= \underline{11.1627^n - 10} \\ &\ 2.0465^n \\ &= \log -111.3. \end{aligned}$$

46.

$$\frac{0.07654}{83.947 \times 0.8395} = \log 0.07654 + \operatorname{colog} 83.947 + \operatorname{colog} 0.8395.$$

$$\begin{aligned} \log \quad 0.07654 &= 8.8839 - 10 \\ \operatorname{colog} \quad 83.947 &= 8.0760 - 10 \\ \operatorname{colog} \quad 0.8395 &= \underline{0.0759 } \\ &\ 7.0358 - 10 \\ &= \log 0.001086. \end{aligned}$$

47.

$$\frac{7564 \times 0.07643}{8093 \times 0.09817}.$$

$$\begin{aligned} \log \quad 7564 &= 3.8787 \\ \log \quad 0.07643 &= 6.8832 - 10 \\ \operatorname{colog} \quad 8093 &= 6.0919 - 10 \\ \operatorname{colog} 0.09817 &= \underline{1.0080 } \\ &\ 9.8618 - 10 \\ &= \log 0.7277. \end{aligned}$$

48.

$$\frac{89 \times 753 \times 0.0097}{36709 \times 0.08497}.$$

$$\begin{aligned} \log \quad 89 &= 1.9494 \\ \log \quad 753 &= 2.8768 \\ \log \quad 0.0097 &= 7.4352 - 10 \\ \operatorname{colog} \quad 36709 &= 5.4352 - 10 \\ \operatorname{colog} 0.08497 &= \underline{1.0708 } \\ &\ 9.3190 - 10 \\ &= \log 0.2084. \end{aligned}$$

49.

$$\frac{413 \times 8.17 \times 3182}{915 \times 728 \times 2.315}.$$

$$\begin{aligned}
\log\ 413 &= 2.6160\\
\log\ 8.17 &= 0.9122\\
\log\ 3182 &= 3.5027\\
\text{colog}\ 915 &= 7.0386 - 10\\
\text{colog}\ 728 &= 7.1379 - 10\\
\text{colog}\ 2.315 &= 9.6354 - 10\\
\hline
& \ 0.8428\\
&= \log 6.963.
\end{aligned}$$

50.

$$\frac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}.$$

$$\begin{aligned}
\log\ 212 &= 2.3263\\
\log\ (-6.12) &= 0.7868^n\\
\log\ (-2008) &= 3.3028^n\\
\text{colog}\ 365 &= 7.4377 - 10\\
\text{colog}\ (-531) &= 7.2749^n - 10\\
\text{colog}\ 2.576 &= 9.5891 - 10\\
\hline
& \ 0.7176^n\\
&= \log -5.219.
\end{aligned}$$

51.

$$\begin{aligned}
\log 6.05 &= 0.7818\\
&\ \ \ \ \ \ \ \ \ 3\\
\log 6.05^3 &= 2.3454\\
&= \log 221.5.
\end{aligned}$$

52.

$$\begin{aligned}
\log 1.051 &= 0.0216\\
&\ \ \ \ \ \ \ \ \ 7\\
\log 1.051^7 &= 0.1512\\
&= \log 1.416.
\end{aligned}$$

53.

$$\begin{aligned}
\log 1.1768 &= 0.0707\\
&\ \ \ \ \ \ \ \ \ 5\\
\log 1.1768^5 &= 0.3535\\
&= \log 2.257.
\end{aligned}$$

54.

$$\begin{aligned}
\log 1.3178 &= 0.1198\\
&\ \ \ \ \ \ \ 10\\
\log 1.3178^{10} &= 1.1980\\
&= \log 15.78.
\end{aligned}$$

55.

$$\begin{aligned}
\log 0.78765 &= 9.8963 - 10\\
&\ \ \ \ \ \ \ \ \ 6\\
\log 0.78765^6 &= 9.3778 - 10\\
&= \log 0.2387.
\end{aligned}$$

56.

$$\begin{aligned}
\log 0.691 &= 9.8395 - 10\\
&\ \ \ \ \ \ \ \ \ 9\\
\log 0.691^9 &= 8.5555 - 10\\
&= \log 0.03593.
\end{aligned}$$

57.

$$\begin{aligned}
\log\left(\tfrac{73}{61}\right)^{11} &= 11(\log 73 + \text{colog } 61)\\
&= 11(1.8633 + 8.2147 - 10)\\
&= 0.8580\\
&= \log 7.212.
\end{aligned}$$

58.

$$\begin{aligned}
\log\left(\tfrac{14}{51}\right)^{7} &= 7(\log 14 + \text{colog } 51)\\
&= 7(1.1461 + 8.2924 - 10)\\
&= 6.0695 - 10\\
&= \log 0.0001174.
\end{aligned}$$

59.

$$\begin{aligned}
\left(10\tfrac{2}{3}\right)^4 &= \left(\tfrac{32}{3}\right)^4.\\
\log\left(\tfrac{32}{3}\right)^4 &= 4(\log 32 + \text{colog } 3)\\
&= 4(1.5051 + 9.5229 - 10)\\
&= 4.1120\\
&= \log 12940.
\end{aligned}$$

60.

$$\begin{aligned}
\left(1\tfrac{7}{9}\right)^8 &= \left(\tfrac{16}{9}\right)^8.\\
\log\left(\tfrac{16}{9}\right)^8 &= 8(\log 16 + \text{colog } 9)\\
&= 8(1.2041 + 9.0458 - 10)\\
&= 1.9992\\
&= \log 99.82
\end{aligned}$$

61. $$\begin{aligned}\log\left(\tfrac{951}{823}\right)^6 &= 6(\log 951 + \operatorname{colog} 823)\\ &= 6(2.9782 + 7.0846 - 10)\\ &= 0.3768\\ &= \log 2.381.\end{aligned}$$

62. $$\begin{aligned}\left(7\tfrac{6}{11}\right)^{0.38} &= \left(\tfrac{83}{11}\right)^{0.38}.\\ \log\left(\tfrac{83}{11}\right)^{0.38} &= 0.38(\log 83 + \operatorname{colog} 11)\\ &= 0.38(1.9191 + 8.9586 - 10)\\ &= 0.3335\\ &= \log 2.155.\end{aligned}$$

63. $$\begin{aligned}\left(3\tfrac{27}{31}\right)^{4.17} &= \left(\tfrac{120}{31}\right)^{4.17}.\\ \log\left(\tfrac{120}{31}\right)^{4.17} &= 4.17(\log 120 + \operatorname{colog} 31)\\ &= 4.17(2.0792 + 8.5086 - 10)\\ &= 2.4511\\ &= \log 282.6.\end{aligned}$$

64. $$\begin{aligned}\left(1\tfrac{2}{11}\right)^{3.2} &= \left(\tfrac{13}{11}\right)^{3.2}.\\ \log\left(\tfrac{13}{11}\right)^{3.2} &= 3.2(\log 13 + \operatorname{colog} 11)\\ &= 3.2(1.1139 + 8.9586 - 10)\\ &= 0.2320\\ &= \log 1.706.\end{aligned}$$

65. $$\begin{aligned}\left(8\tfrac{3}{4}\right)^{2.3} &= \left(\tfrac{35}{4}\right)^{2.3}.\\ \log\left(\tfrac{35}{4}\right)^{2.3} &= 2.3(\log 35 + \operatorname{colog} 4)\\ &= 2.3(1.5441 + 9.3979 - 10)\\ &= 2.1666\\ &= \log 146.8.\end{aligned}$$

66. $$\begin{aligned}\left(5\tfrac{31}{37}\right)^{0.375} &= \left(\tfrac{216}{37}\right)^{0.375}.\\ \log\left(\tfrac{216}{37}\right)^{0.375} &= 0.375(\log 216 + \operatorname{colog} 37)\\ &= 0.375(2.3345 + 8.4318 - 10)\\ &= 0.2874\\ &= \log 1.938.\end{aligned}$$

67. $\log 7 = 0.8451.$

$$\begin{array}{rl} & 3\overline{)0.8451} \\ \log 7^{\frac{1}{3}} = & 0.2817 \\ = & \log 1.913. \end{array}$$

68. $\log 11 = 1.0414.$

$$\begin{array}{rl} & 5\overline{)1.0414} \\ \log 11^{\frac{1}{5}} = & 0.2083 \\ = & \log 1.616. \end{array}$$

69. $\log 783 = 2.8938.$

$$\begin{array}{rl} & 3\overline{)2.8938} \\ \log 783^{\frac{1}{3}} = & 0.9646 \\ = & \log 9.218. \end{array}$$

70. $\log 8379 = 3.9232.$

$$\begin{array}{rl} & 10\overline{)3.9232} \\ \log 8379^{\frac{1}{10}} = & 0.3923 \\ = & \log 2.468 \end{array}$$

71. $\log 906.80 = 2.9575.$

$$\begin{array}{rl} & 4\overline{)2.9575} \\ \log 906.80^{\frac{1}{4}} = & 0.7394 \\ = & \log 5.487. \end{array}$$

72. $\log 8.1904 = 0.9133.$

$$\begin{array}{rl} & 5\overline{)0.9133} \\ \log 8.1904^{\frac{1}{5}} = & 0.1826 \\ = & \log 1.523. \end{array}$$

73. $\log 0.17643 = 9.2466 - 10$

$$\begin{array}{rr} & 5 \\ \hline & 46.2330 - 50 \\ & 10. \qquad - 10 \\ & 6\overline{)56.2330 - 60} \\ \log 0.17643^{\frac{5}{6}} = & 9.3722 - 10 \\ = & \log 0.2356. \end{array}$$

74. $\log 2.5637 = 0.4088$

$$\begin{array}{rr} & 3 \\ & 11\overline{)1.2264} \\ \log 2.5637^{\frac{3}{11}} = & 0.1115 \\ = & \log 1.293. \end{array}$$

75. $\log \left(\frac{431}{788}\right)^{\frac{1}{2}} = \frac{1}{2}(\log 431 + \operatorname{colog} 788).$

$$\begin{array}{lr} \log\ 431 = & 2.6345 \\ \operatorname{colog} 788 = & 7.1035 - 10 \\ \hline & 9.7380 - 10 \\ & 10. \qquad - 10 \\ & 2\overline{)19.7380 - 20} \\ & 9.8690 - 10 = \log 0.7397. \end{array}$$

76. $\log \left(\frac{71}{43406}\right)^{\frac{4}{7}} = \frac{4}{7}(\log 71 + \operatorname{colog} 43406).$

$$\begin{array}{lr} \log\ 71 = & 1.8513 \\ \operatorname{colog} 43406 = & 5.3624 - 10 \\ \hline & 7.2137 - 10 \\ & 4 \\ \hline & 28.8548 - 40 \\ & 30. \qquad - 30 \\ & 7\overline{)58.8548 - 70} \\ & 8.4078 - 10 = \log 0.02558. \end{array}$$

77. $(9\frac{21}{43})^{\frac{1}{5}} = (\frac{408}{43})^{\frac{1}{5}}$.

$$
\begin{aligned}
\log (\tfrac{408}{43})^{\frac{1}{5}} &= \tfrac{1}{5}(\log 408 + \text{colog } 43).\\
\log \quad 408 &= 2.6107\\
\text{colog} \quad 43 &= \underline{8.3665 - 10}\\
5)\underline{0.9772}&\\
0.1954 &= \log 1.568.
\end{aligned}
$$

78. $(11\frac{21}{71})^{\frac{4}{5}} = (\frac{802}{71})^{\frac{4}{5}}$.

$$
\begin{aligned}
\log (\tfrac{802}{71})^{\frac{4}{5}} &= \tfrac{4}{5}(\log 802 + \text{colog } 71).\\
\log \quad 802 &= 2.9042\\
\text{colog} \quad 71 &= \underline{8.1487 - 10}\\
&\ 1.0529\\
\tfrac{4}{5}(1.0529) &= 0.8423 = \log 6.955.
\end{aligned}
$$

79. $\sqrt[5]{\dfrac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285^{\frac{1}{3}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{2}}}}$.

$$
\begin{aligned}
\log 0.0075433^2 &= 5.7552 - 10\\
\log \quad 78.343 &= 1.8940\\
\log \quad 8172.4^{\frac{1}{3}} &= 1.3041\\
\log \quad 0.00052 &= 6.7160 - 10\\
\text{colog} \quad 64285^{\frac{1}{3}} &= 8.3973 - 10\\
\text{colog} \quad 15427^4 &= 1.2468 - 10\\
\text{colog} \quad 0.001 &= 3.0000\\
\text{colog} \quad 586.79^{\frac{1}{2}} &= \underline{8.6158 - 10}\\
5)\underline{36.9292 - 50}&\\
7.3858 - 10 &= \log 0.002431.
\end{aligned}
$$

80. $\sqrt[5]{\dfrac{15.832^3 \times 5793.6^{\frac{1}{3}} \times 0.78426}{0.000327^{\frac{1}{3}} \times 768.94^2 \times 3015.3 \times 0.007^{\frac{1}{2}}}}$.

$$
\begin{aligned}
\log \quad 15.832^3 &= 3.5988\\
\log \quad 5793.6^{\frac{1}{3}} &= 1.2543\\
\log \quad 0.78426 &= 9.8445 - 10\\
\text{colog}\ 0.000327^{\frac{1}{3}} &= 1.1618\\
\text{colog} \quad 768.94^2 &= 4.2282 - 10\\
\text{colog} \quad 3015.3 &= 6.5207 - 10\\
\text{colog} \quad 0.007^{\frac{1}{2}} &= \underline{1.0774 \qquad\ }\\
&\ 27.7357 - 30\\
&\ \underline{20. \qquad - 20}\\
5)\underline{47.7357 - 50}&\\
9.5471 - 10 &= \log 0.3525.
\end{aligned}
$$

81. $\sqrt[5]{\dfrac{7.1895 \times 4764.2^2 \times 0.00326^5}{0.00048953 \times 457^3 \times 5764.4^2}}.$

$$\begin{array}{lrl}
\log & 7.1895 = & 0.8566 \\
\log & 4764.2^2 = & 7.3558 \\
\log & 0.00326^5 = & 7.5660 - 20 \\
\text{colog} & 0.00048953 = & 3.3102 \\
\text{colog} & 457^3 = & 2.0203 - 10 \\
\text{colog} & 5764.4^2 = & \underline{2.4786 - 10} \\
 & & 23.5875 - 40 \\
 & & \underline{10. \quad\quad - 10} \\
 & 5) & \underline{33.5875 - 50} \\
 & & 6.7175 - 10 = \log 0.0005218.
\end{array}$$

82. $\sqrt[5]{\dfrac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{2}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}.$

$$\begin{array}{lrl}
\log & 3.1416 = & 0.4971 \\
\log & 4771.21 = & 3.6786 \\
\log & 2.7183^{\frac{1}{2}} = & 0.2172 \\
\text{colog} & 30.103^4 = & 4.0856 - 10 \\
\text{colog} & 0.4343^{\frac{1}{2}} = & 0.1811 \\
\text{colog} & 69.897^4 = & \underline{2.6220 - 10} \\
 & & 11.2816 - 20 \\
 & & \underline{30. \quad\quad - 30} \\
 & 5) & \underline{41.2816 - 50} \\
 & & 8.2563 - 10 = \log 0.01804.
\end{array}$$

83. $\sqrt[7]{\dfrac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}.$

$$\begin{array}{lrl}
\log & 0.03271^2 = & 7.0292 - 10 \\
\log & 53.429 = & 1.7278 \\
\log & 0.77542^3 = & 9.6688 - 10 \\
\text{colog} & 32.769 = & 8.4845 - 10 \\
\text{colog} & 0.000371^4 = & \underline{13.7224} \\
 & 7) & \underline{10.6327} \\
 & & 1.5190 = \log 33.04.
\end{array}$$

84. $\sqrt[3]{\dfrac{732.056^2 \times 0.0003572^4 \times 89793}{42.2798^3 \times 3.4574 \times 0.0026518^5}}.$

$$\begin{array}{lrl}
\log & 732.056^2 = & 5.7290 \\
\log & 0.0003572^4 = & 6.2116 - 20 \\
\log & 89793 = & 4.9533 \\
\text{colog} & 42.2798^3 = & 5.1217 - 10 \\
\text{colog} & 3.4574 = & 9.4612 - 10 \\
\text{colog} & 0.0026518^5 = & \underline{12.8825} \\
 & 3) & \underline{4.3593} \\
 & & 1.4531 = \log 28.39.
\end{array}$$

85. $\sqrt[3]{\dfrac{7932 \times 0.00657 \times 0.80464}{0.03274 \times 0.6428}}$.

$$\begin{array}{lrl}
\log & 7932 = & 3.8994 \\
\log & 0.00657 = & 7.8176 - 10 \\
\log & 0.80464 = & 9.9056 - 10 \\
\text{colog} & 0.03274 = & 1.4849 - 10 \\
\text{colog} & 0.6428 = & \underline{0.1919\qquad\quad} \\
& 3\,) & \underline{3.2994} \\
& & 1.0998 = \log 12.58.
\end{array}$$

86. $\sqrt[3]{\dfrac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}$.

$$\begin{array}{lrl}
\log & 7.1206 = & 0.8525 \\
\log & \sqrt{0.13274} = & 9.5615 - 10 \\
\log & 0.057389 = & 8.7588 - 10 \\
\text{colog} & \sqrt{0.43468} = & 0.1809 \\
\text{colog} & 17.385 = & 8.7599 - 10 \\
\text{colog} & \sqrt{0.0096372} = & \underline{1.0080\qquad\quad} \\
& 3\,) & \underline{29.1216 - 30} \\
& & 9.7072 - 10 = \log 0.5096.
\end{array}$$

87. $\left\{\dfrac{3.075526^2 \times 5771.2^{\frac{1}{2}} \times 0.0036984^{\frac{1}{5}} \times 7.74}{72258 \times 327.93^3 \times 86.97^5}\right\}^{\frac{3}{5}}$

$$\begin{array}{lrl}
\log & 3.075526^2 = & 0.9758 \\
\log & 5771.2^{\frac{1}{2}} = & 1.8806 \\
\log & 0.0036984^{\frac{1}{5}} = & 9.5136 - 10 \\
\log & 7.74 = & 0.8887 \\
\text{colog} & 72258 = & 5.1412 - 10 \\
\text{colog} & 327.93^3 = & 2.4526 - 10 \\
\text{colog} & 86.97^5 = & \underline{0.3030 - 10} \\
& & 1.1555 - 20 \\
& & \underline{\qquad\qquad\;\, 3} \\
& & 3.4665 - \;\;60 \\
& & \underline{40 \qquad\quad - \;\;40} \\
& 5\,) & \underline{43.4665 - 100} \\
& & 8.6933 - \;\;20 \\
& & = \log 0.000000000004936.
\end{array}$$

EXERCISE CXIII.

1. Write down the ratio compounded of 3 : 5 and 8 : 7. Which of these ratios is increased and which is diminished by the composition?

$$\frac{3}{5} \times \frac{8}{7} = \frac{24}{35}.$$

As $\frac{3}{5} = \frac{21}{35}$,

and $\frac{8}{7} = \frac{40}{35}$.

3 : 5 is increased.

8 : 7 is decreased.

2. Compound the duplicate ratio of 4 : 15 with the triplicate of 5 : 2.

$$\left(\frac{4}{15}\right)^2 = \frac{16}{225},$$
$$\left(\frac{5}{2}\right)^3 = \frac{125}{8},$$
$$\frac{16}{225} \times \frac{125}{8} = \frac{2 \times 5}{9} = \frac{10}{9}.$$

3. Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater or less inequality.

$a^2 : b^2$ is $>$ or $<$ $a : b$.

As $\frac{a^2}{b^2}$ is $>$ or $<$ $\frac{a}{b}$.

As $\frac{a}{b} \times \frac{a}{b}$ is $>$ or $<$ $\frac{a}{b}$.

As $\frac{a}{b}$ is $>$ or $<$ 1.

4. Arrange in order of magnitude the ratios 3 : 4; 23 : 25; 10 : 11; and 15 : 16.

$$3 : 4 = \frac{3300}{4400},$$
$$23 : 25 = \frac{4048}{4400},$$
$$10 : 11 = \frac{4000}{4400},$$
$$15 : 16 = \frac{4125}{4400}.$$

$\therefore$ the order of magnitude is 15 : 16, 23 : 25, 10 : 11, 3 : 4.

5. **Arrange in order of magnitude**

$$a+b : a-b \text{ and } a^2+b^2 : a^2-b^2, \text{ if } a>b.$$

$$a+b : a-b \text{ is} > \text{or} < a^2+b^2 : a^2-b^2.$$

As $\dfrac{a+b}{a-b}$ is $>$ or $<\dfrac{a^2+b^2}{a^2-b^2}$.

As $\dfrac{a^2+2ab+b^2}{a^2-b^2}$ is $>$ or $<\dfrac{a^2+b^2}{a^2-b^2}$.

As $a^2+2ab+b^2$ is $>$ or $< a^2+b^2$.

But $a^2+2ab+b^2$ is $> a^2+b^2$.

$\therefore a+b : a-b$ is $> a^2+b^2 : a^2-b^2$.

6. **Ratio compounded of**

$$3:5;\ 10:21;\ 14:15,$$
$$=\tfrac{3}{5}\times\tfrac{10}{21}\times\tfrac{14}{15}$$
$$=\tfrac{4}{15}$$
$$=4:15.$$

7. **Ratio compounded of**

$$7:9;\ 102:105;\ 15:17,$$
$$=\tfrac{7}{9}\times\tfrac{102}{105}\times\tfrac{15}{17}$$
$$=\tfrac{2}{3}$$
$$=2:3.$$

8. **Ratio compounded of**

$$\frac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3} \text{ and } \frac{a^2-ax+x^2}{a+x}$$
$$=\frac{a^4+a^2x^2+x^4}{a^4-x^4}$$
$$=(a^4+a^2x^2+x^4):(a^4-x^4).$$

9. **Ratio compounded of**

$$\frac{x^2-9x+20}{x^2-6x} \text{ and } \frac{x^2-13x+42}{x^2-5x}$$
$$=\frac{(x-5)(x-4)}{x(x-6)}\times\frac{(x-6)(x-7)}{x(x-5)}$$
$$=(x^2-11x+28):x^2.$$

10. Ratio compounded of

$$a+b:a-b;\ a^2+b^2:(a+b)^2;\ (a^2-b^2)^2:a^4-b^4,$$

$$=\frac{a+b}{a-b}\times\frac{a^2+b^2}{(a+b)^2}\times\frac{(a^2-b^2)^2}{a^4-b^4}$$

$$=\frac{a+b}{a-b}\times\frac{a^2+b^2}{(a+b)(a+b)}\times\frac{(a-b)(a+b)(a-b)(a+b)}{(a^2+b^2)(a+b)(a-b)}$$

$$=1:1.$$

11. Two numbers are in the ratio $2:3$, and if 9 is added to each, they are in the ratio $3:4$. Find the numbers.

Let $2x$ and $3x$ = the numbers.

Then $2x+9:3x+9::3:4$,

$8x+36:9x+27$.

$\therefore x=9$.

Hence, $2x=18$,

and $3x=27$.

12. Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

$$\frac{a}{b}=\left(\frac{a+c}{b+c}\right)^2.$$

This becomes, when $c^2=ab$,

$$\frac{a}{b}=\left(\frac{a+\sqrt{ab}}{b+\sqrt{ab}}\right)^2.$$

$$\frac{a}{b}=\frac{a^2+2a\sqrt{ab}+ab}{b^2+2b\sqrt{ab}+ab}$$

$$=\frac{a(a+2\sqrt{ab}+b)}{b(b+2\sqrt{ab}+a)}$$

$$=\frac{a}{b}.$$

13. Find two numbers in the ratio $3:4$, of which the sum is to the sum of their squares in the ratio of $7:50$.

Let $3x$ = first number,

and $4x$ = second number.

Then $3x+4x$ or $7x$ = sum.

$9x^2+16x^2$ or $25x^2$ = sum of squares.

$\therefore 7x:25x^2::7:50$, or $x:x^2::1:2$, or $1:x::1:2$.

$\therefore x=2$.

The required numbers are 6 and 8.

14. **If five gold coins and four silver ones are worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.**

Let $x =$ value of 1 gold coin,

and $y =$ value of 1 silver coin.

Then $5x + 4y = 3x + 12y$,

$$2x = 8y.$$

$$\therefore x = 4y.$$

That is, $x : y :: 4 : 1$.

15. **If eight gold and nine silver coins are worth as much as six gold and nineteen silver coins, find the ratio of the value of a silver coin to that of a gold one.**

Let $x =$ value of gold coin,

and $y =$ value of silver coin.

Then $8x + 9y = 6x + 19y$,

or $2x = 10y$.

$$\therefore x = 5y.$$

That is, $y : x :: 1 : 5$.

16. **There are two roads from A to B, one of them 14 miles longer than the other; and two roads from B to C, one of them 8 miles longer than the other. The distance from A to B is to the distance from B to C, by the shorter roads, as 1 to 2; by the longer roads, as 2 to 3. Find the distances.**

Let $x =$ shorter road from A to B,

and $x + 14 =$ longer road from A to B.

Then $y =$ shorter road from B to C,

and $y + 8 =$ longer road from B to C.

That is, $x : y :: 1 : 2$,

$$x + 14 : y + 8 :: 2 : 3.$$

$$\therefore 2x = y \qquad (1)$$

And $3x + 42 = 2y + 16$,

or $3x - 2y = -26$.

Substitute $2x$ for y, $x = 26$,

$$x + 14 = 40,$$

$$y = 52,$$

$$y + 8 = 60.$$

17. What must be added to each of the terms of the ratio $m:n$, that it may become equal to the ratio $p:q$?

$$\text{Let} \qquad x = \text{number to be added.}$$

$$\therefore \frac{m+x}{n+x} = \frac{p}{q}.$$

$$mq + qx = pn + px,$$

$$x(q-p) = pn - mq.$$

$$\therefore x = \frac{pn - mq}{q - p}.$$

18. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of 31 : 17. Find its dimensions.

Let $31x =$ number of rods in length,

and $17x =$ number of rods in width.

Then $31x \times 17x =$ number of square rods in area.

But $160 \times 5270 =$ number of square rods in area.

$$\therefore 31x \times 17x = 160 \times 5270.$$

$$527x^2 = 160 \times 5270,$$

$$x^2 = 1600,$$

$$x = 40,$$

$$31x = 1240 \text{ rods},$$

$$17x = 680 \text{ rods}.$$

Exercise CXIV.

1. If $a:b::c:d$,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply by $\frac{m}{n}$,

$$\frac{ma}{nb} = \frac{mc}{nd}.$$

That is,

$$ma:nb::mc:nd.$$

2. $3a+b:b::3c+d:d.$

If $a:b::c:d$, $\frac{a}{b} = \frac{c}{d}.$

Multiply by 3, $\frac{3a}{b} = \frac{3c}{d}.$

Add 1 to each side,

$$\frac{3a}{b} + 1 = \frac{3c}{d} + 1,$$

or $\frac{3a+b}{b} = \frac{3c+d}{d}.$

$$\therefore 3a+b:b::3c+d:d.$$

3. If $a : b :: c : d$,

then $$\frac{a}{b} = \frac{c}{d}.$$

Add 2 to each side,

$$\frac{a}{b} + 2 = \frac{c}{d} + 2.$$

$$\frac{a+2b}{b} = \frac{c+2d}{d}.$$

$$\therefore a + 2b : b :: c + 2d : d.$$

4. Since $a : b :: c : d$,

$$\frac{a}{b} = \frac{c}{d}.$$

Cubing, $$\frac{a^3}{b^3} = \frac{c^3}{d^3}.$$

$$\therefore a^3 : b^3 :: c^3 : d^3.$$

5. $a : a + b :: c : c + d$.

If $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$.

By inversion, $\frac{b}{a} = \frac{d}{c}$.

By composition,

$$\frac{b+a}{a} = \frac{d+c}{c}.$$

By inversion, $$\frac{a}{a+b} = \frac{c}{d+c}.$$

$$\therefore a : a + b :: c : c + d.$$

6. $a : a - b :: c : c - d$.

If $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$.

By inversion, $\frac{b}{a} = \frac{d}{c}$.

By division, $$\frac{a-b}{a} = \frac{c-d}{c}.$$

By inversion, $$\frac{a}{a-b} = \frac{c}{c-d}.$$

$$\therefore a : a - b :: c : c - d$$

7. If $a : b :: c : d$, $$\frac{a}{b} = \frac{c}{d}.$$

Multiply by $\frac{m}{n}$, $$\frac{ma}{nb} = \frac{mc}{nd}.$$

By composition and division,

$$\frac{ma + nb}{ma - nb} = \frac{mc + nd}{mc - nd}.$$

$$\therefore ma + nb : ma - nb :: mc + nd : mc - nd.$$

8. If $a : b :: c : d$, $$\frac{a}{b} = \frac{c}{d}.$$

$$\frac{a}{b} + \frac{3}{2} = \frac{c}{d} + \frac{3}{2},$$

$$\frac{2a + 3b}{2b} = \frac{2c + 3d}{2d},$$

$$\text{or } \frac{2a + 3b}{b} = \frac{2c + 3d}{d} \qquad (1)$$

Also, $$\frac{a}{b} - \frac{4}{3} = \frac{c}{d} - \frac{4}{3},$$

$$\frac{3a-4b}{3b}=\frac{3c-4d}{3d},$$

$$\text{or } \frac{3a-4b}{b}=\frac{3c-4d}{d} \qquad (2)$$

Dividing (1) by (2), $$\frac{2a+3b}{3a-4b}=\frac{2c+3d}{3c-4d}.$$

$$\therefore 2a+3b:3a-4b::2c+3d:3c-4d.$$

9. If $a:b::c:d$, $$\frac{a}{b}=\frac{c}{d}.$$

By squaring, $$\frac{a^2}{b^2}=\frac{c^2}{d^2}.$$

$$\therefore \frac{ma^2}{mb^2}=\frac{nc^2}{nd^2}.$$

Let $$\frac{ma^2}{mb^2}=r.$$

Then $$\frac{nc^2}{nd^2}=r.$$

Hence, $$ma^2=mb^2r, \text{ and } nc^2=nd^2r,$$

$$ma^2+nc^2=(mb^2+nd^2)\,r,$$

$$\text{and } \frac{ma^2+nc^2}{mb^2+nd^2}=r=\frac{a^2}{b^2}.$$

$$\therefore ma^2+nc^2:mb^2+nd^2::a^2:b^2.$$

10. If $a:b::c:d$, by alternation, $a:c::b:d$.

$$\frac{a}{c}=\frac{b}{d}.$$

$$\therefore \frac{a^2}{c^2}=\frac{b^2}{d^2}.$$

Also, $$\frac{a}{c}\times\frac{a}{c}=\frac{b}{d}\times\frac{a}{c}.$$

$$\therefore \frac{a^2}{c^2}=\frac{ab}{cd}=\frac{b^2}{d^2}.$$

$$\therefore \frac{ma^2}{mc^2}=\frac{nab}{ncd}=\frac{pb^2}{pd^2}.$$

$$\therefore \frac{ma^2+nab+pb^2}{mc^2+ncd+pd^2}=\frac{pb^2}{pd^2}=\frac{b^2}{d^2}.$$

$$\therefore ma^2+nab+pb^2:mc^2+ncd+pd^2::b^2:d^2.$$

11. If $\qquad a : b :: b : c,$

by composition, $\qquad a + b : a :: b + c : d;$

by alternation, $\qquad a + b : b + c :: a : b.$

12. If $a : b :: b : c,$ $\qquad \dfrac{a}{b} = \dfrac{b}{c}.$

Multiply by $\dfrac{a}{b}$, $\qquad \dfrac{a^2}{b^2} = \dfrac{ab}{bc},$

or $a^2 : b^2 :: ab : bc.$

By alternation, $\qquad a^2 : ab :: b^2 : bc.$

By composition, $\qquad a^2 + ab : ab :: b^2 + bc : bc.$

By alternation, $\qquad a^2 + ab : b^2 + bc :: ab : bc.$

Cancelling b in the terms of last ratio,

$$a^2 + ab : b^2 + bc :: a : c.$$

13. If $a : b :: b : c,$ $\qquad b^2 = ac.$

Multiply by $(a - c)$, $\qquad ab^2 - b^2c = a^2c - ac^2.$

Add $2abc$ to both sides,

$$ab^2 + 2abc - b^2c = a^2c + 2abc - ac^2.$$

Transpose $-b^2c$ and $-ac^2$,

$$ab^2 + 2abc + ac^2 = a^2c + 2abc + b^2c,$$

$$\text{or } a(b^2 + 2bc + c^2) = c(a^2 + 2ab + b^2),$$

$$\text{or } a(b + c)^2 = c(a + b)^2.$$

Divide by $c(b + c)^2$, $\qquad \dfrac{a}{c} = \dfrac{(a + b)^2}{(b + c)^2},$

or $a : c :: (a + b)^2 : (b + c)^2.$

14. When a, b, and c are proportionals, and a the greatest, show that $a + c > 2b$.

$$a : b :: b : c.$$

Since $\qquad \dfrac{a}{b} = \dfrac{b}{c}$ and $a > b$,

$$\therefore b > c.$$

Also, since by division $\dfrac{a - b}{b} = \dfrac{b - c}{c}$ and $b > c$,

$$\therefore a - b > b - c.$$

By adding, $\qquad b + c = b + c,$

$$a + c > 2b.$$

15. If $\frac{x-y}{l}=\frac{y-z}{m}=\frac{z-x}{n}$, and x, y, z are unequal, then $l+m+n=0$.

Let $$\frac{x-y}{l}=r,\quad \frac{y-z}{m}=r,\quad \frac{z-x}{n}=r.$$

Then
$$x-y=lr,$$
$$y-z=mr,$$
$$z-x=nr.$$
$$x-y+y-z+z-x=(l+m+n)\,r,$$
$$\text{or } 0=(l+m+n)\,r.$$
$$\therefore l+m+n=0.$$

16. Find x when $\quad x+5:2x-3::5x+1:3x-3.$

Equate the product of the means and the product of the extremes,
$$10x^2-13x-3=3x^2+12x-15,$$
$$7x^2-25x=-12,$$
$$196x^2-(\ )+625=289,$$
$$14x-25=\pm 17,$$
$$14x=42 \text{ or } 8.$$
$$\therefore x=3 \text{ or } \tfrac{4}{7}.$$

17. Find x when $\quad x+a:2x-b::3x+b:4x-a.$
$$\frac{x+a}{2x-b}=\frac{3x+b}{4x-a}.$$
Clear of fractions, $4x^2+3ax+a^2=6x^2-bx-b^2,$
$$2x^2-x(3a+b)=(b^2-a^2),$$
$$16x^2-(\ )+(3a+b)^2=a^2+6ab+9b^2,$$
$$4x-(3a+b)=\pm(a+3b),$$
$$4x=4a+4b \text{ or } 2a-2b.$$
$$\therefore x=a+b \text{ or } \frac{a-b}{2}.$$

18. Find x when
$$\sqrt{x}+\sqrt{b}:\sqrt{x}-\sqrt{b}::a:b.$$
$$b\sqrt{x}+b\sqrt{b}=a\sqrt{x}-a\sqrt{b},$$
$$(a-b)\sqrt{x}=(a+b)\sqrt{b},$$
$$\sqrt{x}=\frac{(a+b)\sqrt{b}}{a-b}.$$
$$\therefore x=\frac{(a^2+2ab+b^2)b}{a^2-2ab+b^2}.$$

19. Find x and y when $x:27::y:9$, and $x:27::2:x-y$.

$$x:27::y:9.$$
$$\therefore x=3y \tag{1}$$
$$x:27::2:x-y.$$
$$\therefore x^2-xy=54.$$

Substitute $3y$ for x, $9y^2-3y^2=54$,

$$6y^2=54,$$
$$y^2=9.$$
$$\therefore y=\pm 3.$$

Substitute values of y in (1), $x=\pm 9$.

20. Find x and y when $x+y+1:x+y+2::6:7$, and when $y+2x:y-2x::12x+6y-3:6y-12x-1$.

$$x+y+1:x+y+2::6:7.$$

By division, $x+y+1:1::6:1$.

$$\therefore x+y+1=6,$$
$$\text{or } x+y=5 \tag{1}$$
$$y+2x:y-2x::12x+6y-3:6y-12x-1.$$

By composition and division,

$$2y:4x::12y-4:24x-2,$$
$$\text{or } y:2x::6y-2:12x-1.$$
$$\therefore 12xy-y=12xy-4x.$$
$$\therefore 4x=y \tag{2}$$

From (1) and (2), $x=1$,

and $y=4$.

21. Find x when

$$x^2-4x+2:x^2-2x-1::x^2-4x:x^2-2x-2.$$

By alternation,

$$x^2-4x+2:x^2-4x::x^2-2x-1:x^2-2x-2.$$

By division, $2:x^2-4x::1:x^2-2x-2$.

$$\therefore 2x^2-4x-4=x^2-4x.$$
$$\therefore x^2=4.$$
$$\therefore x=\pm 2.$$

22. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

Let $x =$ rate of the faster train,

and $y =$ rate of the slower train.

Then $x + y : x - y :: 30 : 2.$

By composition and division,

$$2x : 2y :: 32 : 28.$$

$$\therefore x : y :: 8 : 7.$$

23. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock : B's :: 2 : ½. But, if A had gained \$100 and B lost \$85, their stocks would have been as 15 : 3¼. Find the original stock of each.

Let $x =$ original stock of A.,

and $y =$ original stock of B.

Then $x + 200 : y - 50 :: 2 : \frac{1}{2}.$

Simplify, $x + 200 = 4y - 200,$

$$x - 4y = -400 \quad (1)$$

Also, $x + 100 : y - 85 :: 15 : 3\frac{1}{4}.$

Simplify, $13x + 1300 = 60y - 5100,$

$$13x - 60y = -6400 \quad (2)$$

Multiply (1) by 15, $15x - 60y = -6000 \quad (3)$

Subtract (2) from (3), $2x = 400.$

$$\therefore x = 200.$$

$$200 - 4y = -400.$$

$$\therefore y = 150.$$

24. A quantity of milk is increased by watering in the ratio 4 : 5, and then 3 gallons are sold; the remainder is mixed with 3 quarts of water, and is increased in the ratio 6 : 7. How many gallons of milk were there at first?

Let $x =$ number of quarts of milk at first,

and $y =$ number of quarts of water put in at first.

Then $x + y =$ number of quarts of mixture after watering.

$$\therefore x : x + y :: 4 : 5,$$

or

$$\frac{x}{x+y} = \frac{4}{5},$$

$$5x = 4x + 4y,$$

$$x - 4y = 0.$$

$x+y-12=$ number of quarts in remainder before watering.

$x+y-\ \ 9=$ number of quarts in remainder after watering.

$$\therefore \frac{x+y-12}{x+y-9}=\frac{6}{7}$$

$$7x+7y-84=6x+6y-54.$$

$$x-4y=0 \qquad (1)$$

$$x+y=30 \qquad (2)$$

$$5y=30$$

$$\therefore y=6.$$

Substitute value of y in (1),

$$x-24=0.$$

$$\therefore x=24 \text{ quarts or 6 gallons.}$$

25. In a mile race between a bicycle and a tricycle their rates were as 5 : 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rate of each.

Let $x=$ number of yards bicycle goes per minute,

and $y=$ number of yards tricycle goes per minute.

$$x:y::5:4,$$

$$4x=5y.$$

$$\therefore x=\frac{5y}{4}.$$

$\frac{1584}{y}-\frac{1}{2}=$ number of minutes tricycle was going after bicycle started,

$\frac{1760}{x}=$ number of minutes bicycle was going.

$$\frac{1584}{y}-\frac{1}{2}=\frac{1760}{x},$$

$$1584x-\frac{xy}{2}=1760y,$$

$$3168x-xy=3520y.$$

Substitute $\frac{5y}{4}$ for x, $\qquad 5y^2=1760y.$

$$\therefore y=352,$$

$$\text{and } x=440.$$

26. The time which an express-train takes to travel 180 miles is to that taken by an ordinary train as $9:14$. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express-train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

Let $y =$ number of miles ordinary train goes per hour,

and $y + 15 =$ number of miles express-train goes per hour.

Then $\frac{180+30}{y} =$ number of hours required for ordinary train.

Also, $\frac{180}{y+15} + \frac{15}{y} =$ number of hours required for express-train.

$$\therefore \frac{180+30}{y} : \frac{180}{y+15} + \frac{15}{y} :: 14:9.$$

$$\frac{1890}{y} = \frac{2520}{y+15} + \frac{210}{y},$$

$$\frac{1680}{y} = \frac{2520}{y+15},$$

$$1680y + 25200 = 2520y,$$

$$840y = 25200.$$

$$\therefore y = 30,$$

$$\text{and } y + 15 = 45.$$

27. A line is divided into two parts in the ratio $2:3$, and into two parts in the ratio $3:4$; the distance between the points of section is 2. Find the length of the line.

Let $x =$ one part,

and $y =$ the other part.

$$\therefore x:y::2:3.$$

$$3x = 2y,$$

$$3x - 2y = 0 \quad (1)$$

Also, $x + 2 : y - 2 :: 3:4,$

$$4x - 3y = -14 \quad (2)$$

Multiply (1) by 3, $9x - 6y = 0$

Multiply (2) by 2, $8x - 6y = -28$

Subtract, $x = 28$

Substitute value of x in (1), $2y = 84,$

$$y = 42.$$

$$\therefore x + y = 70.$$

28. A railway consists of two sections; the annual expenditure on one is increased this year 5%, and on the other 4%, producing on the whole an increase of $4\frac{3}{10}\%$. Compare the amount expended on the two sections last year, and also this year.

Let $x + y =$ amount expended last year.

$x =$ amount expended on first section last year,

and $y =$ amount expended on second section last year.

Then $\frac{1043x + 1043y}{1000} =$ amount expended on whole this year.

But $x + \frac{5x}{100} =$ amount expended on one part this year.

and $y + \frac{4y}{100} =$ amount expended on other part this year.

Then $$\frac{105x}{100} + \frac{104y}{100} = \frac{1043x + 1043y}{1000}.$$

Simplify, $$7x = 3y.$$

$$\therefore \frac{x}{y} = \frac{3}{7},$$

and $$\frac{105x}{104y} = \frac{315}{728} = 3\tfrac{3}{20} : 7\tfrac{7}{25}.$$

29. When a, b, c, d are proportional and unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

If $a : b :: c : d$, $$ad = bc;$$

and if $$a + x : b + x :: c + x : d + x,$$

$$ad + dx + ax + x^2 = bc + cx + bx + x^2.$$

Transpose, and cancel x^2,

$$ax - bx - cx + dx = bc - ad.$$

But $$ad = bc.$$

$$\therefore x(a - b - c + d) = 0.$$

$$\therefore x = 0.$$

Exercise CXV.

1. If $A \propto B$, and $A = 4$ when $B = 5$, find A when $B = 12$.

Here $$A = mB,$$
$$\text{or } m = \frac{A}{B}.$$
$$\therefore m = \tfrac{4}{5}.$$

And if $\frac{4}{5}$ and 12 be substituted for m and B,
$$A = \tfrac{4}{5} \times 12.$$
$$\therefore A = 9\tfrac{3}{5}.$$

2. If $A \propto B$, and when $B = \frac{1}{2}$, $A = \frac{1}{3}$, find A when $B = \frac{1}{3}$.

Here $$A = mB.$$
$$m = \frac{A}{B}.$$
$$\therefore m = \tfrac{2}{3}.$$

Substitute $\frac{2}{3}$ for m and $\frac{1}{3}$ for B,
$$A = \tfrac{2}{3} \times \tfrac{1}{3}.$$
$$\therefore A = \tfrac{2}{9}.$$

3. If A varies jointly as B and C, and 3, 4, 5 are simultaneous values of A, B, C, find A when $B = C = 10$.

Here $$A = mBC.$$
$$m = \frac{A}{BC}.$$

Substitute 3 for A, 4 and 5 for B and C,
$$m = \tfrac{3}{20}.$$
Then $$A = \tfrac{3}{20} \times 10 \times 10.$$
$$\therefore A = 15.$$

4. If $A \propto \dfrac{1}{B}$, and when $A = 10$, $B = 2$, find the value of B when $A = 4$.

Here $$A = \frac{m}{B},$$
$$m = AB.$$
$$\therefore m = 20.$$

Substitute values of m and A,
$$4 = \frac{20}{B},$$
$$4B = 20.$$
$$\therefore B = 5.$$

5. If $A \propto \frac{B}{C}$, and when $A = 6$, $B = 4$, and $C = 3$, find the value of A when $B = 5$ and $C = 7$.

Here
$$A = \frac{mB}{C},$$
$$mB = AC,$$
$$4m = 18.$$
$$\therefore m = 4\tfrac{1}{2}.$$
Substitute value of B, C, and m,
$$A = \frac{4\frac{1}{2} \times 5}{7}.$$
$$\therefore A = 3\tfrac{3}{14}.$$

6. If the square of X varies as the cube of Y, and $X = 3$ when $Y = 4$, find the equation between X and Y.

Here
$$X^2 = mY^3,$$
$$m = \frac{X^2}{Y^3}.$$
$$\therefore m = \tfrac{9}{64}.$$
Substitute value of m,
$$X^2 = \tfrac{9}{64}Y^3,$$
$$64X^2 = 9Y^3.$$

7. If the square of X varies inversely as the cube of Y, and $X = 2$ when $Y = 3$, find the equation between X and Y.

Here
$$X^2 = \frac{m}{Y^3},$$
$$m = X^2Y^3.$$
$$\therefore m = 108.$$
Substitute value of m,
$$X^2 = \frac{108}{Y^3}.$$

8. If Z varies as X directly and Y inversely, and if when $Z = 2$, $X = 3$, and $Y = 4$, find the value of Z when $X = 15$ and $Y = 8$.

$$Z \propto \frac{X}{Y}.$$
Here
$$Z = \frac{mX}{Y},$$
$$m = \frac{ZY}{X}.$$
$$\therefore m = \tfrac{8}{3} = 2\tfrac{2}{3}.$$
Substitute values of m, X, and Y,
$$Z = \frac{2\frac{2}{3} \times 15}{8}.$$
$$\therefore Z = 5.$$

9. If $A \propto B + c$ where c is constant, and if $A = 2$ when $B = 1$, and if $A = 5$ when $B = 2$, find A when $B = 3$.

As $\qquad A = mB + c.$

Substitute first values of A and B,

$$2 = m + c \qquad (1)$$

Substitute second values of A and B,

$$5 = 2m + c \qquad (2)$$

Subtract (1) from (2), $\qquad m = 3.$

Whence, from (1), $\qquad c = -1.$

But $\qquad A = mB + c.$

Substitute for m, B, and c their values 3, 3, and -1,

$$A = 8.$$

10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it is found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance at the end of 5 seconds.

Let $\qquad v = \text{velocity},$

$\qquad T = \text{time},$

$\qquad d = \text{distance}.$

Then $\qquad v \propto T,$

and $\qquad d \propto T^2.$

Let $\qquad v = mT.$

Substitute $96\frac{2}{3}$ for v and 3 for T,

$$96\tfrac{2}{3} = 3m.$$

$$\therefore m = \tfrac{290}{9}.$$

When $T = 5$, $\qquad v = \frac{290}{9} \times 5 = 161\frac{1}{9}.$

Let $\qquad d = mT^2.$

$$\therefore 145 = 3^2 m.$$

$$m = \tfrac{145}{9}.$$

When $T = 5$, $\qquad d = \frac{145}{9} \times 5^2 = 402\frac{7}{9}.$

11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

Let $x =$ heavy weight,

$y =$ light weight,

$z =$ space.

$$z \propto \frac{x-y}{x+y}.$$

$$z = \frac{(x-y)m}{x+y},$$

$$m = \frac{z(x+y)}{x-y}.$$

Substitute values, $$m = \frac{(7+9)8}{9-7}.$$

$$\therefore m = 64.$$

$$64 = \frac{(12+9)z}{12-9},$$

$$64 = \frac{21z}{3},$$

$$7z = 64.$$

$$\therefore z = 9\tfrac{1}{7}.$$

12. The space will vary also as the square of the time. Find the space in Example 11, if the time in the latter case is 3 seconds.

We have from last example, $9\frac{1}{7}$ feet for 2 seconds.

Since space varies as square of time, we have

$$9\tfrac{1}{7} : x :: 2^2 : 3^2.$$

$$\therefore 4x = 9 \times \tfrac{64}{7},$$

$$x = 9 \times \tfrac{16}{7}$$

$$= \tfrac{144}{7}$$

$$= 20\tfrac{4}{7}.$$

$20\frac{4}{7}$ feet. *Ans.*

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{10}$ ounces.

Let $x =$ required weight.

9 inches $= \frac{3}{4}$ of a foot.

If $\frac{3}{4}$ of a foot weigh 31.9 ounces, $\frac{1}{4}$ of a foot would weigh $10.03\frac{1}{3}$ ounces, and $10\frac{1}{2}$ feet would weigh 446.60 ounces.

And, as equal volumes of iron and copper weigh 77 and 89 ounces respectively,

$$77 : 89 :: 446\frac{3}{5} : x.$$

$$\therefore x = 516\frac{1}{5} \text{ ounces.}$$

14. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find in days the time of Mercury's revolution.

Let $x =$ time of Mercury's revolution.

$$91^3 : 35^3 :: 1^2 : x^2,$$

$$13^3 : 5^3 :: 1 : x^2.$$

Whence, $x^2 = 0.056895.$

$$\therefore x = 0.238, \text{ time in years,}$$

$$= 87.1, \text{ time in days.}$$

15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

Let $D =$ diameter of shell,

$d =$ diameter of sphere required to fill the shell,

and 1 represent the weight of iron sphere having diameter $= D$.

Then $1 - \frac{91}{216}$ will represent the weight of iron sphere having diameter $= d$.

Now the weights vary as the cubes of their diameters,

$$\therefore D^3 : d^3 :: 1 : 1 - \frac{91}{216}.$$

That is, $D^3 : d^3 :: 1 : \frac{125}{216}$;

or, by extracting the cube root of each term,

$$D : d :: 1 : \frac{5}{6},$$

$$\text{or } d = \frac{5}{6}D.$$

Since the thickness of the shell $= \frac{1}{2}(D - d)$,

the thickness of the shell $= \frac{1}{2}(1 - \frac{5}{6}) = \frac{1}{12}$.

Hence, the thickness of the shell is $\frac{1}{12}$ of a foot, $= 1$ inch.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

Let x = volume of first sphere,
and y = sum of volume of other three.
Then $x : y :: (6)^3 : (3)^3 + (4)^3 + (5)^3,$
$x : y :: 216 : 216.$
Therefore, the ratio is a ratio of equality.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a singular circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

Let a_1 = area of gold plate 6 inches in diameter,
a_2 = area of gold plate 8 inches in diameter,
a_3 = area of gold plate formed from the other two plates,
and x = diameter required.
Then $a_1 + a_2 : a_3 :: 6^2 + 8^2 : x^2.$
Since the first ratio is a ratio of equality, the second is also.
Therefore, $x^2 = 6^2 + 8^2 = 100.$
$\therefore x = 10.$

18. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet, is found to contain 10 cubic yards. What must be the height of a pyramid upon a base 3 feet square, in order that it may contain 2 cubic yards?

Let v = volume,
b = area of base,
and a = altitude.
Then $v \propto ba,$
$$v = mba \qquad (1)$$
$$m = \frac{v}{ba}.$$
When $v = 10$ cubic yards $= 270$ cubic feet,
$b = 9 \times 9 = 81$ square feet,
and $a = 10$ feet.
Then $m = \dfrac{v}{ba} = \frac{270}{810} = \frac{1}{3}.$
From (1), $a = \dfrac{v}{mb}.$
When $m = \frac{1}{3},$
$v = 2$ cubic yards $= 54$ cubic feet,
$b = 3 \times 3 = 9$ square feet.
Then $a = \dfrac{v}{mb} = \frac{54}{3} = 18.$

Let $\qquad x$ and y = first and last terms.

Then $\qquad \dfrac{2xy}{x+y}$ = middle term.

Hence,
$$x+y+\frac{2xy}{x+y}=11 \qquad (1)$$

and
$$x^2+y^2+\frac{4x^2y^2}{(x+y)^2}=49 \qquad (2)$$

Square (1), and subtract (2) from the result,
$$6xy=72, \text{ and } xy=12 \qquad (3)$$

Substitute 12 for xy in (1), and clear of fractions,
$$(x+y)^2-11(x+y)=-24.$$

Complete the square and extract the root,
$$x+y=8 \qquad (4)$$

21. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

Let $\qquad x$ = first digit,

rx = second digit,

and $\qquad r^2x$ = third digit.

$$x+rx+r^2x=13 \qquad (1)$$
$$100x+10rx+r^2x+792=100r^2x+10rx+x,$$
$$-99r^2x+99x=-792.$$
$$r^2x-x=\ 8 \qquad (2)$$
$$x+rx+r^2x=13$$

Subtract,
$$2x+rx=\ 5$$
$$\therefore x=\frac{5}{2+r}.$$

Substitute value of x in (2),
$$\frac{5r^2}{2+r}-\frac{5}{2+r}=8,$$
$$5r^2-5=16+8r,$$
$$5r^2-8r=21,$$
$$100r^2-(\ )+64=484,$$
$$10r-8=\pm 22,$$
$$10r=30.$$
$$\therefore r=3.$$

From (1),
$$x+3x+9x=13.$$
$$\therefore x=1.$$

Hence, the number is 139.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

Let $x =$ volume of first sphere,

and $y =$ sum of volume of other three.

Then $x : y :: (6)^3 : (3)^3 + (4)^3 + (5)^3,$

$x : y :: 216 : 216.$

Therefore, the ratio is a ratio of equality.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a singular circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the

$$\therefore \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8. \qquad = \frac{2}{\frac{5}{3}} = \frac{6}{5}.$$

(2) $a = \frac{1}{2},$
$r = \frac{2}{3}.$
$$\therefore \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{2}{3}} = 1\frac{1}{2}.$$

(7) $a = 0.1,$
$r = 0.1.$
$$\therefore \frac{a}{1-r} = \frac{0.1}{1-0.1} = \frac{1}{9}.$$

(3) $a = \frac{1}{4},$
$r = -\frac{1}{4}.$
$$\therefore \frac{a}{1-r} = \frac{\frac{1}{4}}{1-(-\frac{1}{4})} = \frac{\frac{1}{4}}{1+\frac{1}{4}} = \frac{1}{5}.$$

(8) $a = 0.86,$
$r = 0.01.$
$$\therefore \frac{a}{1-r} = \frac{0.86}{1-0.01} = \frac{86}{99}.$$

(4) $a = 1,$
$r = -\frac{2}{5}.$
$$\therefore \frac{a}{1-r} = \frac{1}{1-(-\frac{2}{5})} = \frac{1}{1+\frac{2}{5}} = \frac{5}{7}.$$

(9) $a = \frac{4}{100},$
$r = \frac{1}{10}.$
$$\therefore \frac{a}{1-r} = \frac{\frac{4}{100}}{\frac{9}{10}} = \frac{4}{90},$$
$$\frac{4}{90} + \frac{5}{10} = \frac{49}{90}.$$

(5) $a = \frac{1}{5},$
$r = \frac{1}{3}.$
$$\therefore \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{3}} = \frac{\frac{1}{5}}{\frac{2}{3}} = \frac{3}{10}.$$

(10) $a = \frac{86}{1000},$
$r = \frac{1}{100}.$
$$\therefore \frac{a}{1-r} = \frac{\frac{86}{1000}}{\frac{99}{100}} = \frac{2}{55},$$
$$\frac{2}{55} + \frac{8}{10} = \frac{46}{55}.$$

Let x and $y =$ first and last terms.

Then $\dfrac{2xy}{x+y} =$ middle term.

Hence, $$x + y + \frac{2xy}{x+y} = 11 \qquad (1)$$

and $$x^2 + y^2 + \frac{4x^2y^2}{(x+y)^2} = 49 \qquad (2)$$

Square (1), and subtract (2) from the result,

$$6xy = 72, \text{ and } xy = 12 \qquad (3)$$

Substitute 12 for xy in (1), and clear of fractions,

$$(x+y)^2 - 11(x+y) = -24.$$

Complete the square and extract the root,

$$x + y = 8 \qquad (4)$$

Square (4), $x^2 + 2xy + y^2 = 64$
From (3), $4xy \quad = 48$

$$x^2 - 2xy + y^2 = 16$$

$$x - y = \pm 4 \qquad (5)$$

From (4) and (5), $x = 6$, and $y = 2$.

Hence, the numbers are 6, 3, 2.

11. When a, b, c are in harmonical progression, show that $a : c :: a - b : b - c$.

If a, b, c are a harmonical series,

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Multiply by abc, $ac - bc = ab - ac$,

or $c(a-b) = a(b-c)$,

or $a : c :: a - b : b - c$.

Exercise CXIX.

1. How many different permutations can be made of the letters in the word *ecclesiastical*, taken all together?

The word contains t once, a, e, i, l, and s each twice, and c three times; in all 14 letters. Hence (Rule V) the number of different permutations of the letters is

$$\frac{\underline{|14}}{\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|2}\,\underline{|3}} = 454,053,600.$$

2. Of all the numbers that can be formed with four of the digits

5, 6, 7, 8, 9, how many will begin with 56?

The last two digits of the numbers may be selected in any way from 7, 8, 9. This is possible in

$$\frac{\lfloor 3}{\lfloor 1} = 6 \text{ ways.}$$

Hence 6 numbers can be formed of the required kind.

3. If the number of permutations of n things, taken 4 together, is equal to 12 times the permutations of n things, taken 2 together, find n.

By the given conditions

$$\frac{\lfloor n}{\lfloor n-4} = 12 \frac{\lfloor n}{\lfloor n-2},$$

or $\quad n(n-1)(n-2)(n-3)$
$$= 12\, n(n-1).$$

Hence $(n-2)(n-3) = 12$

$$n^2 - 5n + 6 = 12$$

$$n^2 - 5n - 6 = 0$$

$$(n+1)(n-6) = 0$$

$$n = -1 \text{ or } 6.$$

Therefore $\qquad n = 6.$

4. With 3 consonants and 2 vowels, how many words of 3 letters can be found, beginning and ending with a consonant, and having a vowel for a middle letter?

The 2 consonants can be chosen from 3 in 6 ways, and the vowel can be chosen from 2 in 2 ways. Hence the number of words of the required kind that can be found is $6 \times 2 = 12$. *Ans.*

5. Out of 20 men, in how many different ways can 4 be chosen to be on guard? In how many of these would one particular man be taken, and from how many would he be left out?

(1) Four men can be selected from 20 in $\frac{\lfloor 20}{\lfloor 4 \lfloor 16} = 4845$ ways.

(2) If one particular man is to be included, the remaining 3 can be selected from the remaining 19 in $\frac{\lfloor 19}{\lfloor 3 \lfloor 16} = 969$ ways.

(3) If a particular man is to be excluded, the 4 can be selected from the remaining 19 in

$$\frac{\lfloor 19}{\lfloor 4 \lfloor 15} = 3876 \text{ ways.}$$

$$3876 + 969 = 4845.$$

Hence the total number of selections is 4845; including a particular man, 969; excluding a particular man, 3876.

6. Of 12 books of the same size, a shelf will hold 5. How many different arrangements on the shelf may be made?

The number of arrangements is

$$\frac{\lfloor 12}{\lfloor 7} = 95{,}040.$$

7. Of 8 men forming a boats' crew, 'one is selected as stroke. How many arrangements of the rest are possible? When the 4 who row on each side are decided on, how many arrangements are still possible?

(1) The 7 remaining men can be arranged in the 7 remaining seats in $\underline{|7} = 5040$ ways.

(2) The 3 men, besides the stroke, who row on the stroke side can be seated in $\underline{|3} = 6$ ways. The 4 men on the other side can be seated in $\underline{|4} = 24$ ways. Hence the total number of arrangements under the given conditions is

$$6 \times 24 = 144.$$

8. How many signals may be made with 6 flags of different colors, which can be hoisted either singly or any number at a time?

A set of r flags can be selected in $\dfrac{\underline{|6}}{\underline{|r}\,\underline{|6-r}}$ ways, and each set can be hoisted in $\underline{|r}$ different orders. Hence the number of different signals with r flags is $\dfrac{\underline{|6}}{\underline{|6-r}}$.

The number of signals, therefore,

with 1 flag is 6			=	6
"	2 flags is	6×5	=	30
"	3 "	$6 \times 5 \times 4$	=	120
"	4 "	$6 \times 5 \times 4 \times 3$	=	360
"	5 "	$6 \times 5 \times 4 \times 3 \times 2$	=	720
"	6 "	$6 \times 5 \times 4 \times 3 \times 2 \times 1$	=	720
		In all,		1956

9. How many signals may be made with 8 flags of different colors, which can be hoisted either singly or any number at a time one above another?

The number of signals

with 1 flag is 8			=	8
"	2 flags is	8×7	=	56
"	3 "	$8 \times 7 \times 6$	=	336
"	4 "	$8 \times 7 \times 6 \times 5$	=	1,680
"	5 "	$8 \times 7 \times 6 \times 5 \times 4$	=	6,720
"	6 "	$8 \times 7 \times 6 \times 5 \times 4 \times 3$	=	20,160
"	7 "	$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$	=	40,320
"	8 "	$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$	=	40,320
		In all,		109,600

10. How many different signals can be made with 10 flags, of which 3 are white, 2 red, and the rest blue, always hoisted all together and one above another?

The number of different signals is equal to the number of arrangements of 10 things, of which 2 are alike, 3 are alike, and 5 are alike. This number is (Rule V)

$$\frac{|\underline{10}}{|\underline{2}\,|\underline{3}\,|\underline{5}} = 2520.$$

11. How many signals can be made with seven flags, of which 2 are red, 1 white, 3 blue, and 1 yellow, always displayed all together and one above another?

By Rule V the number is

$$\frac{|\underline{7}}{|\underline{2}\,|\underline{3}} = 420.$$

12. In how many ways may the 8 men serving a field-gun be arranged so that the same man may always lay the gun?

The 7 remaining men may be arranged in $|\underline{7} = 5040$ ways.

13. Find the number of signals which can be made with 4 lights of different colors when displayed any number at a time, arranged above one another, side by side, or diagonally.

The number of arrangements in vertical, horizontal, or either diagonal line is evidently the same. But the case where only one light is used must be counted only once; all other cases four times. The number of signals in a single line

with	1 light	is	4	$= 4$
"	2 lights	is	4×3	$= 12$
"	3	"	$4 \times 3 \times 2$	$= 24$
"	4	"	$4 \times 3 \times 2 \times 1$	$= 24$

In all, $60 + 4$.

Hence the total number of signals is $4 \times 60 + 4 = 244$.

14. From 10 soldiers and 8 sailors, how many different parties of 3 soldiers and 3 sailors can be formed?

The 3 soldiers can be selected in $\frac{|\underline{10}}{|\underline{3}\,|\underline{7}} = 120$ ways, and the 3 sailors in $\frac{|\underline{8}}{|\underline{3}\,|\underline{5}} = 56$ ways. Hence the party of 3 soldiers and 3 sailors can be selected in

$$120 \times 56 = 6720 \text{ ways.}$$

15. How many signals can be made with 3 blue and 2 white flags, which can be displayed either singly or any number at a time one above another?

A single flag may be either blue or white. 2 ways.

Two flags may be both blue or both white, or 1 blue and 1 white. In the last case they may be arranged in two ways.

In all, 4 ways.

Three flags may be all blue, or 2 blue and 1 white or 2 white

and 1 blue. In the last two cases they may be arranged in 3 ways. In all, 7 ways.

Four flags may be 3 blue and 1 white, 4 ways; or 2 blue and 2 white, $\frac{4!}{2!\,2!} = 6$ ways (Rule V). In all, 10 ways.

Five flags must be 3 blue and 2 white, $\frac{5!}{3!\,2!} = 10$ ways (Rule V). In all, 10 ways.

There are, therefore, in all, $2 + 4 + 7 + 10 + 10 = 33$ possible signals.

16. In how many ways can a party of 6 take their places at a round table?

One person may take any one of the 6 seats; the other 5 can then be seated in $5! = 120$ different orders. Hence the answer is

$$6 \times 120 = 720.$$

17. Out of 12 Democrats and 16 Republicans, how many different committees can be formed, each consisting of 3 Democrats and 4 Republicans?

The 3 Democrats can be selected from 12 in $\frac{12!}{3!\,9!} = 220$ ways; the 4 Republicans from 16 in $\frac{16!}{4!\,12!}$ $= 1820$ ways. The committee, consisting of any 3 Democrats and any 4 Republicans, can therefore be formed in

$$220 \times 1820 = 400{,}400 \text{ ways.}$$

18. From 12 soldiers and 8 sailors, how many different parties of 3 soldiers and 2 sailors can be found?

The 3 soldiers can be selected from 12 in $\frac{12!}{3!\,9!} = 220$ ways; the 2 sailors from 8 in $\frac{8!}{2!\,6!} = 28$ ways. The party can therefore be formed in

$$220 \times 28 = 6160 \text{ ways.}$$

19. Find the number of combinations of 100 things, 97 together.

By Rule VIII the required number is

$$\frac{100!}{97!\,3!} = \frac{100 \times 99 \times 98}{3 \times 2}$$
$$= 161{,}700.$$

20. With 20 consonants and 5 vowels, how many different words can be formed consisting of 3 different consonants and two different vowels, any arrangement of letters being considered a word?

The 3 consonants can be selected from 20 in $\frac{20!}{3!\,17!} = 1140$ ways; the 2 vowels from 5 in $\frac{5!}{2!\,3!} = 10$ ways. The number of combinations of 3 different consonants and two different vowels is therefore $10 \times 1140 = 11{,}400.$

The five letters of each combination can then be arranged in $\underline{|5} = 120$ ways. Hence the total number of words of the required kind is

$$120 \times 11{,}400 = 1{,}368{,}000.$$

21. Of 30 things, how many must be taken together in order that, having that number for selection, there may be the greatest possible variety of choice?

By § 417, the number taken must be

$$\frac{30}{2} = 15.$$

22. There are m things of one kind and n of another. How many different sets can be made containing r of the first and s of the second?

The r of the first kind can be selected from m in $\dfrac{\underline{|m}}{\underline{|r}\,\underline{|m-r}}$ ways; the s of the second kind from n in $\dfrac{\underline{|n}}{\underline{|s}\,\underline{|n-s}}$. The total number of sets of r of the first kind and s of the second is therefore

$$\frac{\underline{|m}\,\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|m-r}\,\underline{|n-s}}.$$

23. In how many ways may 10 persons be seated at a round table so that in no two of the arrangements may every one have the same neighbors?

Only the order of seating being of account, one person may take any seat; the other 9 can then be seated in $\underline{|9} = 362{,}880$ different orders. For each order of the 10 persons and the reverse order, each person has the same neighbors, and this is true only in this case. Hence the number of arrangements required is

$$\frac{362{,}880}{2} = 181{,}440.$$

24. The number of combinations of n things taken r together, is 3 times the number taken $r-1$ together, and half the number taken $r+1$ together. Find n and r.

By the given conditions

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} = 3\,\frac{\underline{|n}}{\underline{|r-1}\,\underline{|n-r+1}}$$

$$= \tfrac{1}{2}\,\frac{\underline{|n}}{\underline{|r+1}\,\underline{|n-r-1}}.$$

$$\therefore \frac{\underline{|n-r+1}}{\underline{|n-r}} = 3\,\frac{\underline{|r}}{\underline{|r-1}}, \qquad (1)$$

$$\frac{\underline{|n-r}}{\underline{|n-r-1}} = 2\,\frac{\underline{|r+1}}{\underline{|r}}. \qquad (2)$$

From (1),

$$n - r + 1 = 3r,$$
$$n + 1 = 4r.$$

From (2),

$$n - r = 2(r+1),$$
$$n = 3r + 2.$$
$$\therefore 3r + 2 = 4r - 1,$$
$$r = 3,$$
$$n = 3r + 2$$
$$= 11.$$

Therefore,

$$n = 11,\ r = 3.$$

25. In how many ways may 12 things be divided into 3 sets of 4?

Here the sets are indifferent, and the answer is

$$\frac{\underline{|12}}{\underline{|3}\,\underline{|4}\,\underline{|4}\,\underline{|4}} = 5775.$$

26. How many words of 6 letters may be formed of 3 vowels and 3 consonants, the vowels always having the even places?

The 3 consonants can be arranged in the 3 odd places in $\underline{|3} = 6$ ways, and the 3 vowels can be arranged in the even places in 6 ways. The total number of different words of the required kind is therefore $6 \times 6 = 36$.

27. From a company of 90 men, 20 are detached for mounting guard each day. How long will it be before the same 20 men are on guard together, supposing the men to be changed as much as possible; and how many times will each man have been on guard?

20 men can be selected from 90 in $\dfrac{\underline{|90}}{\underline{|20}\,\underline{|70}}$ ways; and this is the number of days that will elapse before the same 20 men go on guard the second time.

The number of guard details which include a particular man is

$$\frac{\underline{|89}}{\underline{|19}\,\underline{|70}}.$$

28. Supposing that a man can place himself in 3 distinct attitudes, how many signals can be made by 4 men placed side by side?

Since each man can place himself in 3 attitudes, the total number of distinct attitudes of the group of 4 men is $3^4 = 81$.

But one is the attitude of rest. Therefore $3^4 - 1 = 80$ signals.

29. How many different arrangements can be made of 11 cricketers, supposing the same 2 always to bowl?

The 9 remaining cricketers can be arranged in $\underline{|9} = 362,880$ ways. And since the two who bowl may change places, the total number of arrangements is

$$2 \times 362,880 = 725,760.$$

30. Five flags of different colors can be hoisted either singly or any number at a time one above another. How many different signals can be made with them?

The number of signals

with 1 flag	is 5		= 5
" 2 flags	is 5×4		= 20
" 3	"	$5 \times 4 \times 3$	= 60
" 4	"	$5 \times 4 \times 3 \times 2$	= 120
" 5	"	$5 \times 4 \times 3 \times 2 \times 1$	= 120
		In all,	325

31. How many signals can be made with 5 lights of different colors, which can be displayed either singly, or any number at a time side by side, or one above another?

The number of signals with the lights side by side is equal to that with the lights one above another. Every possible arrangement must therefore be counted twice; but if only one light is used, this can be counted only once.

The number of arrangements in a single line

with 1 light	is	5	$= 5$
" 2 lights	is	5×4	$= 20$
" 3	"	$5 \times 4 \times 3$	$= 60$
" 4	"	$5 \times 4 \times 3 \times 2$	$= 120$
" 5	"	$5 \times 4 \times 3 \times 2 \times 1$	$= 120$

In all, $320 + 5$.

The total number of possible signals is therefore

$$2 \times 320 + 5 = 645.$$

32. The number of permutations of n things, 3 at a time, is 6 times the number of combinations, four at a time. Find n.

By the conditions of the problem,

$$\frac{\underline{|n}}{\underline{|n-3}} = 6\,\frac{\underline{|n}}{\underline{|4}\,\underline{|n-4}}.$$

$$\therefore \frac{\underline{|4}}{6} = \frac{\underline{|n-3}}{\underline{|n-4}}$$

$$4 = n - 3$$

$$n = 7.$$

33. At a game of cards, 3 being dealt to each person, any one can have 425 times as many hands as there are cards in the pack. How many cards are there?

Let $n =$ the number of cards;

then $\dfrac{\underline{|n}}{\underline{|3}\,\underline{|n-3}} =$ the number of hands one can hold.

Hence $\dfrac{\underline{|n}}{\underline{|3}\,\underline{|n-3}} = 425\,n$

$$n\,(n-1)\,(n-2) = 425\,n\,\underline{|3}$$

$$(n-1)\,(n-2) = 2550$$

$$n^2 - 3\,n - 2548 = 0$$

$$(n-52)\,(n+49) = 0$$

$$\therefore n = 52.$$

There are 52 cards in the pack.

Exercise CXX.

1. $(1 + 2\,x)^5$

$$= (1)^5 + 5\,(1)^4(2\,x) + 10\,(1)^3(2\,x)^2 + 10\,(1)^2(2\,x)^3 + 5\,(1)\,(2\,x)^4 + (2\,x)^5$$

$$= 1 + 10\,x + 40\,x^2 + 80\,x^3 + 80\,x^4 + 32\,x^5.$$

2. $(x-3)^8 = x^8 - 8\,x^7(3) + 28\,x^6(3)^2 - 56\,x^5(3)^3 + 70\,x^4(3)^4$
$\qquad - 56\,x^3(3)^5 + 28\,x^2(3)^6 - 8\,x(3)^7 + (3)^8$

$$= x^8 - 24\,x^7 + 252\,x^6 - 1512\,x^5 + 5670\,x^4$$
$$- 13608\,x^3 + 20412\,x^2 - 17496\,x + 6561.$$

3. $(2x-3y)^4$
$= (2x)^4 - 4(2x)^3(3y) + 6(2x)^2(3y)^2 - 4(2x)(3y)^3 + (3y)^4$
$= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$

4. $(2-x)^3$
$= (2)^3 - 3(2)^2(x) + 3(2)(x)^2 - (x)^3$
$= 8 - 12x + 6x^2 - x^3.$

5. $\left(1-\frac{3y}{4}\right)^5$

$$= (1)^5 - 5(1)^4\left(\frac{3y}{4}\right) + 10(1)^3\left(\frac{3y}{4}\right)^2 - 10(1)^2\left(\frac{3y}{4}\right)^3 + 5(1)\left(\frac{3y}{4}\right)^4 - \left(\frac{3y}{4}\right)^5$$

$$= 1 - \frac{15y}{4} + \frac{45y^2}{8} - \frac{135y^3}{32} + \frac{405y^4}{256} - \frac{243y^5}{1024}.$$

6. $\left(1-\frac{x}{3}\right)^9$

$$= 1 - 9\left(\frac{x}{3}\right) + 36\left(\frac{x}{3}\right)^2 - 84\left(\frac{x}{3}\right)^3 + 126\left(\frac{x}{3}\right)^4 - 126\left(\frac{x}{3}\right)^5 + 84\left(\frac{x}{3}\right)^6 - 36\left(\frac{x}{3}\right)^7 + 9\left(\frac{x}{3}\right)^8 - 1\left(\frac{x}{3}\right)^9$$

$$= 1 - 3x + 4x^2 - \frac{28x^3}{9} + \frac{14x^4}{9} - \frac{14x^5}{27} + \frac{28x^6}{243} - \frac{4x^7}{243} + \frac{x^8}{729} - \frac{x^9}{19683}.$$

7. **The fourth term of $(2x-5y)^{12}$.**

Substitute in formula values of n and r,

$$\frac{n(n-1)\,.....\,(n-r+2)}{1\times 2\,.....\,(r-1)}a^{n-r+1}x^{r-1}$$

$$= \frac{12\times 11\times 10}{1\times 2\times 3}(2x)^9(5y)^3$$

$$= -14080000x^9y^3.$$

8. The seventh term of $\left(\frac{x}{2}+\frac{y}{3}\right)^{10}$

$$= \frac{10\times 9\times 8\times 7}{1\times 2\times 3\times 4}\left(\frac{x}{2}\right)^4\left(\frac{y}{3}\right)^6$$

$$= \frac{35x^4y^6}{1944}.$$

15. The rth term from the end of $(2a+x)^n$

$$=\frac{n(n-1)\ldots\ldots(n-r+2)}{1\times 2\ldots\ldots(r-1)}(2a)^{r-1}x^{n-r+1}$$

16. The $(r+4)th$ term of $(a+x)^n$

$$=\frac{n(n-1)\ldots\ldots(n-r-2)}{1\times 2\ldots\ldots(r+3)}a^{n-r-3}x^{r+3}.$$

17. The middle term of $(a+x)^{2n}$

$$=\frac{2n(2n-1)\ldots\ldots(2n-(n+1)+2)}{1\times 2\ldots\ldots(n+1-1)}a^{2n-n-1+1}x^n$$

$$=\frac{2n(2n-1)\ldots\ldots(n+1)}{1\times 2\ldots\ldots n}a^n x^n.$$

Multiply both terms by $\underline{|n}$,

$$(n+1)th \text{ term} = \frac{\underline{|2n}}{(\underline{|n})^2}a^n x^n.$$

18. Expand $(2a+x)^{12}$, and find the sum of the terms if $a=1$, $x=-2$.

$$(2a)^{12}+12(2a)^{11}x+66(2a)^{10}x^2+220(2a)^9x^3+495(2a)^8x^4$$
$$+792(2a)^7x^5+924(2a)^6x^6+792(2a)^5x^7+495(2a)^4x^8$$
$$+220(2a)^3x^9+66(2a)^2x^{10}+12(2a)x^{11}+x^{12}.$$

Substitute 1 for a and -2 for x,

$$(2-2)^{12}=0.$$

EXERCISE CXXI.

1. $(1+x)^{\frac{1}{2}}$ to four terms

$$=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3}x^3$$

$$=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots\ldots$$

2. $(1+x)^{\frac{2}{3}}$ to four terms

$$=1-\tfrac{2}{3}x+\frac{\frac{2}{3}(\frac{2}{3}-1)}{2}x^2-\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{1\times 2\times 3}x^3$$

$$=1+\tfrac{2}{3}x-\tfrac{1}{9}x^2+\tfrac{4}{81}x^3-\ldots\ldots$$

3. $(a+x)^{\frac{3}{4}}$ to four terms

$$=a^{\frac{3}{4}}+\tfrac{3}{4}a^{\frac{3}{4}-1}x+\frac{\frac{3}{4}(\frac{3}{4}-1)}{2}a^{\frac{3}{4}-2}x^2+\frac{\frac{3}{4}(\frac{3}{4}-1)(\frac{3}{4}-2)}{\lfloor 3}a^{\frac{3}{4}-3}x^3+\ldots..$$

$$=a^{\frac{3}{4}}+\frac{3x}{4a^{\frac{1}{4}}}-\frac{3x^2}{32a^{\frac{5}{4}}}+\frac{5x^3}{128a^{\frac{9}{4}}}-\ldots..$$

$$=a^{\frac{3}{4}}\left(1+\frac{3x}{4a}-\frac{3x^2}{3a^2}+\frac{5x^3}{128a^3}-\ldots..\right).$$

4. $(1-x)^{-4}$ to four terms

By substituting 1 for a and $-x$ for x in the formula $(a+x)^n$

$$=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)(n-3)}{\lfloor 3}ax^3+\ldots..$$

$$=1+4x+10x^2+20x^3+\ldots..$$

5. $(a^2-x^2)^{\frac{5}{2}}$ to four terms

$$=(a^2)^{\frac{5}{2}}-\tfrac{5}{2}(a^2)^{\frac{5}{2}-1}+\frac{\frac{5}{2}(\frac{5}{2}-1)}{\lfloor 2}(a^2)^{\frac{5}{2}-2}x$$

$$-\frac{\frac{5}{2}(\frac{5}{2}-1)(\frac{5}{2}-2)}{\lfloor 3}(a^2)^{\frac{5}{2}-3}x^3+\ldots..$$

$$=a^5-\frac{5a^3x^2}{2}+\frac{15ax^4}{8}-\frac{5x^6}{16a}+\ldots..$$

6. $(x^2+xy)^{-\frac{3}{2}}$ to four terms

$$=(x^2)^{-\frac{3}{2}}-\tfrac{3}{2}(x^2)^{-\frac{3}{2}-1}(xy)+\frac{-\frac{3}{2}(-\frac{3}{2}-1)}{2}(x^2)^{-\frac{3}{2}-2}(xy)^2$$

$$+\frac{-\frac{3}{2}(-\frac{3}{2}-1)(-\frac{3}{2}-2)}{\lfloor 3}x^{2-\frac{3}{2}-3}(xy)^3+\ldots..$$

$$=x^{-3}-\tfrac{3}{2}x^{-4}y+\frac{15x^{-5}y^2}{8}-\frac{35x^{-6}y^3}{16}+\ldots..$$

7. $(2x-3y)^{-\frac{1}{4}}$ to four terms

$$=(2x)^{-\frac{1}{4}}-(-\tfrac{1}{4})(2x)^{-\frac{5}{4}}(3y)+\frac{(-\frac{1}{4})(-\frac{5}{4})}{2}(2x)^{-\frac{9}{4}}(3y)^2$$

$$-\frac{(-\frac{1}{4})(-\frac{5}{4})(-\frac{9}{4})}{2\times 3}(2x)^{-\frac{13}{4}}(3y)^3+\ldots..$$

$$=2x^{-\frac{1}{4}}\left\{1+\tfrac{3}{8}x^{-1}y+\tfrac{45}{128}x^{-2}y^2+\tfrac{405}{1024}x^{-3}y^3+\ldots..\right\}$$

8. $\sqrt[5]{1-5x}$ to four terms

$$=(1-5x)^{\frac{1}{5}}$$
$$=1-\tfrac{1}{5}(5x)+\frac{\frac{1}{5}(\frac{1}{5}-1)}{1\times 2}(5x)^2-\frac{\frac{1}{5}(\frac{1}{5}-1)(\frac{1}{5}-2)}{1\times 2\times 3}(5x)^3$$
$$=1-x-2x^2-6x^3-\cdots\cdots$$

9. $\dfrac{1}{\sqrt{(4a^2-3ax)^3}}$ to four terms

$$=(4a^2-3ax)^{-\frac{3}{2}}$$
$$=(4a^2)^{-\frac{3}{2}}-(-\tfrac{3}{2})(4a^2)^{-\frac{5}{2}}(3ax)+\frac{(-\frac{3}{2})(-\frac{5}{2})(4a^2)^{-\frac{7}{2}}(3ax)^2}{1\times 2}$$
$$-\frac{(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{1\times 2\times 3}(4a^2)^{-\frac{9}{2}}(3ax)^3$$
$$=\tfrac{1}{8}a^{-3}+\tfrac{9}{64}a^{-4}x+\tfrac{135}{1024}a^{-5}x^2+\tfrac{945}{8192}a^{-6}x^3$$
$$=\frac{1}{8a^3}\left\{1+\frac{9x}{8a}+\frac{135x^2}{128a^2}+\frac{945x^3}{1024a^3}+\cdots\cdots\right\}.$$

10. $\sqrt[6]{\dfrac{1}{(1-3y)^5}}$ to four terms

$$=(1-3y)^{-\frac{5}{6}}$$
$$=1-(-\tfrac{5}{6})(3y)+\frac{(-\frac{5}{6})(-\frac{11}{6})}{1\times 2}(3y)^2-\frac{(-\frac{5}{6})(-\frac{11}{6})(-\frac{17}{6})}{1\times 2\times 3}(3y)^3$$
$$=1+\frac{5y}{2}+\frac{55y^2}{8}+\frac{935y^3}{48}+\cdots\cdots$$

11. $(1+x+x^2)^{\frac{2}{3}}$ to four terms

$$=[1+(x+x^2)]^{\frac{2}{3}}$$
$$=1+\tfrac{2}{3}(x+x^2)+\frac{\frac{2}{3}(\frac{2}{3}-1)}{2}(x+x^2)^2+\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{\underline{|3}}(x+x^2)$$
$$=1+\tfrac{2}{3}x+\tfrac{5}{9}x^2-\tfrac{14}{81}x^3+\cdots\cdots$$

12. $(1-x+x^2)^{\frac{3}{2}}$ to four terms

$$=[1-(x-x^2)]^{\frac{3}{2}}$$
$$=1-\tfrac{3}{2}(x-x^2)+\frac{\frac{3}{2}(\frac{3}{2}-1)}{1\times 2}(x-x^2)^2-\frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{1\times 2\times 3}(x-x^2)^3$$
$$=1-\tfrac{3}{2}x+\tfrac{3}{2}x^2+(\tfrac{3}{8}x^2-\tfrac{6}{8}x^3)-(-\tfrac{1}{16}x^3)$$
$$=1-\tfrac{3}{2}x+\tfrac{15}{8}x^2-\tfrac{11}{16}x^3+\cdots\cdots$$

... term of $(a+x)^{\frac{1}{2}}$

$$=\frac{\frac{1}{2}(\frac{1}{2}-1)\ldots\ldots(\frac{1}{2}-r+2)}{1\times2\ldots\ldots(r-1)}a^{\frac{1}{2}-r+1}x^{r-1}$$

$$=\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\cdots\left(\frac{5-2r}{2}\right)}{\lfloor r-1}a^{\frac{3-2r}{2}}x^{r-1}$$

$$=(-1)^r\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\cdots\left(\frac{2r-5}{2}\right)}{\lfloor r-1}a^{\frac{3-2r}{2}}x^{r-1}.$$

Multiply both terms by 2^{r-1}

$$=(-1)^r\frac{1\times3\times5\ldots\ldots(2r-5)}{\lfloor r-1\times2^{r-1}}a^{\frac{3}{2}-\frac{2r}{2}}x^{r-1}$$

14. The *rth* term of $(a-x)^{-3}$

$$=\frac{(-3)(-4)\ldots\ldots(-r-1)}{1\times2\times3\ldots\ldots r-1}a^{-r-2}x^{r-1}$$

$$=(-1)^r\frac{3\times4\ldots\ldots(r-1)(r)(r+1)}{1\times2\times3\times4\ldots\ldots(r-1)}a^{-r-2}x^{r-1}$$

$$=(-1)^r\frac{r(r+1)}{1\times2}a^{-r-2}x^{r-1}.$$

15. $\sqrt{65}$ to five decimal places

$$=\{64(1+\tfrac{1}{64})\}^{\frac{1}{2}}$$
$$=8(1+\tfrac{1}{64})^{\frac{1}{2}}$$
$$=8\{1+(\tfrac{1}{2})(\tfrac{1}{64})+\frac{\frac{1}{2}(\frac{1}{2}-2)}{1\times2}(\tfrac{1}{64})^2\}$$
$$=8(1+\tfrac{1}{128}-\tfrac{1}{32768})$$
$$=8(1+0.00781-0.00003)$$
$$=8.06224\ldots\ldots$$

16. $\sqrt[3]{1\frac{1}{30}}$ to five decimal places

$$=\sqrt[3]{\tfrac{31}{30}}$$
$$=\{1(1+\tfrac{1}{30})^{\frac{1}{3}}\}$$
$$=1+\tfrac{1}{3}\times\tfrac{1}{30}+\frac{\frac{1}{3}(\frac{1}{3}-1)}{1\times2}(\tfrac{1}{30})^2$$
$$=1+0.01111-0.00012$$
$$=1.01099.$$

17. $\sqrt[7]{129}$ to six decimal places

$$=\{128(1+\tfrac{1}{128})\}^{\frac{1}{7}}$$
$$=\{2^7(1+\tfrac{1}{128})\}^{\frac{1}{7}}$$
$$\therefore \sqrt[7]{129}=2(1+\tfrac{1}{128})^{\frac{1}{7}}$$
$$=2\{1+\tfrac{1}{7}\times\tfrac{1}{128}+\frac{\frac{1}{7}(\frac{1}{7}-1)}{2}(\tfrac{1}{128})^2\}+\cdots\cdots$$
$$=2(1+0.001116-0.000004)$$
$$=2.002224.$$

18. $(1-2x+3x^2)^{-\frac{2}{5}}$ to four terms

$$=\{1-(2x-3x^2)\}^{-\frac{2}{5}}$$
$$=1-(-\tfrac{2}{5})(2x-3x^2)+\frac{(-\frac{2}{5})(-\frac{7}{5})}{1\times 2}(2x-3x^2)^2$$
$$-\frac{(-\frac{2}{5})(-\frac{7}{5})(-\frac{12}{5})}{1\times 2\times 3}(2x-3x^2)^3$$
$$=1+\left(\frac{4x}{5}-\frac{6x^2}{5}\right)+\left(\frac{28x^2}{25}-\frac{84x^3}{25}+\frac{63x^4}{25}\right)+\left(\frac{224x^3}{125}-\cdots\cdots\right)$$
$$=1+\frac{4x}{5}-\frac{2x^2}{25}-\frac{196x^3}{125}-\cdots\cdots$$

19. $\dfrac{(1+2x)^2}{(1+3x)^3}$ to coefficient of x^4

$$=(1+2x)^2(1+3x)^{-3}$$
$$=(1+4x+4x^2)(1-9x+54x^2-270x^3+1215x^4).$$

The terms containing x^4 will be

$$1215x^4-4x(270x^3)+4x^2(54x^2)$$
$$=351x^4.$$

20. $(1+x)^{\frac{1}{2}}$ expanded

$$1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{\lfloor 3}x^3$$
$$+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{\lfloor 4}x^4\cdots\cdots$$
$$=1+\tfrac{1}{2}x-\frac{1}{2\times 2^2}x^2+\frac{1\times 3}{2\times 3\times 2^3}x^3-\frac{1\times 3\times 5}{2\times 3\times 4\times 2^4}x^4\cdots\cdots$$

When $x=1$, this becomes $(1+1)^{\frac{1}{2}}$

$$=1+\frac{1}{2}-\frac{1}{2\times 2^2}+\frac{1\times 3}{2\times 3\times 2^2}-\frac{1\times 3\times 5}{2\times 3\times 4\times 2^4}+\cdots\cdots$$

Exercise CXXII.

1. If I throw a single die, what is the chance that it will turn up

(i.) An ace?

(ii.) An ace or a two?

(iii.) Neither an ace nor a two?

Since a die has 6 faces,

(i.) The chance of an ace is $\frac{1}{6}$.

(ii.) The chance of an ace is $\frac{1}{6}$, and the chance of a two is $\frac{1}{6}$. Hence the chance that either an ace or a two turns up is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

(iii.) There being 4 faces besides the ace and the two, the chance that neither an ace nor a two turns up is $\frac{4}{6} = \frac{2}{3}$.

2. The chance of a plan succeeding is $\frac{1}{4}$. What is the chance that it fails?

Since there is one chance in 4 that it succeeds, there are 3 chances in 4 that it fails. Hence the chance of failure is $\frac{3}{4}$.

3. If the odds are 10 to 1 against an event, what is the probability of its happening?

Out of 11 chances there are 10 that the event will not happen and 1 that it will. Hence the probability of its happening is 1 in 11 or $\frac{1}{11}$.

4. If the odds are 5 to 2 in favor of the success of an experiment, what are the respective chances of success or failure?

Out of 7 chances, 5 are favorable and 2 unfavorable. Hence the chance of success is $\frac{5}{7}$, and of failure $\frac{2}{7}$.

5. The chance of an event is $\frac{2}{9}$. Find the odds for or against the event.

Out of 9 chances, 2 are favorable for the event and 7 unfavorable. Hence the odds against the event are 7 to 2.

6. What is the chance of a year, not a leap year, having 53 Sundays?

If a year of 365 days has 53 Sundays, it must begin on Sunday. But, as the year may begin on any day of the week, the chance that it will begin on Sunday is $\frac{1}{7}$.

7. Two numbers are chosen at random. Find the chance that their sum is even.

The numbers may be

(i.) both even,

(ii.) both odd,

(iii.) the first odd, the second even,

(iv.) the first even, the second odd;

and all these four cases are equally likely to occur. But the sum is even only in the first two cases, and the chance that one of these will happen is $\frac{2}{4} = \frac{1}{2}$.

8. If 4 cards are drawn from a pack, what is the chance that they will all be hearts?

Four cards can be selected from the 52 in the pack in $\frac{52!}{4!\,48!}$ ways.

Four hearts can be selected from the 13 hearts in $\frac{13!}{4!\,9!}$ ways.

Since any set of four cards is as likely to be drawn as any other, the chance that they will all be hearts is

$$\frac{13!}{4!\,9!} \div \frac{52!}{4!\,48!} = \frac{13!\,48!}{9!\,52!}$$
$$= \frac{13 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49}$$
$$= \frac{11}{4165}.$$

9. If 10 persons stand in a line, what is the chance that 2 assigned persons will stand together?

The total number of pairs of places among the 10 is $\frac{10!}{2!\,8!} = 45$. Of these, 9 are adjacent; viz., the first and second, second and third, etc., up to the ninth and tenth. Hence the chance that the two assigned persons will stand together is $\frac{9}{45} = \frac{1}{5}$.

10. If 10 persons form a ring, what is the chance that 2 assigned persons will stand together?

As in Ex. 9, the total number of pairs of places among the 10 is 45. But the number of adjacent places is now 10, since the first and tenth are now adjacent. Hence the chance that the two assigned persons will stand together is $\frac{10}{45} = \frac{2}{9}$.

11. Show that if n persons sit down at a round table, the odds against 2 particular persons sitting next each other are $n - 3$ to 2.

One of the two persons being seated, there are 2 seats next to him and $n - 3$ not next to him. The second person being equally likely to take any seat, the odds that he does not sit next the first one are $n - 3$ to 2.

12. If two letters are selected at random out of the alphabet, what is the chance that both will be vowels?

With the 26 letters of the alphabet, $\frac{26!}{2!\,24!} = 325$ pairs can be formed and from the 5 vowels $\frac{5!}{2!\,3!} = 10$ pairs. Hence the chance that a random pair should be vowels is $\frac{10}{325} = \frac{2}{65}$.

13. Five men, A, B, C, D, E, speak at a meeting, and it is known that A speaks before B. What is the chance that A speaks *immediately* before B?

The five men can be assigned to speak in $5! = 120$ different

orders. In one-half of these A will precede B. Also there are 4 ways in which B can immediately follow A; viz., A may speak first and B second, or A second and B third, or A third and B fourth, or A fourth and B last. The 3 remaining speakers may be arranged in each case in 6 different orders.

Hence the chance that A speaks immediately before B is

$$\frac{4 \times 6}{60} = \frac{2}{5}.$$

14. A, B, C have equal claims for a prize. A says to B, "You and I will draw lots, and the winner shall draw lots with C for the prize." Is this fair?

The chance that A will win from B is $\frac{1}{2}$. If he wins from B, the chance that he will win from C is also $\frac{1}{2}$. To get the prize he must win from B and then from C, and his chance of doing this is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Similarly, B's chance of winning is $\frac{1}{4}$; but C's chance, since he is sure of a trial and has to win only once, is $\frac{1}{2}$.

Originally A, B, C had each a chance of $\frac{1}{3}$. A's proposal is equally unfair to A and B, and more than fair to C.

15. A person is allowed to draw 2 tickets from a bag containing 40 blank tickets, and 10 tickets each entitling the holder to a prize of \$100. What is his expectation?

From the 50 tickets in the bag, 2 can be drawn in $\frac{|\underline{50}}{|\underline{2}\,|\underline{48}} = 1225$ ways.

Two tickets entitling to prizes can be drawn from the ten in $\frac{|\underline{10}}{|\underline{2}\,|\underline{8}} = 45$ ways, and the expectation from drawing 2 prizes is $\frac{45}{1225} \times \$200 = \$\frac{360}{49}$.

One ticket entitling to a prize and one blank can be drawn in $40 \times 10 = 400$ ways, and the expectation from the event is $\frac{400}{1225} \times \$100 = \$\frac{1600}{49}$.

Hence the total expectation is

$$\$\frac{360 + 1600}{49} = \$40.$$

16. One of two events must happen. If the chance of one is $\frac{2}{3}$ of the chance of the other, find the odds on the first.

Since one of the two events must happen, the sum of their probabilities is 1; and as the chance of the first is $\frac{2}{3}$ of the chance of the second, their respective chances are $\frac{2}{5}$ and $\frac{3}{5}$. Hence the odds on the first are 2 to 3.

17. There are 3 events, A, B, C, one of which must happen. The odds are 3 to 8 on A, and 2 to 5 on B. Find the odds on C.

The chance that A happens is $\frac{3}{11}$; that B happens, $\frac{2}{7}$. Hence (Rule V), since A, B, or C must

happen, the chance that C happens is $1-(\frac{2}{7}+\frac{3}{11})=\frac{34}{77}$; and the odds on C are 34 to 43.

18. In a bag are 7 white and 5 red balls. Find the chance that if one is drawn it will be (i.) white or (ii.) red; or, if two are drawn, that they will be (i.) both white, (ii.) both red, or (iii.) one white and the other red.

Since there are 12 balls in all, the chance of drawing a white ball is $\frac{7}{12}$; of drawing a red ball, $\frac{5}{12}$.

From the 12 balls, 2 can be drawn in $\frac{|\underline{12}}{|\underline{2}\,|\underline{10}}=66$ ways. Two white balls can be drawn from the 7 in $\frac{|\underline{7}}{|\underline{2}\,|\underline{5}}=21$ ways; two red balls from the 5 in $\frac{|\underline{5}}{|\underline{2}\,|\underline{3}}=10$ ways; one white ball and one red ball in $7\times 5=35$ ways. Hence the chance of drawing 2 white balls is $\frac{21}{66}=\frac{7}{22}$; 2 red balls, $\frac{10}{66}=\frac{5}{33}$; one white ball and 1 red, $\frac{35}{66}$.

19. If 3 cards are drawn from a pack, what is the chance that they will be king, queen, and knave of the same suit?

Three cards can be drawn from the 52 in the pack in $\frac{|\underline{52}}{|\underline{3}\,|\underline{49}}=22{,}100$ ways. King, queen, and knave of the same suit can be drawn in 4 ways.

Hence the chance of drawing king, queen, and knave of the same suit is $\frac{4}{22100}=\frac{1}{5525}$.

20. A general orders 2 men by lot out of 100 mutineers to be shot, the real leaders of the mutiny being ten in number. Find the chance (i.) that 1 only, (ii.) that two, of the leaders will be shot.

Two men can be selected from 100 in $\frac{|\underline{100}}{|\underline{2}\,|\underline{98}}=4950$ ways. One leader and one follower can be selected in $10\times 90=900$ ways; two leaders in $\frac{|\underline{10}}{|\underline{2}\,|\underline{8}}=45$ ways.

Hence the chance that only one leader will be shot is $\frac{900}{4950}=\frac{2}{11}$; that two leaders will be shot,

$$\frac{45}{4950}=\frac{1}{110}.$$

21. Show that the odds are 8 to 1 against throwing 9 in a single throw with 2 dice.

The total number of throws with 2 dice is $6\times 6=36$. The number of ways of throwing 9 is 4; viz., 3 and 6, 6 and 3, 4 and 5, 5 and 4. Hence the odds against throwing 9 is 32 to 4, or 8 to 1.

22. Show that in a throw with 3 dice the chance of either a triplet or a doublet is $\frac{4}{9}$.

The total number of throws with 3 dice is $6\times 6\times 6=216$. The number of ways of throwing

a triplet is 6, since it may be three 1's, three 2's, etc. A doublet may happen with any 2 of the three dice, and accordingly the number of ways of throwing a doublet is equal to the number of doublets, 6, multiplied by the numbers of pairs of dice, 3, and multiplied further by the number of ways the odd die may fall. This last number is 5, since the case of a triplet has already been considered.

Hence a triplet can be thrown in 6 ways, and a doublet, but not a triplet, in $6 \times 3 \times 5 = 90$ ways. Either a doublet or a triplet can be thrown in $6 + 90 = 96$ ways. The chance of throwing a doublet or triplet is therefore $\frac{96}{216} = \frac{4}{9}$.

23. In a bag are 5 white and 4 black balls. If drawn out one by one, what is the chance that the first will be white, the second black, and so on, alternately?

The chance that the first ball will be white is $\frac{5}{9}$.

The first ball having been drawn, if it is white, the chance that the second one is black is $\frac{4}{8} = \frac{1}{2}$.

The chance that the third ball is white is then $\frac{4}{7}$; that the fourth is white, $\frac{3}{6} = \frac{1}{2}$; that the fifth is black, $\frac{3}{5}$; that the sixth is white, $\frac{2}{4} = \frac{1}{2}$; and so on.

Hence the chance that the balls will be alternately white and black is $\frac{5}{9} \times \frac{1}{2} \times \frac{4}{7} \times \frac{1}{2} \times \frac{3}{5} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{126}$.

24. A bag contains 2 white balls, 3 black balls, and 5 red balls. If 4 balls are drawn, find the chance that there will be among them:

(i.) Both the white balls.

(ii.) Two *only* of the black balls.

(iii.) Two *at least* of the red balls.

Four balls can be drawn from the 10 in the bag in $\frac{\lfloor 10}{\lfloor 4 \lfloor 6} = 210$ ways.

(i.) If the 4 balls are to include the 2 white ones, the other 2 can be drawn from the remaining 8 in $\frac{\lfloor 8}{\lfloor 2 \lfloor 6} = 28$ ways. Hence the chance that the 2 white balls are drawn among the 4 is $\frac{28}{210} = \frac{2}{15}$.

(ii.) Two black balls can be drawn from the 3 black balls in $\frac{\lfloor 3}{\lfloor 2 \lfloor 1} = 3$ ways. If exactly 2 black balls are to be included in the 4 drawn, the other 2 can be selected from the 7 white and red balls in $\frac{\lfloor 7}{\lfloor 2 \lfloor 5} = 21$ ways. Hence the chance of drawing exactly 2 black balls is $\frac{3 \times 21}{210} = \frac{3}{10}$.

(iii.) If *no* red ball is drawn, the 4 balls can be selected from the 5 white and black balls in $\frac{\lfloor 5}{\lfloor 4 \lfloor 1} = 5$ ways. Hence the chance that *no* red ball is drawn is $\frac{5}{210} = \frac{1}{42}$. If exactly *one* red ball is to be drawn, it can be selected from the 5 red balls in 5 ways. The 3 remaining balls can be selected from the 5 white and black balls in $\frac{\lfloor 5}{\lfloor 3 \lfloor 2} = 10$ ways. Hence the chance that exactly 1 red ball is drawn is $\frac{5 \times 10}{210} = \frac{5}{21}$. The chance of drawing either no red ball or exactly 1 is $\frac{1}{42} + \frac{5}{21} = \frac{11}{42}$; and the chance of drawing more than one red ball is $1 - \frac{11}{42} = \frac{31}{42}$.

Exercise CXXIII.

1. The chance that A can solve a certain problem is $\frac{1}{4}$, and the chance that B can solve it is $\frac{2}{3}$. What is the chance that the problem will be solved, if both try?

The chance that A will fail is $\frac{3}{4}$, and the chance that B will fail is $\frac{1}{3}$. Hence the chance that both will fail is $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$; and this is the chance that the problem will not be solved. The chance that it will be solved is, therefore, $1 - \frac{1}{4} = \frac{3}{4}$.

2. What is the chance of throwing at least one ace in 2 throws with one die?

The chance of not throwing an ace is $\frac{5}{6}$ each time. Hence the chance of not throwing an ace both times is $\frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$; and the chance of throwing at least one ace is $1 - \frac{25}{36} = \frac{11}{36}$.

3. If n coins are tossed up, what is the chance that one, and only one, will turn up head?

The chance that any given one will be head is $\frac{1}{2}$, and the chance that the rest will be tails is $(\frac{1}{2})^{n-1}$. Hence the chance that a given one will be head and the others tails is $\frac{1}{2} \times (\frac{1}{2})^{n-1} = (\frac{1}{2})^n$. And the chance that *some* one will be head and the rest tails is $n(\frac{1}{2})^n$.

4. What is the chance of throwing double sixes at least once in 3 throws with 2 dice?

The chance of throwing double sixes any given time is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, and the chance of not throwing them any given time is $1 - \frac{1}{36} = \frac{35}{36}$. Hence the chance of not throwing them in any of the 3 throws is $(\frac{35}{36})^3$; and the chance that they will be thrown at least once is $1 - (\frac{35}{36})^3 = \frac{3781}{46656}$.

5. A copper is tossed 3 times. Find the odds that it will fall:

(i.) Head and 2 tails without regard to order.

(ii.) Head, tail, head.

The coin may fall head three times, which can happen in only 1 way; or head twice and tail once, which can happen in 3 ways; or head once and tail twice, 3 ways; or tails three times, 1 way. All these 8 ways are equally likely to happen. Hence the chance that it will fall head and two tails is $\frac{3}{8}$.

The chance that it will fall head, tail, head is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

6. If a copper is tossed 4 times, find the odds that it will fall 2 heads and 2 tails sooner than 4 heads.

The total number of ways it can fall is $2^4 = 16$. It can fall head every time in only 1 way. It can fall head twice and tail twice in as many ways as 2 selections can be made from 4; viz.,

$$\frac{|\underline{4}}{|\underline{2}\,|\underline{2}} = 6.$$

Hence the odds are 6 to 1 that it will fall head twice and tail twice rather than head four times.

7. If from a lottery of 30 tickets, marked 1, 2, 3, ... four tickets are drawn, what is the chance that 1 and 2 will be among them?

Four tickets can be drawn from 30 in $\frac{|\underline{30}}{|\underline{4}\,|\underline{26}}$ ways. If 1 and 2 are to be among them, the other two can be drawn from the remaining 28 in $\frac{|\underline{28}}{|\underline{2}\,|\underline{26}}$ ways. Hence the chance that 1 and 2 will be drawn is

$$\frac{|\underline{28}}{|\underline{2}\,|\underline{26}} \div \frac{|\underline{30}}{|\underline{4}\,|\underline{26}} = \frac{|\underline{28}\,|\underline{4}}{|\underline{30}\,|\underline{2}} = \frac{2}{145}.$$

8. If 2 coppers are tossed 3 times, find the odds that they will fall 2 heads and 4 tails.

There are 3 ways of getting 2 heads and 4 tails. *First*, the first copper may fall head twice, all the other cases giving tails; *second*, each copper may fall head exactly once; and *third*, the second copper may fall head twice, all other cases giving tails.

The chance that the first copper will fall head twice and tail once is $3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$, since there are three ways of getting two heads and one tail. The chance that the second copper will fall tail every time is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. Hence the chance that the first copper gives two heads and a tail, and the second copper 3 tails, is $\frac{3}{8} \times \frac{1}{8} = \frac{3}{64}$; and the chance that one of the two coppers gives two heads and a tail, and the other three tails, is $2 \times \frac{3}{64} = \frac{3}{32}$.

The chance that each copper gives 1 head and 2 tails is $\frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$.

Hence the chance of tossing two heads and four tails is

$$\tfrac{3}{32} + \tfrac{9}{64} = \tfrac{15}{64}.$$

9. There are 10 tickets, five of which are numbered 1, 2, 3, 4, 5, and the other five are blank. Find the chance that the sum of the numbers on the tickets drawn in 3 trials will be 10, one ticket being drawn and then replaced at each trial.

The numbers drawn must be one of the five sets:

5, 5, 0; 5, 4, 1; 5, 3, 2;
4, 4, 2; 4, 3, 3.

Of these the first can be drawn as 5, 5, 0 or 5, 0, 5 or 0, 5, 5, and the 0 may be any one of the five blanks, giving $3 \times 5 = 15$ ways. The fourth and fifth sets can be drawn in 3 ways each. The other two sets can be drawn in 6 ways each, making in all $15 + 6 + 12 = 33$ ways of drawing a sum of 10.

The total number of possible drawings is $10 \times 10 \times 10 = 1000$.

Hence the chance of drawing a sum of 10 is $\frac{33}{1000}$.

10. Find the chance in Ex. 9 if the tickets are not replaced.

If the tickets are not replaced, the only combinations which give 10 are 5, 4, 1 and 5, 3, 2. 5, 4, 1 can be drawn in 6 ways. The chance that 5 will be drawn the first time is $\frac{1}{10}$. 5 not being replaced, the chance of drawing 4 the second time is $\frac{1}{9}$. And the chance of drawing 1 the third time is $\frac{1}{8}$. Hence the chance of drawing 5, 4, 1 in this order is $\frac{1}{10} \times \frac{1}{9} \times \frac{1}{8} = \frac{1}{720}$. Therefore the chance of drawing 5, 4, 1 in any order is $6 \times \frac{1}{720} = \frac{1}{120}$. And similarly, the chance of drawing 5, 3, 2 is $\frac{1}{120}$. Hence the chance of drawing a total of 10 is

$$\tfrac{1}{120} + \tfrac{1}{120} = \tfrac{1}{60}.$$

11. A bag contains 4 white and 6 red balls. A, B, and C draw each a ball, in order, replacing. Find the chance that they have drawn

(i.) Each a white ball.
(ii.) A and B white, C red.
(iii.) Two white and 1 red.

(i.) The chance that one draws a white ball being $\frac{4}{10} = \frac{2}{5}$, the chance that they all draw white balls is $(\frac{2}{5})^3 = \frac{8}{125}$.

(ii.) The chance that A and B draw white, and C red, is

$$(\tfrac{2}{5})^2 \times \tfrac{3}{5} = \tfrac{12}{125}.$$

(iii.) Multiplying the last result by 3, we have $\frac{36}{125}$ as the chance that some one of the three draws red and the other white.

12. Find the answer to Ex. 11 if the balls are not replaced.

The chance that A, B, and C successively draw white balls is now $\frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$.

The chance that A and B draw white, and C red is $\frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} = \frac{1}{10}$.

The chance that A draws white, B red, and C white is $\frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{1}{10}$; and the chance that A draws red, and B and C white is $\frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{10}$. Hence the chance that two white balls and one red are drawn is $\frac{3}{10}$.

13. A draws 4 times from a bag containing 2 white and 8 black balls, replacing. Find the chance that he has drawn

(i.) Two white, two black.
(ii.) Not less than two white.
(iii.) Not more than two white.
(iv.) One white, three black.

(i.) Two white balls and 2 black can be drawn in $\frac{|\underline{4}}{|\underline{2}\,|\underline{2}} = 6$ different orders. The chance of drawing a white ball is at every trial $\frac{2}{10} = \frac{1}{5}$, and the chance of drawing a black one is $\frac{4}{5}$. Hence the chance of drawing 2 white balls and 2 black is

$$6 \times (\tfrac{1}{5})^2 \times (\tfrac{4}{5})^2 = \tfrac{96}{625}.$$

(ii.) The chance of drawing a white ball 4 times is $(\frac{1}{5})^4$; of drawing a white ball 3 times and a black ball once, $4 \times (\frac{1}{5})^3 + \frac{4}{5}$; of drawing 2 white balls and 2 black ones, $\frac{96}{625}$, as just shown. Hence the chance of drawing not less than two white balls is

$$(\tfrac{1}{5})^4 + 4 \times (\tfrac{1}{5})^3 \times \tfrac{4}{5} + \tfrac{96}{625} = \tfrac{113}{625}.$$

(iii.) The chance of drawing 2 white balls and 2 black is $\frac{96}{625}$; of drawing 1 white and 3 black, $4 \times \frac{1}{5} \times (\frac{4}{5})^3$; of drawing 4 black, $(\frac{4}{5})^4$. Hence the chance of drawing not more than 2 white balls is

$$\tfrac{96}{625} + 4 \times \tfrac{1}{5} \times (\tfrac{4}{5})^3 + (\tfrac{4}{5})^4 = \tfrac{608}{625}.$$

(iv.) The chance of drawing 1 white and 3 black is

$$4 \times \tfrac{1}{5} \times (\tfrac{4}{5})^3 = \tfrac{256}{625}.$$

14. Find the odds against throwing one of the two numbers 7 or 11 in a single throw with 2 dice.

To give 7 or 11, the number thrown must be one of the sets

$$6,\ 1;\quad 5,\ 2;\quad 4,\ 3;\quad 6,\ 5;$$

and each of these can be thrown in two ways; for example, 6, 1 as 6, 1 or as 1, 6.

The total number of throws with 2 dice is 36. Hence the chance of throwing 7 or 11 is $\frac{4 \times 2}{36} = \frac{2}{9}$, and the odds against it are 7 to 2.

15. If a copper is tossed 5 times, what is the chance that it will fall head either 2 times or else 3 times?

The number of ways it can fall either head twice and tail three times, or tail twice and head

three times is $\frac{|\underline{5}}{|\underline{2}\,|\underline{3}} = 10$. The chance that it will fall head twice and tail three times in given order is $(\frac{1}{2})^5$. Hence the chance that it will fall head twice and tail three times in any order is $10 \times (\frac{1}{2})^5 = \frac{5}{16}$; and the chance that it will fall head exactly twice or exactly 3 times is

$$2 \times \tfrac{5}{16} = \tfrac{5}{8}.$$

16. Find the same chance if the copper is tossed 6 times.

The number of ways the copper can fall head exactly twice is now $\frac{|\underline{6}}{|\underline{2}\,|\underline{4}} = 15$, and the number of ways it can fall head exactly 3 times is $\frac{|\underline{6}}{|\underline{3}\,|\underline{3}} = 20$.

The chance that it will fall head exactly twice, in one order or another, is $15 \times (\frac{1}{2})^6$, and the chance that it will fall head exactly three times, in any order, is $20 \times (\frac{1}{2})^6$.

Hence the chance that it will fall head exactly twice or exactly three times is

$$15 \times (\tfrac{1}{2})^6 + 20 \times (\tfrac{1}{2})^6 = \tfrac{35}{64}.$$

17. In one bag are 10 balls and in another 6; and in each bag the balls are marked 1, 2, 3, etc. What is the chance that on drawing one ball from each bag the two balls will have the same number?

The number of pairs of balls that can be drawn is $6 \times 10 = 60$, and there are 6 pairs which have the same number. Hence the chance of drawing such a pair is

$$\tfrac{6}{60} = \tfrac{1}{10}.$$

18. A bag contains n balls. A person takes out one ball and then replaces it. He does this n times. What is the chance that he has had in his hand every ball in the bag?

The number of sets that can be drawn in n times is n^n. The number of orders in which the n balls can be arranged without repetitions is $|\underline{n}$. Hence the chance that there are no repetitions in the drawing is $\frac{|\underline{n}}{n^n}$.

19. If on an average 9 ships out of 10 return safe to port, what is the chance that out of 5 ships expected, at least 3 will return?

The chance that all the 5 ships will return is $(\frac{9}{10})^5$; that 4 will return and one be lost is $5 \times (\frac{9}{10})^4 \times \frac{1}{10}$; that 3 will return and 2 be lost is $10 \times (\frac{9}{10})^3 \times (\frac{1}{10})^2$. Hence the chance that at least 3 will return is

$$(\tfrac{9}{10})^5 + 5 \times (\tfrac{9}{10})^4 \times \tfrac{1}{10} + 10 \times (\tfrac{9}{10})^3 \times (\tfrac{1}{10})^2$$
$$= (\tfrac{9}{10})^3 (\tfrac{81}{100} + \tfrac{45}{100} + \tfrac{10}{100})$$
$$= \tfrac{12393}{12500}.$$

20. What is the chance of throwing double sixes at least once in 3 throws with a pair of dice?

The chance that they will be thrown any particular time is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$, and the chance that they will not be thrown any particular time is $1 - \frac{1}{36} = \frac{35}{36}$. Hence the chance that they will not be thrown any time is $(\frac{35}{36})^3$, and the chance that they will be thrown is $1 - (\frac{35}{36})^3 = \frac{3781}{46656}$.

21. What is the chance of throwing 15 in one throw with 3 dice?

The sum of 15 can be thrown as

6, 6, 3	3 ways,
6, 5, 4	6 ways,
5, 5, 5	1 way,
	10 ways.

The 3 dice can fall in $6 \times 6 \times 6 = 216$ ways. Hence the chance of throwing 15 is $\frac{10}{216} = \frac{5}{108}$.

22. In 5 throws with a single die what is the chance of throwing an ace

(i.) Three times exactly?
(ii.) Not less than three times?
(iii.) Not more than three times?

(i.) An ace can be thrown 3 times out of 5 in $\frac{\underline{|5}}{\underline{|3}\,\underline{|2}} = 10$ ways. The chance that it will be thrown any particular time is $\frac{1}{6}$, and the chance that it will be thrown 3 particular times and not thrown the other times is $(\frac{1}{6})^3 \times (\frac{5}{6})^2$. Hence the chance that it will be thrown exactly 3 times out of the 5 is $10 \times (\frac{1}{6})^3 \times (\frac{5}{6})^2 = \frac{125}{3888}$.

(ii.) The chance that an ace will not be thrown at all is $(\frac{5}{6})^5$; that an ace will be thrown only once is $5 \times (\frac{5}{6})^4 \times \frac{1}{6} = (\frac{5}{6})^5$; that an ace will be thrown exactly twice is $10 \times (\frac{5}{6})^3 \times (\frac{1}{6})^2$. Hence the chance that an ace will be thrown three times or more is

$$1-[2\times(\tfrac{5}{6})^5+10\,(\tfrac{5}{6})^3\times(\tfrac{1}{6})^2]=\tfrac{23}{648}.$$

(iii.) The chance that an ace will be thrown exactly 4 times is $5 \times (\frac{1}{6})^4 \times \frac{5}{6}$, and that it will be thrown 5 times is $(\frac{1}{6})^5$. Hence the chance that it will be thrown more than 3 times is $5 \times (\frac{1}{6})^4 \times \frac{5}{6} + (\frac{1}{6})^5 = \frac{13}{3888}$, and the chance that an ace will be thrown not more than 3 times is $\frac{3875}{3888}$.

23. In a bag are 3 white, 5 red, and 7 black balls, and a person draws three times, replacing. Find the chance that he has drawn:

(i.) A ball of each color.
(ii.) Two white, one red.
(iii.) Three red.
(iv.) Two red, one black.

(i.) In this case the balls can be drawn in 6 different orders. The chance of drawing one ball of each color in a particular order is $\frac{3}{15} \times \frac{5}{15} \times \frac{7}{15} = \frac{7}{225}$, and the

chance of drawing a ball of each color in any order is $6 \times \frac{7}{225} = \frac{14}{75}$.

(ii.) Here the balls can be drawn in only 3 different orders, and the chance is

$$3 \times (\tfrac{3}{15})^2 \times \tfrac{5}{15} = \tfrac{1}{25}.$$

(iii.) The chance of drawing 3 red balls is $(\frac{5}{15})^3 = \frac{1}{27}$.

(iv.) The chance of drawing 2 red and 1 black is

$$3 \times (\tfrac{5}{15})^2 \times \tfrac{7}{15} = \tfrac{7}{45}.$$

24. A and B play at chess, and A wins on an average 2 games out of 3. Find the chance of A's winning exactly 4 games out of the first 6, drawn games being disregarded.

A may win 4 games out of the first 6 in $\frac{\underline{|6}}{\underline{|4}\,\underline{|2}} = 15$ ways. His chance of winning any particular game is $\frac{2}{3}$, and of winning any 4 particular games and losing the other 2 is $(\frac{2}{3})^4 \times (\frac{1}{3})^2 = \frac{16}{729}$. Hence his chance of winning some 4 of the 6 games and losing the other 2 is $15 \times \frac{16}{729} = \frac{80}{243}$.

25. A and B engage in a game in which A's skill is to B's as $2:3$. Find the chance of A's winning at least 2 games out of the first 5, drawn games not being counted.

A's chance of losing any particular game is $\frac{3}{5}$. His chance of losing all the games is $(\frac{3}{5})^5$, and of losing just 4 games out of the 5 is $5 \times (\frac{3}{5})^4 \times \frac{2}{5}$. Hence his chance of losing 3 games or less, that is, his chance of winning 2 games or more, is

$$1 - [(\tfrac{3}{5})^5 + 5 \times (\tfrac{3}{5})^4 \times \tfrac{2}{5}]$$
$$= 1 - \tfrac{1053}{3125} = \tfrac{2072}{3125}.$$

26. The skill of A is double that of B. Find the odds against A's winning 4 games before B wins 2.

In order that A may win 4 games before B wins 2 it is necessary and sufficient that A should win at least 4 games out of the first 5.

The chance that A will lose any particular game is $\frac{1}{3}$, and the chance that he will lose some one of the first five games and win the rest is $5 \times \frac{1}{3} \times (\frac{2}{3})^4 = \frac{80}{243}$. The chance that he will win all the games is $(\frac{2}{3})^5 = \frac{32}{243}$. Hence the chance that he will win 4 games out of the first 5 is $\frac{80 + 32}{243} = \frac{112}{243}$, and the odds against his doing it is $131 : 112$.

27. If B's skill in a certain game is equal to three-fifths of A's, find A's chance of winning 5 games out of 8.

A's chance of winning a single game is $\frac{5}{8}$. His chance of winning exactly 5 games out of 8 is

$$\frac{\underline{|8}}{\underline{|5}\,\underline{|3}} \times \left(\frac{5}{8}\right)^5 \times \left(\frac{3}{8}\right)^3 = 56 \times \frac{5^5 \times 3^3}{8^8}.$$

His chance of winning exactly 6 games is

$$\frac{\underline{|8}}{\underline{|6}\,\underline{|2}} \times \left(\frac{5}{8}\right)^6 \times \left(\frac{3}{8}\right)^2 = 28 \times \frac{5^6 \times 3^2}{8^8}.$$

His chance of winning exactly 7 games is

$$8 \times \left(\frac{5}{8}\right)^7 \times \frac{3}{8} = 8 \times \frac{5^7 \times 3}{8^8}.$$

His chance of winning all the games is $\left(\frac{5}{8}\right)^8 = \frac{5^8}{8^8}.$

Hence his chance of winning at least 5 games out of 8 is

$$56 \times \frac{5^5 \times 3^3}{8^8} + 28 \times \frac{5^6 \times 3^2}{8^8} + 8 \times \frac{5^7 \times 3}{8^8} + \frac{5^8}{8^8}$$

$$= \frac{5^5}{8^8}(56 \times 27 + 28 \times 45 + 24 \times 25 + 125)$$

$$= \frac{5^5}{8^8} \times 3497.$$

28. A bag contains 4 red balls and 2 others, each of which is equally likely to be red or white. Three times in succession a ball is drawn and replaced. Find the chance that all the drawn balls are red.

The chance that the 2 unidentified balls are both red or both white is in each case $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; that one is red and the other white is $\frac{1}{2}$.

If the balls are all red, the drawn balls are of course all red. The chance of this is $\frac{1}{4}$.

If two of the balls were white, the chance that the drawn balls are all red would be $(\frac{2}{3})^3 = \frac{8}{27}$. Hence the chance that 2 balls are white and the drawn balls all red is $\frac{1}{4} \times \frac{8}{27} = \frac{2}{27}$.

If only 1 ball were white, the chance that the drawn balls are all red would be $(\frac{5}{6})^3 = \frac{125}{216}$. Hence the chance that there is only 1 white ball and that the drawn balls are all red is $\frac{1}{2} \times \frac{125}{216} = \frac{125}{432}$.

Therefore the chance that the drawn balls are all red, whatever the 2 unidentified balls may be, is

$$\frac{1}{4} + \frac{2}{27} + \frac{125}{432} = \frac{265}{432}.$$

29. A man has left his umbrella in one of three shops which he visited in succession. He is in the habit of leaving it, on an average, once in every four times that he goes to a shop. Find the chance that he left it in the first, second, and third shops respectively.

The chance that he would leave it in the first shop is $\frac{1}{4}$.

The chance that he would not leave it in the first shop, but would leave it in the second, is $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$.

The chance that he would not leave it in the first or second shop, but would leave it in the third, is $(\frac{3}{4})^2 \times \frac{1}{4} = \frac{9}{64}$.

Hence the probability that he would leave it in one of the three shops is $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} = \frac{37}{64}$.

But as the umbrella was certainly left in one of the shops, all

the preceding chances must be increased in the ratio of 64 : 37 (Rule IX.).

Hence the chance that it was left in the first shop is $\frac{64}{37} \times \frac{1}{4} = \frac{16}{37}$; in the second shop, $\frac{64}{37} \times \frac{3}{16} = \frac{12}{37}$; in the third shop, $\frac{64}{37} \times \frac{9}{64} = \frac{9}{37}$. Hence, 16 : 12 : 9.

30. A bets B \$10 to \$1 that he will throw head at least once in 3 trials. What is B's expectation? What would have been a fair bet?

The chance that A will not throw head at all in 3 trials is $(\frac{1}{2})^3 = \frac{1}{8}$; and the chance that he will throw head at least once is $1 - \frac{1}{8} = \frac{7}{8}$. B's expectation is therefore $\frac{1}{8}$ of gaining \$10, and $\frac{7}{8}$ of losing \$1; that is, his expectation is $\frac{10}{8} - \frac{7}{8} = \$\frac{3}{8}$.

A fair bet would have been in the ratio of the chances of winning and losing, that is, 7 to 1.

31. A draws 5 times (replacing) from a bag containing 3 white and 7 black balls. Every time he draws a white ball he is to receive \$1, and every time he draws a black ball he is to pay 50 cents. What is his expectation?

A's expectation is the same for each drawing, viz., $\frac{3}{10} \times \$1 - \frac{7}{10} \times \$\frac{1}{2} = -\$\frac{1}{20}$. His expectation from 5 drawings is therefore $-\$\frac{1}{4}$, The chances are that he will lose 25 cents.

32. From a bag containing 2 eagles, 3 dollars, and 3 quarter-dollars A is to draw one coin and then B three coins; and A, B, and C are to divide equally the value of the remainder. What are their expectations?

In the final division A, B, and C receive equal shares.

Consider their expectations from this division.

There are \$23.75 in the bag.

A and B are to draw out 4 coins at first.

4 coins can be selected from 8 in $\frac{\lfloor 8}{\lfloor 4 \lfloor 4} = 70$ ways.

A and B may draw:	Probability.	Amount remaining	Expectation for the division.
(1) 2 eagles, 2 dollars	$\frac{3}{70}$	\$1.75	$\frac{3}{70} \times \$1.75 = \$\frac{21}{280}$
(2) 2 eagles, 1 dollar, 1 quarter	$\frac{9}{70}$	\$2.50	$\frac{7}{90} \times \$2.50 = \$\frac{9}{280}$
(3) 2 eagles, 2 quarters	$\frac{3}{70}$	\$3.25	$\frac{3}{70} \times \$3.25 = \$\frac{39}{280}$
(4) 1 eagle, 3 dollars	$\frac{2}{70}$	\$10.75	$\frac{2}{70} \times \$10.75 = \$\frac{43}{140}$
(5) 1 eagle, 2 dollars, 1 quarter	$\frac{18}{70}$	\$11.50	$\frac{18}{70} \times \$11.50 = \$\frac{207}{70}$
(6) 1 eagle, 1 dollar, 2 quarters	$\frac{18}{70}$	\$12.25	$\frac{18}{70} \times \$12.25 = \$\frac{63}{20}$
(7) 1 eagle, 3 quarters	$\frac{2}{70}$	\$13.00	$\frac{2}{70} \times \$13.00 = \$\frac{13}{35}$
(8) 3 dollars, 1 quarter	$\frac{3}{70}$	\$20.50	$\frac{3}{70} \times \$20.50 = \$\frac{123}{140}$
(9) 2 dollars, 2 quarters	$\frac{9}{70}$	\$21.25	$\frac{9}{70} \times \$21.25 = \$\frac{153}{56}$
(10) 1 dollar, 3 quarters	$\frac{3}{70}$	\$22.00	$\frac{3}{70} \times \$22.00 = \$\frac{33}{35}$

Total expectation for the division $\$\frac{3244}{280}$ or $\$11\frac{41}{70}$.

Hence the expectation of each from the division is $\frac{1}{3}$ of $\$11\frac{41}{70}$, or $\$3.86\frac{4}{21}$.

Hence C's entire expectation is $\$3.86\frac{4}{21}$.

Again, A may draw an eagle. The probability of this is $\frac{2}{8}$, or $\frac{1}{4}$. The corresponding expectation is \$2.50.

Or, he may draw a dollar. The probability of this is $\frac{3}{8}$. The corresponding expectation of this is $37\frac{1}{2}$ cents.

Or, he may draw a quarter. The probability of this is $\frac{3}{8}$. The corresponding expectation is $9\frac{3}{8}$ cents.

Hence A's expectation from his draw is $\$2.96\frac{7}{8}$.

And A's total expectation is $\$6.83\frac{11}{168}$.

A and C's expectations together are $\$10.69\frac{43}{168}$.

Hence B's expectation is $\$23.75 - \$10.69\frac{43}{168} = \$13.05\frac{125}{168}$.

33. A, B, and C, staking each \$5, draw from a bag in which are 4 white and 6 black balls, each drawing in order, and the whole sum is to be received by him who first draws a white ball. What are their expectations:

(i.) Replacing the balls?
(ii.) Not replacing the balls?

(i.) The chance that A draws a white ball the first time is $\frac{2}{5}$; that A, B, and C fail the first time and that A draws a white ball the second time is $(\frac{3}{5})^3 \times \frac{2}{5}$; that they all fail twice and that A draws a white ball the third time is $(\frac{3}{5})^6 \times \frac{2}{5}$; and so on.

Hence A's chance of drawing a white ball first is

$$\frac{2}{5} + \frac{2}{5}(\frac{3}{5})^3 + \frac{2}{5}(\frac{3}{5})^6 + \cdots$$
$$= \frac{2}{5} \div [1 - (\frac{3}{5})^3] = \frac{25}{49}. \quad (\S\ 395.)$$

The chance that A fails the first time and that B draws a white ball is $\frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$; that A, B, and C all fail the first time, and A the second time, and that B draws a white ball the second time is $(\frac{3}{5})^4 \times \frac{2}{5} = (\frac{3}{5})^3 \times \frac{6}{25}$; and so on.

Hence B's chance of drawing a white ball first is

$$\frac{6}{25} + \frac{6}{25} \times (\frac{3}{5})^3 + \frac{6}{25} \times (\frac{3}{5})^6 + \cdots$$
$$= \frac{6}{25} \div [1 - (\frac{3}{5})^3] = \frac{15}{49}.$$

C's chance of winning is therefore $1 - \frac{25}{49} - \frac{15}{49} = \frac{9}{49}$.

Hence the expectations of A, B, and C respectively are

$$\frac{25}{49} \times \$15 = \$7\frac{32}{49},$$
$$\frac{15}{49} \times \$15 = \$4\frac{29}{49},$$
and
$$\frac{9}{49} \times \$15 = \$2\frac{37}{49}.$$

The chances are that A will gain $\$2\frac{32}{49}$, that B will lose $\$\frac{20}{49}$, and that C will lose $\$2\frac{12}{49}$.

(ii.) The chance that A draws a white ball the first time is, as before, $\frac{2}{5}$. The chance that A, B, C all draw black balls the first time, and that A draws a

white ball the second time, is $\frac{3}{5} \times \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{2}{21}$. The chance that A, B, C fail each time is $\frac{3}{5} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} = \frac{1}{210}$, and A *must* then draw a white ball. Hence A's chance of drawing a white ball first is

$$\frac{2}{5} + \frac{2}{21} + \frac{1}{210} = \frac{1}{2}.$$

The chance that A fails the first time and that B draws a white ball is $\frac{3}{5} \times \frac{4}{9} = \frac{4}{15}$. The chance that A, B, and C fail the first time, and A the second time, and that B draws a white ball the second time is

$$\frac{3}{5} \times \frac{5}{9} \times \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6} = \frac{1}{21}.$$

Hence B's chance of drawing a white ball first is $\frac{4}{15} + \frac{1}{21} = \frac{11}{35}$, and C's chance is $1 - \frac{1}{2} - \frac{11}{35} = \frac{13}{70}$.

Hence the expectations of A, B, and C respectively are

$$\frac{1}{2} \times \$15 = \$7\tfrac{1}{2},$$
$$\frac{11}{35} \times \$15 = \$4\tfrac{5}{7},$$
and
$$\frac{13}{70} \times \$15 = \$2\tfrac{11}{14}.$$

The chances are that A will gain $\$2\frac{1}{2}$, that B will lose $\$\frac{2}{7}$, and that C will lose $\$2\frac{3}{14}$.

Exercise CXXIV.

Note. — Four-place logarithms do not give very accurate results in some of the following problems.

1. In how many years will \$100 amount to \$1050 at 5 per cent compound interest?

$$P = 100,\ R = 1.05,\ A = 1050.$$
$$100 \times (1.05)^n = 1050,$$
$$(1.05)^n = 10.5,$$
$$n \log 1.05 = \log 10.5,$$
$$n = \frac{\log 10.5}{\log 1.05}$$
$$= \frac{1.0212}{0.0212}$$
$$= 48, \text{ nearly}.$$

The time is 48 years, nearly.

2. In how many years will $\$A$ amount to $\$B$ (i.) at simple interest, (ii.) at compound interest, r and R being used in their usual sense?

(i.) $\frac{B-A}{Ar}$ years.

(ii.) $B = AR^n$.

$$\therefore n \log R = \log B - \log A,$$
$$n = \frac{\log B - \log A}{\log R}.$$

The time is $\frac{\log B - \log A}{\log R}$ years.

3. Find the difference (to five places of decimals) between the amount of \$1 in 2 years, at 6 per cent compound interest, according as the interest is due yearly or monthly.

If the interest is due yearly,

$$A = (1.06)^2 \times 1$$
$$= 1.1236.$$

If the interest is due monthly,

$$A = (1.005)^{24} \times 1$$
$$= 1.12923.$$

Hence the difference is

\$0.00563.

4. At 5 per cent, find the amount of an annuity A which has been left unpaid for 4 years.

By § 452, the amount due is

$$\frac{S(R^n-1)}{r}=\frac{A[(1.05)^4-1]}{0.05}$$
$$=\frac{A(0.2155)}{0.05}$$
$$=4.31A\text{, nearly.}$$

5. Find the present value of an annuity of \$100 for 5 years, reckoning interest at 4 per cent.

$$P=\frac{S}{R-1}\times\frac{R^n-1}{R^n}$$
$$=\frac{100}{0.04}\times\frac{1.04^5-1}{1.04^5}$$
$$=444.$$

The present value is \$444.

6. A perpetual annuity of \$1000 is to be purchased, to begin in 10 years. If interest is reckoned at $3\frac{1}{2}$ per cent, what should be paid for it?

$$P=\frac{S}{R^n(R-1)}$$
$$=\frac{1000}{1.035^{10}\times 0.035}$$
$$=20{,}270\text{, nearly.}$$

The amount paid should be \$20,270.

7. A debt of \$1850 is discharged by two payments of \$1000 each, at the end of one and two years. Find the rate of interest paid.

$$\begin{aligned}\text{Amount of \$1850 for 2 years} &= 1850R^2,\\ \text{Amount of \$1000 for 1 year} &= 1000R,\\ \text{Balance due} &= 1000.\\ \therefore 1850R^2-1000R &= 1000,\\ 37R^2-20R &= 20,\\ R &= \frac{10+\sqrt{840}}{37}\\ &= 1.0535.\end{aligned}$$

Hence the rate of interest is 5.35 per cent.

8. Reckoning interest at 4 per cent, what annual premium should be paid for 30 years in order to secure \$2000 to be paid at the end of that time, the premium being due at the beginning of each year?

$$P=\frac{Ar}{R(R^n-1)}$$
$$=\frac{2000\times 0.04}{1.04(1.04^{30}-1)}$$
$$=\frac{80}{1.04\times 2.2426}$$
$$=34.402.$$

The annual premium should be \$34.40.

9. An annual premium of \$150 is paid to a life insurance company for insuring \$5000. If money is worth 4 per cent, for how many years must the pre-

mium be paid in order that the company may sustain no loss?

$$A = \frac{S(R^n - 1)}{r},$$

$$R^n = 1 + \frac{Ar}{S},$$

$$(1.04)^n = 1 + \frac{5000 \times 0.04}{150}$$

$$= 1 + \frac{200}{150}$$

$$= \frac{7}{3}.$$

$$n \log 1.04 = \log 7 - \log 3,$$

$$n = \frac{\log 7 - \log 3}{\log 1.04}$$

$$= \frac{0.36798}{0.01703}$$

$$= 22 \text{ nearly.}$$

The premium must be paid for 22 years.

10. What may be paid for bonds due in 10 years, and bearing semi-annual coupons of 4 per cent each, in order to realize 3 per cent semi-annually, if money is worth 3 per cent semi-annually?

By § 454,
$$P(1+x)^n = S + \frac{Sr[(1+q)^n - 1]}{q}$$

$$= S\left[\frac{q + r(1+q)^n - r}{q}\right]$$

$$\frac{P}{S} = \frac{q + r(1+q)^n - r}{q(1+x)^n}$$

$$= \frac{0.03 + 0.04 \times 1.03^{20} - 0.04}{0.03 \times (1.03)^{20}}$$

$$= \frac{0.03 + 0.04 \times 1.8063 - 0.04}{0.03 \times 1.8063}$$

$$= \frac{0.062252}{0.054189}$$

$$= 1.15, \text{ nearly.}$$

The price paid should be 115, nearly.

11. When money is worth 2 per cent semi-annually, if bonds having 12 years to run and bearing semi-annual coupons of $3\frac{1}{2}$ per cent each are bought at $114\frac{1}{8}$, what per cent is realized on the investment?

$$1 + x = \left(\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right)^{\frac{1}{n}}$$

$$= \left(\frac{1}{1.14125} \times \frac{0.02 + 0.035(1.02^{24} - 1)}{0.02}\right)^{\frac{1}{24}}$$

$$= \left(\frac{1}{1.14125} \times \frac{0.02 + 0.035 \times 0.6084}{0.02}\right)^{\frac{1}{24}}$$
$$= \left(\frac{2.0647}{1.14125}\right)^{\frac{1}{24}}$$
$$= 1.025.$$

The per cent realized is $2\frac{1}{2}$ semi-annually; that is, 5 per cent.

12. If \$126 is paid for bonds due in 12 years and yielding $3\frac{1}{2}$ per cent semi-annually, what per cent is realized on the investment, provided money is worth 2 per cent semi-annually?

$$1 + x = \left(\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right)^{\frac{1}{n}}$$
$$= \left(\frac{1}{1.26} \times \frac{0.02 + 0.035(1.02^{24} - 1)}{0.02}\right)^{\frac{1}{24}}$$
$$= \left(\frac{1}{1.26} \times \frac{0.02 + 0.035 \times 0.6084}{0.02}\right)^{\frac{1}{24}}$$
$$= \left(\frac{2.0647}{1.26}\right)^{\frac{1}{24}}$$
$$= 1.0207.$$

The per cent realized is 4.2, nearly.

13. A person borrows \$600.25. How much must he pay annually that the whole debt may be discharged in 35 years, allowing simple interest at 4 per cent?

Let $P =$ amount of debt.
$S =$ annual payment.

Then $$PR^n = \frac{S(R^n - 1)}{r}.$$

$$\therefore S = \frac{PrR^n}{R^n - 1},$$

$P = 600.25,$
$r = 0.04,$
$R = 1.04,$
$n = 35,$

$$S = \frac{600.25 \times 0.04 \times 1.04^{35}}{1.04^{35} - 1}$$
$$= \frac{94.5}{2.935} = 32.3.$$

He must pay \$32.30 per year.

14. A perpetual annuity of \$100 a year is sold for \$2500. At what rate is the interest reckoned?

\$100 is the interest on \$2500 for 1 year. Hence the rate is 4 per cent.

15. A perpetual annuity of \$320, to begin 10 years hence, is to be purchased. If interest is reckoned at $3\frac{1}{5}$ per cent, what should be paid for it?

Let P denote the amount paid. Then at the end of 10 years P is worth $P \times 1.032^{10} = P \times 1.37025$. The interest on this amount at $3\frac{1}{5}$ per cent should be \$320. Hence

$$P \times 1.37025 \times 0.032 = 320$$
$$P = \frac{10000}{1.37025}$$
$$= 7298.$$

The price paid should be \$7298.

16. A sum of \$10,000 is loaned at 4 per cent. At the end of the first year a payment of \$400 is made, and at the end of each following year a payment is made greater by 30 per cent than the preceding payment. Find in how many years the debt will be paid.

The amount of the original loan at the end of n years is

$$10{,}000 \times 1.04^n.$$

The amount of the several payments at the end of the n years is

$$400 \times 1.04^{n-1} + 400 \times 1.30 \times 1.04^{n-2}$$
$$+ 400 \times 1.30^2 \times 1.04^{n-3} + \cdots\cdots$$
$$= \frac{400 \times 1.04^{n-1}\left[\left(\frac{1.30}{1.04}\right)^n - 1\right]}{\frac{1.30}{1.04} - 1}$$
$$= \frac{400 \times 1.04^{n}\left[\left(\frac{1.30}{1.04}\right)^n - 1\right]}{0.26}.$$

When the debt is paid,

$$10{,}000 \times 1.04^n = \frac{400 \times 1.04^{n}\left[\left(\frac{1.30}{1.04}\right)^n - 1\right]}{0.26},$$

$$\frac{10000 \times 0.26}{400} = \left(\frac{1.30}{1.04}\right)^n - 1,$$

$$\left(\frac{1.30}{1.04}\right)^n = 1 + \frac{2600}{400}$$

$$= 7.5,$$

$$n \log \frac{1.30}{1.04} = \log 7.5,$$

$$n = \frac{\log 7.5}{\log 1.30 - \log 1.04}$$

$$= \frac{0.87506}{0.11394 - 0.01703}$$

$$= 9, \text{ nearly.}$$

The debt will be paid in 9 years.

17. A man with a capital of $100,000 spends every year $9000. If the current rate of interest is 5 per cent, in how many years will he be ruined?

The amount of the original capital at the end of n years is

$$100.000 \times 1.05^n.$$

The sums spent yearly amount at the end of the n years to

$$9000 \times 1.05^{n-1} + 9000 \times 1.05^{n-2} + 9000 \times 1.05^{n-3} + \cdots\cdots$$

$$= \frac{9000 \times 1.05^{n-1}(1.05^{-n} - 1)}{1.05^{-1} - 1}.$$

When the money is all spent,

$$100,000 \times 1.05^n = \frac{9000 \times 1.05^n(1.05^{-n} - 1)}{-0.05},$$

$$100,000 = \frac{9000(1 - 1.05^{-n})}{0.05},$$

$$1 - 1.05^{-n} = \frac{5}{9},$$

$$1.05^{-n} = \frac{4}{9},$$

$$n = \frac{\log 9 - \log 4}{\log 1.05}$$

$$= \frac{0.3521}{0.0212}$$

$$= 17, \text{ nearly.}$$

He will be ruined in 17 years.

18. Find the amount of \$365 at compound interest for 20 years at 5 per cent.

The amount is

$$\$365 \times 1.05^{20} = \$969.$$

19. In how many years will \$20 amount to \$150 at 4 per cent compound interest?

$$20 \times 1.04^n = 150,$$

$$n = \frac{\log 150 - \log 20}{\log 1.04}$$

$$= \frac{2.17609 - 1.30103}{0.01703}$$

$$= 51, \text{ nearly.}$$

The required time is 51 years.

20. At what rate per cent, compound interest, will \$2500 amount to \$3450 in 7 years?

$$2500\,R^7 = 3450,$$

$$R = \sqrt[7]{\frac{3450}{2500}}$$

$$= \sqrt[7]{1.38}$$

$$= 1.047.$$

The rate is $4\frac{7}{10}$ per cent.

21. If the population of a State increases in 10 years from 2,009,000 to 2,487,000, find the yearly rate of increase.

$$2{,}009{,}000\,R^{10} = 2{,}487{,}000,$$

$$R = \sqrt[10]{\frac{2487}{2009}}$$

$$= 1.0216.$$

The annual rate of increase is

$2\frac{1}{6}$ per cent, nearly.

22. The population of a State now is 1,918,600, and the yearly rate of increase is 2.38 per cent. Determine its population 10 years hence.

The population 10 years hence will be

$$1{,}918{,}600 \times 1.0238^{10} = 2{,}428{,}000.$$

23. A banker borrows a sum of money at $3\frac{1}{2}$ per cent, interest payable annually, and loans the same at 5 per cent, interest payable quarterly. If his annual gain is \$441, determine the sum borrowed.

If the sum is A, it will amount in one year, at $3\frac{1}{2}$ per cent annually, to

$$A \times 1.035.$$

At 5 per cent, payable quarterly, it will amount to

$$A \times 1.0125^4 = A \times 1.051.$$

Hence

$$A\,(1.051 - 1.035) = 441,$$

$$A \times 0.016 = 441,$$

$$A = \frac{441}{0.016}$$

$$= 27{,}563.$$

The sum borrowed was \$27,563.

Exercise CXXV.

1. Find continued fractions for $\frac{123}{157}$; $\frac{159}{47}$; $\sqrt{5}$; $\sqrt{11}$; $4\sqrt{6}$; and find the fifth convergent to each.

(i.) $$\frac{123}{157}=\cfrac{1}{1+\cfrac{34}{123}}=\cfrac{1}{1+\cfrac{1}{3+\cfrac{21}{34}}}$$

$$=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{13}{21}}}}=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{8}{13}}}}}$$

$$=\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}}}}}$$

Fifth convergent $=\frac{11}{14}$.

(ii.) $$\frac{159}{47}=3+\frac{18}{47}$$

$$=3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}}}$$

Fifth convergent $=3\frac{5}{13}$.

(iii.) Let
$$\sqrt{5} = 2 + \frac{1}{x},$$

then
$$x = \frac{1}{\sqrt{5} - 2}$$
$$= \sqrt{5} + 2.$$

Let
$$\sqrt{5} + 2 = 4 + \frac{1}{y},$$

then
$$y = \frac{1}{\sqrt{5} - 2}$$
$$= x.$$

Hence
$$\sqrt{5} = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \text{etc.}}}}$$
$$= 2 + \frac{\dot{1}}{4}.$$

Fifth convergent $= 2\frac{305}{1292}$.

(iv.) Let
$$\sqrt{11} = 3 + \frac{1}{x},$$

then
$$x = \frac{1}{\sqrt{11} - 3}$$
$$= \frac{\sqrt{11} + 3}{2}.$$

Let
$$\frac{\sqrt{11} + 3}{2} = 3 + \frac{1}{y},$$

then
$$y = \frac{2}{\sqrt{11} - 3}$$
$$= \sqrt{11} + 3.$$

Let
$$\sqrt{11} + 3 = 6 + \frac{1}{z},$$

then
$$z = \frac{1}{\sqrt{11} - 3}$$
$$= x.$$

Hence
$$\sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \text{etc.}}}}$$
$$= 3 + \frac{\dot{1}}{3+}\,\frac{\dot{1}}{6}.$$

Fifth convergent $= 3\frac{379}{1197}$.

(v.) Let
$$4\sqrt{6} = 9 + \frac{1}{x},$$
$$\therefore x = \frac{1}{4\sqrt{6}-9}$$
$$= \frac{4\sqrt{6}+9}{15}.$$

Let
$$\frac{4\sqrt{6}+9}{15} = 1 + \frac{1}{y}.$$
$$\therefore y = \frac{15}{4\sqrt{6}-6}$$
$$= \frac{2\sqrt{6}+3}{2},$$

Let
$$\frac{2\sqrt{6}+3}{2} = 3 + \frac{1}{z},$$
$$\therefore z = \frac{2}{2\sqrt{6}-3}$$
$$= \frac{4\sqrt{6}+6}{15}.$$

Let
$$\frac{4\sqrt{6}+6}{15} = 1 + \frac{1}{u}.$$
$$\therefore u = \frac{15}{4\sqrt{6}-9}$$
$$= 4\sqrt{6}+9$$
$$= 18 + \frac{1}{x},$$
$$\therefore 4\sqrt{6} = 9 + \frac{\dot{1}}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{\dot{1}}{18}.$$

Fifth convergent is $\frac{970}{99}$.

2. Find the continued fraction for $\frac{47}{257}$; $\frac{457}{204}$; $\frac{2065}{4626}$; $\frac{2991}{568}$; and find the third convergent to each.

(i.)

$$\frac{47}{257}=\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{7+\cfrac{1}{3}}}}$$

Third convergent $= \frac{15}{82}$.

(iii.)

$$\frac{2065}{4626}=\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{8+\cfrac{1}{10}}}}}.$$

Third convergent $= \frac{25}{56}$.

(ii.)

$$\frac{457}{204}=2+\cfrac{1}{4+\cfrac{1}{6+\cfrac{1}{8}}}$$

Third convergent $= \frac{457}{204}$.

(iv.)

$$\frac{2991}{568}=5+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7}}}}}$$

Third convergent $= 5\frac{4}{15}$.

3. Find continued fractions for $\sqrt{21}$; $\sqrt{22}$; $\sqrt{33}$; $\sqrt{55}$.

(i.)

Let $\sqrt{21} = 4 + \frac{1}{x}$,

then $x = \frac{1}{\sqrt{21}-4}$

$= \frac{\sqrt{21}+4}{5}$,

Let $\frac{\sqrt{21}+4}{5} = 1 + \frac{1}{y}$,

then $y = \frac{5}{\sqrt{21}-1}$

$= \frac{\sqrt{21}+1}{4}$.

Let $\frac{\sqrt{21}+1}{4} = 1 + \frac{1}{z}$,

then $z = \frac{4}{\sqrt{21}-3}$

$= \frac{\sqrt{21}+3}{3}$.

Let $\frac{\sqrt{21}+3}{3} = 2 + \frac{1}{u}$,

then $u = \frac{3}{\sqrt{21}-3}$

$= \frac{\sqrt{21}+3}{4}$,

Let $\frac{\sqrt{21}+3}{4} = 1 + \frac{1}{v}$,

then $v = \frac{4}{\sqrt{21}-1}$

$= \frac{\sqrt{21}+1}{5}$.

Let $\frac{\sqrt{21}+1}{5} = 1 + \frac{1}{w}$,

then $$w = \frac{5}{\sqrt{21}-4} = \sqrt{21}+4.$$

Let $$\sqrt{21}+4 = 8 + \frac{1}{t},$$

then $$t = \frac{1}{\sqrt{21}-4} = x.$$

Hence

$$\sqrt{21} = 4 + \frac{\dot{1}}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{\dot{1}}{8}.$$

(ii.)

Let $$\sqrt{22} = 4 + \frac{1}{x},$$

then $$x = \frac{1}{\sqrt{22}-4} = \frac{\sqrt{22}+4}{6}.$$

Let $$\frac{\sqrt{22}+4}{6} = 1 + \frac{1}{y},$$

then $$y = \frac{6}{\sqrt{22}-2} = \frac{\sqrt{22}+2}{3}.$$

Let $$\frac{\sqrt{22}+2}{3} = 2 + \frac{1}{z},$$

then $$z = \frac{3}{\sqrt{22}-4} = \frac{\sqrt{22}+4}{2}.$$

Let $$\frac{\sqrt{22}+4}{2} = 4 + \frac{1}{u},$$

then $$u = \frac{2}{\sqrt{22}-4} = \frac{\sqrt{22}+4}{3}.$$

Let $$\frac{\sqrt{22}+4}{3} = 2 + \frac{1}{v},$$

then $$v = \frac{3}{\sqrt{22}-2} = \frac{\sqrt{22}+2}{6}.$$

Let $$\frac{\sqrt{22}+2}{6} = 1 + \frac{1}{w},$$

then $$w = \frac{6}{\sqrt{22}-4} = \sqrt{22}+4.$$

Let $$\sqrt{22}+4 = 8 + \frac{1}{t},$$

then $$t = \frac{1}{\sqrt{22}-4} = x.$$

Hence

$$\sqrt{22} = 4 + \frac{\dot{1}}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{\dot{1}}{8}.$$

(iii.)

Let $$\sqrt{33} = 5 + \frac{1}{x},$$

then $$x = \frac{1}{\sqrt{33}-5} = \frac{\sqrt{33}+5}{8}.$$

Let $$\frac{\sqrt{33}+5}{8} = 1 + \frac{1}{y},$$

then $$y = \frac{8}{\sqrt{33}-3} = \frac{\sqrt{33}+3}{3}.$$

Let $$\frac{\sqrt{33}+3}{3} = 2 + \frac{1}{z},$$

then $$z = \frac{3}{\sqrt{33}-3}$$
$$= \frac{\sqrt{33}+3}{8},$$

Let $$\frac{\sqrt{33}+3}{8} = 1 + \frac{1}{u},$$

then $$u = \frac{8}{\sqrt{33}-5}$$
$$= \sqrt{33}+5.$$

Let $$\sqrt{33}+5 = 10 + \frac{1}{v},$$

then $$v = \frac{1}{\sqrt{33}-5}$$
$$= x.$$

Hence
$$\sqrt{33} = 5 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{10}$$

(iv.)

Let $$\sqrt{55} = 7 + \frac{1}{x},$$

then $$x = \frac{1}{\sqrt{55}-7}$$
$$= \frac{\sqrt{55}+7}{6}.$$

Let $$\frac{\sqrt{55}+7}{6} = 2 + \frac{1}{y},$$

then $$y = \frac{6}{\sqrt{55}-5}$$
$$= \frac{\sqrt{55}+5}{5}.$$

Let $$\frac{\sqrt{55}+5}{5} = 2 + \frac{1}{z},$$

then $$z = \frac{5}{\sqrt{55}-5}$$
$$= \frac{\sqrt{55}+5}{6}.$$

Let $$\frac{\sqrt{55}+5}{6} = 2 + \frac{1}{u},$$

then $$u = \frac{6}{\sqrt{55}-7}$$
$$= \sqrt{55}+7.$$

Let $$\sqrt{55}+7 = 14 + \frac{1}{v},$$

then $$v = \frac{1}{\sqrt{55}-7}$$
$$= x.$$

Hence
$$\sqrt{55} = 7 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{14}.$$

4. Obtain convergents, with only two figures in the denominator, that approach nearest to the value of $\sqrt{10}$; $\sqrt{15}$; $\sqrt{17}$; $\sqrt{18}$; $\sqrt{20}$.

(i.)

Let $$\sqrt{10} = 3 + \frac{1}{x},$$

then $$x = \frac{1}{\sqrt{10}-3}$$
$$= \sqrt{10}+3.$$

Let $$\sqrt{10}+3 = 6 + \frac{1}{y},$$

then $$y = \frac{1}{\sqrt{10}-3}$$
$$= x.$$

Hence $$\sqrt{10} = 3 + \frac{1}{6},$$

Convergents $= \frac{3}{1}, \frac{19}{6}, \frac{117}{37}, \ldots..$

The required convergent of $\sqrt{10}$ is $\frac{117}{37}$.

(ii.)

Let $\sqrt{15} = 3 + \frac{1}{x}$,

then $x = \frac{1}{\sqrt{15} - 3}$

$= \frac{\sqrt{15} + 3}{6}$.

Let $\frac{\sqrt{15} + 3}{6} = 1 + \frac{1}{y}$,

then $y = \frac{6}{\sqrt{15} - 3}$

$= \sqrt{15} + 3$.

Let $\sqrt{15} + 3 = 6 + \frac{1}{z}$,

then $z = \frac{1}{\sqrt{15} - 3}$

$= x$.

Hence $\sqrt{15} = 3 + \frac{\dot{1}}{1+} \frac{\dot{1}}{6}$.

Quotients $= 1, 6, 1, 6, \ldots\ldots$

Convergents $= \frac{3}{1}, \frac{4}{1}, \frac{27}{7}, \frac{31}{8}, \frac{213}{55}, \frac{244}{63}, \ldots\ldots$

The required convergent of $\sqrt{15}$ is $\frac{244}{63}$.

(iii.)

Let $\sqrt{18} = 4 + \frac{1}{x}$,

then $x = \frac{1}{\sqrt{18} - 4}$

$= \frac{\sqrt{18} + 4}{2}$.

Let $\frac{\sqrt{18} + 4}{2} = 4 + \frac{1}{y}$,

then $y = \frac{2}{\sqrt{18} - 4}$

$= \sqrt{18} + 4$.

Let $\sqrt{18} + 4 = 8 + \frac{1}{z}$,

then $z = \frac{1}{\sqrt{18} - 4}$

$= x$.

Hence $\sqrt{18} = 4 + \frac{\dot{1}}{4+} \frac{\dot{1}}{8}$.

Quotients $= 4, 8, 4, 8, \ldots\ldots$

Convergents $= \frac{4}{1}, \frac{17}{4}, \frac{140}{33}, \ldots\ldots$

The required convergent of $\sqrt{18}$ is $\frac{140}{33}$.

(iv.)

Let $\sqrt{20} = 4 + \frac{1}{x}$,

then $x = \frac{1}{\sqrt{20} - 4}$

$= \frac{\sqrt{20} + 4}{4}$.

Let $\frac{\sqrt{20} + 4}{4} = 2 + \frac{1}{y}$,

then $y = \frac{4}{\sqrt{20} - 4}$

$= \sqrt{20} + 4$.

Let $\sqrt{20} + 4 = 8 + \frac{1}{z}$,

then $z = \frac{1}{\sqrt{20} - 4}$

$= x$.

Hence $\sqrt{20} = 4 + \frac{\dot{1}}{2+} \frac{\dot{1}}{8}$,

Quotients $= 2, 8, 2, 8, \ldots\ldots$

Convergents $= \frac{4}{1}, \frac{9}{2}, \frac{76}{17}, \frac{161}{36}, \ldots\ldots$

The required convergent of $\sqrt{20}$ is $\frac{161}{36}$.

5. Find the proper fraction which, if converted into a continued fraction, will have quotients 1, 7, 5, 2.

Quotients $= 1, 7, 5, 2.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{7}{8}, \frac{36}{41}, \frac{79}{90}.$

The fraction is $\frac{79}{90}$.

6. Find the next convergent when the two preceding convergents are $\frac{3}{14}$ and $\frac{19}{89}$, and the next quotient is 5.

Quotients $= \ldots\ldots, 5, \ldots\ldots$

Convergents $= \ldots\ldots, \frac{3}{14}, \frac{19}{89}, \frac{98}{459}, \ldots\ldots$

The next convergent is $\frac{98}{459}$.

7. If the pound troy is the weight of 22.8157 inches of water, and the pound avoirdupois of 27.7274 inches, find a fraction with denominator < 100 which shall differ from their ratio by < 0.0001.

$$\frac{22.8157}{27.7274} = \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1} + \text{etc.}}}}}}}}$$

Quotients $= 1, 4, 1, 1, 1, 4, 1, 1, \ldots\ldots$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{4}{5}, \frac{5}{6}, \frac{9}{11}, \frac{14}{17}, \frac{65}{79}, \frac{79}{96}, \frac{144}{175}, \ldots\ldots$

The nearest convergent with denominator < 100 is therefore $\frac{79}{96}$. The difference between it and the actual value of the ratio is

$$< \frac{1}{96 \times 175} \text{ or } < 0.0001.$$

8. The ratio of the diagonal to the side of a square being $\sqrt{2}$, find a fraction with a denominator < 100 which shall differ from their ratio by < 0.0001.

Let $$\sqrt{2} = 1 + \frac{1}{x},$$

then $$x = \frac{1}{\sqrt{2} - 1}$$
$$= \sqrt{2} + 1.$$

Let $$\sqrt{2} + 1 = 2 + \frac{1}{y},$$

then
$$y = \frac{1}{\sqrt{2}-1} \qquad\qquad \text{Hence} \quad \sqrt{2} = 1 + \frac{\dot{1}}{\dot{2}}.$$
$$= x.$$

Quotients $= 2, 2, 2, 2, 2, 2, \ldots\ldots$

Convergents $= \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \ldots\ldots$

The nearest convergent with denominator < 100 is therefore $\frac{99}{70}$. The difference between it and $\sqrt{2}$ is $< \frac{1}{70 \times 169}$ or < 0.0001.

9. The ratio of the circumference of a circle to its diameter being 3.14159265, find the first three convergents, and determine to how many decimal places each may be depended upon as agreeing with the true value.

$$3.14159265 = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{288 + \text{etc.}}}}}$$

Quotients $= 7, 15, 1, 288, \ldots\ldots$

Convergents $= \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{102573}{32657}, \ldots\ldots$

The first convergent, $\frac{22}{7}$, differs from the true value by $< \frac{1}{7 \times 106}$ or < 0.01; the second, $\frac{333}{106}$, by $< \frac{1}{106 \times 113}$ or < 0.0001; the third, $\frac{355}{113}$, by $< \frac{1}{113 \times 32657}$ or < 0.000001.

10. Two scales whose zero-points coincide have the distance between consecutive divisions of the one to those of the other as $1 : 1.06577$. Find what division points most nearly coincide.

$$\frac{100000}{106577} = \cfrac{1}{1 + \cfrac{1}{15 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{11 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}}}}}$$

Quotients $= 1, 15, 4, 1, 8, 11, 2, \ldots\ldots$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{15}{16}, \frac{61}{65}, \frac{76}{81}, \frac{669}{713}, \frac{7435}{7924}, \ldots\ldots$

Hence the 15th, 61st, 76th, ... divisions of the first scale nearly coincide with the 16th, 65th, 81st, ... divisions of the other, the coincidence becoming closer as the series proceeds.

11. Find the surd values of

$$3+\frac{\dot{1}}{1+}\,\frac{\dot{1}}{6};\quad \frac{\dot{1}}{3+}\,\frac{1}{1+}\,\frac{\dot{1}}{6};\quad 1+\frac{\dot{1}}{2+}\,\frac{1}{3+}\,\frac{\dot{1}}{4}.$$

(i.)
$$x=\frac{\dot{1}}{1+}\,\frac{\dot{1}}{6}$$
$$=\cfrac{1}{1+\cfrac{1}{6+x}}$$
$$=\frac{6+x}{7+x},$$
$$x^2+6x=6,$$
$$x^2+6x+9=15,$$
$$x+3=\pm\sqrt{15},$$
$$\therefore\ x=-3\pm\sqrt{15}.$$

Hence the required value is
$$3-3\pm\sqrt{15}=\sqrt{15}.$$

(ii.)
$$x=\frac{\dot{1}}{3+}\,\frac{1}{1+}\,\frac{\dot{1}}{6}$$
$$=\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{6+x}}}$$
$$=\cfrac{1}{3+\cfrac{6+x}{7+x}}$$
$$=\frac{7+x}{27+4x},$$
$$27x+4x^2=7+x,$$
$$4x^2+26x=7,$$
$$\therefore\ x=\frac{-13+\sqrt{197}}{4}.$$

(iii.)
$$x=\frac{\dot{1}}{2+}\,\frac{1}{3+}\,\frac{\dot{1}}{4}$$
$$=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+x}}}$$
$$=\cfrac{1}{2+\cfrac{4+x}{13+3x}}$$
$$=\frac{13+3x}{30+7x},$$
$$30x+7x^2=13+3x,$$
$$7x^2+27x=13,$$
$$\therefore\ x=-\frac{27}{14}\pm\frac{\sqrt{1093}}{14}.$$

The required value is
$$1-\frac{27}{14}+\frac{\sqrt{1093}}{14}$$
$$=\frac{\sqrt{1093}-13}{14}.$$

The surd values are
$$\sqrt{15};\quad \frac{-13+\sqrt{197}}{4};\ \text{and}\ \frac{\sqrt{1093}-13}{14}.$$

12. Show that the ratio of the diagonal of a cube to its edge may be nearly expressed by 97 : 56. Find the limit of the error made in taking this ratio for the true value.

The true ratio is $\sqrt{3}$.

By § 463, $$\sqrt{3} = 1 + \frac{\dot{1}}{1+}\,\frac{\dot{1}}{2}.$$

$$\text{Quotients} = 1, 2, 1, 2, 1, 2, 1, 2, \ldots..$$

$$\text{Convergents} = \tfrac{0}{1}, \tfrac{1}{1}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{8}{11}, \tfrac{11}{15}, \tfrac{30}{41}, \tfrac{41}{56}, \tfrac{112}{153}.$$

The 7th convergent is $1\frac{41}{56}$ or $\frac{97}{56}$, and it differs from $\sqrt{3}$ by $< \dfrac{1}{56 \times 153}$ or $< \frac{1}{8568}$.

13. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours,

$$5 \text{ hrs. } 48 \text{ min. } 51 \text{ sec.} = 20{,}931 \text{ sec.}$$

$$24 \text{ hrs.} = 86{,}400 \text{ sec.}$$

$$\frac{20931}{86400} = \cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}}}}}}}$$

$$\text{Quotients} = 4, 7, 1, 4, 1, 1, 1, 1, 3, 2, 1, 2.$$

$$\text{Convergents} = \tfrac{0}{1}, \tfrac{1}{4}, \tfrac{7}{29}, \tfrac{8}{33}, \tfrac{39}{161}, \tfrac{47}{194}, \tfrac{86}{355}, \tfrac{133}{549}, \tfrac{219}{904}, \tfrac{790}{3261}, \tfrac{1799}{7426}, \tfrac{2589}{10687}, \tfrac{6977}{28800}.$$

14. Find a series of fractions converging to the ratio of a cubic yard to a cubic meter, if 1 cubic yard = 0.76453 of a cubic meter.

$$0.76453 = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{19+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}}}}}}}}}$$

Quotients $= 1, 3, 4, 19, 2, 3, 1, 1, 5, 1, 2.$

Convergents $= \frac{0}{1}, \frac{1}{1}, \frac{3}{4}, \frac{13}{17}, \frac{250}{327}, \frac{513}{671}, \frac{1789}{2340}, \frac{2302}{3011}, \frac{4091}{5351}, \frac{22757}{29766}, \frac{26848}{35117}, \frac{76453}{100000}.$

Exercise CXXVI.

1. Expand $\dfrac{1}{2-3x}$ to four terms in ascending powers of x.

Let $\dfrac{1}{2-3x} = A + Bx + Cx^2 + Dx^3 + \cdots,$

then
$$\begin{aligned} 1 = 2A &+ (2B-3A)x + (2C-3B)x^2 \\ &+ (2D-3C)x^3 + \cdots \end{aligned}$$

$$\begin{aligned} \therefore 2A &= 1, & A &= \tfrac{1}{2}, \\ 2B - 3A &= 0, & B &= \tfrac{3}{4}, \\ 2C - 3B &= 0, & C &= \tfrac{9}{8}, \\ 2D - 3C &= 0, & D &= \tfrac{27}{16}, \end{aligned}$$

. . . . . .

Hence, $\dfrac{1}{2-3x} = \frac{1}{2} + \frac{3}{4}x + \frac{9}{8}x^2 + \frac{27}{16}x^3 + \cdots$

2. Expand $\dfrac{1+x}{2+3x}$ to four terms in ascending powers of x.

Let $\dfrac{1+x}{2+3x} = A + Bx + Cx^2 + Dx^3 + \cdots$

then $1 + x = 2A + (2B + 3A)x + (2C + 3B)x^2$
$+ (2D + 3C)x^3 + \cdots\cdots$

$$\begin{aligned} \therefore 2A &= 1, & A &= \tfrac{1}{2}, \\ 2B + 3A &= 1, & B &= -\tfrac{1}{4}, \\ 2C + 3B &= 0, & C &= \tfrac{3}{8}, \\ 2D + 3C &= 0 & D &= -\tfrac{9}{16}, \end{aligned}$$

. . . .

Hence, $\dfrac{1+x}{2+3x} = \tfrac{1}{2} - \tfrac{1}{4}x + \tfrac{3}{8}x^2 - \tfrac{9}{16}x^3 + \cdots\cdots$

3. Expand $\dfrac{3-2x}{4-3x}$ to four terms in ascending powers of x.

Let $\dfrac{3-2x}{4-3x} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots$

then $3 - 2x = 4A + (4B - 3A)x$
$+ (4C - 3B)x^2 + (4D - 3C)x^3 + \cdots\cdots$

$$\begin{aligned} \therefore 4A &= 3, & A &= \tfrac{3}{4}, \\ 4B - 3A &= -2, & B &= \tfrac{1}{16}, \\ 4C - 3B &= 0, & C &= \tfrac{3}{64}, \\ 4D - 3C &= 0, & D &= \tfrac{9}{256}, \end{aligned}$$

. . . .

Hence, $\dfrac{3-2x}{4-3x} = \tfrac{3}{4} + \tfrac{1}{16}x + \tfrac{3}{64}x^2 + \tfrac{9}{256}x^3 + \cdots\cdots$

4. Expand $\dfrac{1-x}{1-x+x^2}$ to four terms in ascending powers of x.

Let $\dfrac{1-x}{1-x+x^2} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots$

then $1 - x = A + (B - A)x + (C - B + A)x^2$
$+ (D - C + B)x^3 + \cdots\cdots$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B - A &= -1, & B &= 0, \\ C - B + A &= 0, & C &= -1, \\ D - C + B &= 0, & D &= -1, \end{aligned}$$

. . . .

Hence $\dfrac{1-x}{1-x+x^2} = 1 - x^2 - x^3 + \cdots\cdots$

5. Expand $\frac{1}{1-2x+3x^2}$ to four terms in ascending powers of x.

Let $$\frac{1}{1-2x+3x^2} = A + Bx + Cx^2 + Dx + \cdots\cdots$$

then $$1 = A + (B-2A)x + (C-2B+3A)x^2 + (D-2C+3B)x^3 + \cdots\cdots$$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B-2A &= 0, & B &= 2, \\ C-2B+3A &= 0, & C &= 1, \\ D-2C+3B &= 0, & D &= -4, \end{aligned}$$

. . . .

Hence $$\frac{1}{1-2x+3x^2} = 1 + 2x + x^2 - 4x^3 - \cdots\cdots$$

6. Expand $\frac{5-2x}{1+3x-x^2}$ to four terms in ascending powers of x.

Let $$\frac{5-2x}{1+3x-x^2} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

then $$5-2x = A + (B+3A)x + (C+3B-A)x^2 + (D+3C-B)x^3 + \cdots\cdots$$

$$\begin{aligned} \therefore A &= 5, & A &= 5, \\ B+3A &= -2, & B &= -17, \\ C+3B-A &= 0, & C &= 56, \\ D+3C-B &= 0, & D &= -185, \end{aligned}$$

. . . .

Hence $$\frac{5-2x}{1+3x-x^2} = 5 - 17x + 56x^2 - 185x^3 + \cdots\cdots$$

7. Expand $\frac{4x-6x^2}{1-2x+3x^2}$ to four terms in ascending powers of x.

Let $$\frac{4x-6x^2}{1-2x+3x^2} = A + Bx + Cx^2 + Dx^3 + \cdots\cdots$$

then $$4x-6x^2 = A + (B-2A)x + (C-2B+3A)x^2 + (D-2C+3B)x^3 + \cdots\cdots$$

$$\begin{aligned} \therefore A &= 0, & A &= 0, \\ B-2A &= 4, & B &= 4, \\ C-2B+3A &= -6, & C &= 2, \\ D-2C+3B &= 0, & D &= -8, \end{aligned}$$

. . . .

Hence $$\frac{4x-6x^2}{1-2x+3x^2} = 4x + 2x^2 - 8x^3 - \cdots\cdots$$

8. Revert the series $y = x + x^2 + x^3 + \cdots$

Let $x = Ay + By^2 + Cy^3 + Dy^4 + \cdots$

then
$$\begin{aligned} y = Ay + By^2 + Cy^3 + Dy^4 + \cdots \\ + A^2y^2 + 2ABy^3 + (B^2 + 2AC)y^4 + \cdots \\ + A^3y^3 + 3A^2By^4 + \cdots \\ + A^4y^4 + \cdots \end{aligned}$$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B + A^2 &= 0, & B &= -1, \\ C + 2AB + A^3 &= 0, & C &= 1, \\ D + B^2 + 2AC + 3A^2B + A^4 &= 0, & D &= -1, \end{aligned}$$

. . . .

Hence $x = y - y^2 + y^3 - y^4 + \cdots$

9. Revert the series $y = x - 2x^2 + 3x^3 - \cdots$

Let $x = Ay + By^2 + Cy^3 + Dy^4 + \cdots$

then
$$\begin{aligned} y = Ay + By^2 + Cy^3 + Dy^4 + \cdots \\ - 2A^2y^2 - 4ABy^3 - (2B^2 + 4AC)y^4 + \cdots \\ + 3A^3y^3 + 9A^2By^4 + \cdots \\ - 4A^4y^4 + \cdots \end{aligned}$$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B - 2A^2 &= 0, & B &= 2, \\ C - 4AB + 3A^3 &= 0, & C &= 5, \\ D - 2B^2 - 4AC + 9A^2B - 4A^4 &= 0, & D &= 14, \end{aligned}$$

. . . .

Hence $x = y + 2y^2 + 5y^3 + 14y^4 + \cdots$

10. Revert the series $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

Let $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + \cdots$

then
$$\begin{aligned} y = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + \cdots \\ - \tfrac{1}{3}A^3y^3 - A^2By^4 - AB^2y^5 - \cdots \\ + \tfrac{1}{5}A^5y^5 + \cdots \end{aligned}$$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B &= 0, & B &= 0, \\ C - \frac{A^3}{3} &= 0, & C &= \tfrac{1}{3}, \\ D - A^2B &= 0, & D &= 0, \\ E - AB^2 + \tfrac{1}{5}A^5 &= 0, & E &= -\tfrac{1}{5}. \end{aligned}$$

Hence $x = y + \frac{1}{3}y^3 - \frac{1}{5}y^5 + \cdots$

11. Revert the series $y = x + \dfrac{x^2}{1 \times 2} + \dfrac{x^3}{1 \times 2 \times 3} + \cdots\cdots$

Let $\quad x = Ay + By^2 + Cy^3 + Dy^4 + \cdots\cdots$

then
$$\begin{aligned} y = Ay + By^2 + Cy^3 + Dy^4 + \cdots\cdots \\ + \tfrac{1}{2}A^2y^2 + ABy^3 + (\tfrac{1}{2}B^2 + AC)\,y^4 + \cdots\cdots \\ + \tfrac{1}{6}A^3y^3 + \tfrac{1}{2}A^2By^4 + \cdots\cdots \\ + \tfrac{1}{24}A^4y^4 + \cdots\cdots \end{aligned}$$

$$\begin{aligned} \therefore A &= 1, & A &= 1, \\ B + \tfrac{1}{2}A^2 &= 0, & B &= -\tfrac{1}{2}, \\ C + AB + \tfrac{1}{6}A^3 &= 0, & C &= \tfrac{1}{3}, \\ D + \tfrac{1}{2}B^2 + AC + \tfrac{1}{2}A^2B + \tfrac{1}{24}A^4 &= 0, & D &= -\tfrac{1}{4}, \end{aligned}$$

. . . .

Hence $\quad x = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \cdots\cdots$

12. Find the fractions in the form $\dfrac{a + bx}{p + qx + rx^2}$ whose expansions produce the series:

$$1 + 3x + 2x^2 - x^3 - \cdots\cdots$$
$$3 + 2x + 3x^2 + 7x^3 + \cdots\cdots$$
$$\tfrac{3}{4} - \tfrac{3}{16}x + \tfrac{63}{64}x^2 - \tfrac{123}{256}x^3 + \cdots\cdots$$

(i.) $\quad \dfrac{a + bx}{p + qx + rx^2} = 1 + 3x + 2x^2 - x^3 - \cdots\cdots$

$$\begin{aligned} a + bx = p + (3p + q)\,x + (2p + 3q + r)\,x^2 \\ + (-p + 2q + 3r)\,x^3 + \cdots\cdots \end{aligned}$$

$$\begin{aligned} \therefore a &= p, & b &= 3p + q, \\ 0 &= 2p + 3q + r, & 0 &= -p + 2q + 3r. \end{aligned}$$

Eliminating r from the last two equations,

$$\begin{aligned} 0 &= 7p + 7q. \\ \therefore q &= -p, \\ r &= p, \\ a &= p, \\ b &= 2p. \end{aligned}$$

Hence
$$\frac{a + bx}{p + qx + r} = \frac{p + 2px}{p - px + px^2} = \frac{1 + 2x}{1 - x + x^2}.$$

(ii.) $$\frac{a+bx}{p+qx+rx^2} = 3 + 2x + 3x^2 + 7x^3 + \cdots\cdots$$

$$\begin{aligned} a + bx = 3p &+ (2p+3q)x + (3p+2q+3r)x^2 \\ &+ (7p+3q+2r)x^3 + \cdots\cdots \end{aligned}$$

$$\therefore a = 3p, \qquad b = 2p + 3q,$$
$$0 = 3p + 2q + 3r, \qquad 0 = 7p + 3q + 2r.$$

Eliminating r from the last two equations,

$$\begin{aligned} 0 &= -15p - 5q. \\ \therefore q &= -3p, \\ r &= p, \\ a &= 3p, \\ b &= -7p. \end{aligned}$$

Hence $$\frac{a+bx}{p+qx+rx^2} = \frac{3p - 7px}{p - 3px + px^2}$$
$$= \frac{3 - 7x}{1 - 3x + x^2}.$$

(iii.) $$\frac{a+bx}{p+qx+rx^2} = \tfrac{3}{4} - \tfrac{3}{16}x + \tfrac{63}{64}x^2 - \tfrac{123}{256}x^3 + \cdots\cdots$$

$$\begin{aligned} a + bx = \tfrac{3}{4}p &+ (-\tfrac{3}{16}p + \tfrac{3}{4}q)x \\ &+ (\tfrac{63}{64}p - \tfrac{3}{16}q + \tfrac{3}{4}r)x^2 \\ &+ (-\tfrac{123}{256}p + \tfrac{63}{64}q - \tfrac{3}{16}r)x^3 \\ &+ \cdots\cdots \end{aligned}$$

$$\therefore a = \tfrac{3}{4}p, \qquad b = -\tfrac{3}{16}p + \tfrac{3}{4}q,$$
$$0 = \tfrac{63}{64}p - \tfrac{3}{16}q + \tfrac{3}{4}r, \qquad 0 = -\tfrac{123}{256}p + \tfrac{63}{64}q - \tfrac{3}{16}r.$$

Eliminating r from the last two equations,

$$\begin{aligned} 0 &= \tfrac{60}{256}p + \tfrac{60}{64}q. \\ \therefore q &= \tfrac{1}{4}p, \\ r &= -\tfrac{5}{4}p, \\ a &= \tfrac{3}{4}p, \\ b &= 0. \end{aligned}$$

Hence $$\frac{a+bx}{p+qx+rx^2} = \frac{\tfrac{3}{4}p}{p + \tfrac{1}{4}px - \tfrac{5}{4}px^2}$$
$$= \frac{3}{4 + x - 5x^2}.$$

13. Resolve $\frac{7x+1}{(x+4)(x-5)}$ into partial fractions.

Let $$\frac{7x+1}{(x+4)(x-5)} = \frac{A}{x+4} + \frac{B}{x-5},$$

then
$$\begin{aligned} 7x+1 &= A(x-5) + B(x+4) \\ &= (A+B)x - 5A + 4B. \end{aligned}$$
$$\begin{aligned} \therefore A + B &= 7, \\ -5A + 4B &= 1, \\ A &= 3, \\ B &= 4. \end{aligned}$$

Hence $$\frac{7x+1}{(x+4)(x-5)} = \frac{3}{x+4} + \frac{4}{x-5}.$$

14. Resolve $\frac{6}{(x+3)(x+4)}$ into partial fractions.

Let $$\frac{6}{(x+3)(x\times 4)} = \frac{A}{x+3} + \frac{B}{x+4},$$

then
$$\begin{aligned} 6 &= A(x+4) + B(x+3) \\ &= (A+B)x + 4A + 3B. \end{aligned}$$
$$\begin{aligned} \therefore A + B &= 0, \\ 4A + 3B &= 6, \\ A &= 6, \\ B &= -6. \end{aligned}$$

Hence $$\frac{6}{(x+3)(x+4)} = \frac{6}{x+3} - \frac{6}{x+4}.$$

15. Resolve $\frac{5x-1}{(2x-1)(x-5)}$ into partial fractions.

Let $$\frac{5x-1}{(2x-1)(x-5)} = \frac{A}{2x-1} - \frac{B}{x-5},$$

then
$$\begin{aligned} 5x-1 &= A(x-5) + B(2x-1) \\ &= (A+2B)x - (5A+B). \end{aligned}$$
$$\begin{aligned} \therefore A + 2B &= 5, \\ 5A + B &= 1, \\ A &= -\tfrac{1}{3}, \\ B &= \tfrac{8}{3}. \end{aligned}$$

Hence $$\begin{aligned} \frac{5x-1}{(2x-1)(x-5)} &= \frac{-\frac{1}{3}}{2x-1} + \frac{\frac{8}{3}}{x-5} \\ &= \frac{8}{3(x-5)} - \frac{1}{3(2x-1)}. \end{aligned}$$

16. Resolve $\dfrac{x-2}{x^2-3x-10}$ into partial fractions.

Let
$$\frac{x-2}{x^2-3x-10}=\frac{A}{x-5}+\frac{B}{x+2},$$
then
$$\begin{aligned} x-2 &= A(x+2)+B(x-5) \\ &= (A+B)x+2A-5B. \end{aligned}$$
$$\begin{aligned} \therefore A+B &= 1, \\ 2A-5B &= -2, \\ A &= \tfrac{3}{7}, \\ B &= \tfrac{4}{7}. \end{aligned}$$
Hence
$$\begin{aligned} \frac{x-2}{x^2-3x-10} &= \frac{\frac{3}{7}}{x-5}+\frac{\frac{4}{7}}{x+2} \\ &= \frac{3}{7(x-5)}+\frac{4}{7(x+2)}. \end{aligned}$$

17. Resolve $\dfrac{3}{x^3-1}$ into partial fractions.

Let
$$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1},$$
then
$$\begin{aligned} 3 &= A(x^2+x+1)+(Bx+C)(x-1) \\ &= (A+B)x^2+(A-B+C)x+A-C. \end{aligned}$$
$$\begin{array}{rlrl} \therefore A+B &= 0, & A &= 1, \\ A-B+C &= 0, & B &= -1, \\ A-C &= 3, & C &= -2. \\ \hline 3A &= 3, & & \end{array}$$
Hence
$$\frac{3}{x^3-1}=\frac{1}{x-1}-\frac{x+2}{x^2+x+1}.$$

18. Resolve $\dfrac{x^2-x-3}{x(x^2-4)}$ into partial fractions.

Let
$$\frac{x^2-x-3}{x(x^2-4)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2},$$
then
$$\begin{aligned} x^2-x-3 &= A(x^2-4)+Bx(x-2)+Cx(x+2) \\ &= (A+B+C)x^2+(2C-2B)x-4A. \end{aligned}$$
$$\begin{array}{rlrl} \therefore A+B+C &= 1, & A &= \tfrac{3}{4}, \\ 2B-2C &= 1, & B &= \tfrac{3}{8}, \\ 4A &= 3, & C &= -\tfrac{1}{8}. \end{array}$$
Hence
$$\frac{x^2-x-3}{x(x^2-4)}=\frac{3}{4x}+\frac{3}{8(x+2)}-\frac{1}{8(x-2)}.$$

19. Resolve $\dfrac{3x^2-4}{x^2(x+5)}$ into partial fractions.

Let $$\frac{3x^2-4}{x^2(x+5)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+5},$$

then $$\begin{aligned}3x^2-4&=Ax(x+5)+B(x+5)+Cx^2\\&=(A+C)x^2+(5A+B)x+5B.\end{aligned}$$

$$\begin{aligned}\therefore A+C&=3, & B&=\tfrac{4}{5},\\ 5A+B&=0, & A&=\tfrac{4}{25},\\ 5D&=-4, & C&=\tfrac{71}{25}.\end{aligned}$$

Hence $$\frac{3x^2-4}{x^2(x+5)}=\frac{4}{25x}-\frac{4}{5x^2}+\frac{71}{25(x+5)}.$$

20. Resolve $\dfrac{7x^2-x}{(x-1)^2(x+2)}$ into partial fractions.

Let $$\frac{7x^2-x}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2},$$

then $$\begin{aligned}7x^2-x&=A(x-1)(x+2)+B(x+2)\\&\quad+C(x-1)^2\\&=(A+C)x^2+(A+B-2C)x\\&\quad-2A+2B+C.\end{aligned}$$

$$\begin{aligned}\therefore A+C&=7, & B&=2,\\ A+B-2C&=-1, & A&=\tfrac{11}{3},\\ -2A+2B+C&=0, & C&=\tfrac{10}{3}.\\ \hline 3B&=6,\end{aligned}$$

Hence $$\frac{7x^2-x}{(x-1)^2(x+2)}=\frac{11}{3(x-1)}+\frac{2}{(x-1)^2}+\frac{10}{3(x+2)}.$$

21. Resolve $\dfrac{2x^2-7x+1}{x^3+1}$ into partial fractions.

Let $$\frac{2x^2-7x+1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1},$$

then $$\begin{aligned}2x^2-7x+1&=A(x^2-x+1)+(Bx+C)(x+1)\\&=(A+B)x^2+(-A+B+C)x+A+C.\end{aligned}$$

$$\begin{aligned}\therefore A+B&=2, & A&=\tfrac{10}{3},\\ -A+B+C&=-7, & B&=-\tfrac{4}{3},\\ A+C&=1, & C&=-\tfrac{7}{3}.\end{aligned}$$

Hence $$\frac{2x^2-7x+1}{x^3+1}=\frac{10}{3(x+1)}-\frac{4x+7}{3(x^2-x+1)}.$$

Exercise CXXVII.

1. Find the fiftieth term of 1, 3, 8, 20, 43, ...

$$\begin{array}{lcccccccccccc}
\text{Series} & = & 1 & & 3 & & 8 & & 20 & & 43 & & \ldots\ldots \\
\text{1st diff.} & = & & 2 & & 5 & & 12 & & 23 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 3 & & 7 & & 11 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 4 & & 4 & & \ldots\ldots & & & \\
\text{4th diff.} & = & & & & & 0 & & \ldots\ldots & & & &
\end{array}$$

$$\therefore a = 1, \quad a_1 = 2, \quad a_3 = 3, \quad a_4 = 4, \quad a_5 = 0.$$

Substituting in formula, we have:

$$\begin{aligned}
\text{Fiftieth term} &= 1 + 49 \times 2 + \frac{49 \times 48}{2} \times 3 \\
&\quad + \frac{49 \times 48 \times 47}{1 \times 2 \times 3} \times 4 \\
&= 1 + 98 + 3528 + 73696 \\
&= 77323.
\end{aligned}$$

2. Find the sum of the series 4, 12, 29, 55, ... to 20 terms.

$$\begin{array}{lcccccccccc}
\text{Series} & = & 4 & & 12 & & 29 & & 55 & & \ldots\ldots \\
\text{1st diff.} & = & & 8 & & 17 & & 26 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 9 & & 9 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 0 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 4, \quad a_1 = 8, \quad a_2 = 9, \quad a_3 = 0.$$

Sum to 20 terms

$$\begin{aligned}
&= 20\left\{4 + \tfrac{19}{2} \times 8 + \frac{19 \times 18}{1 \times 2 \times 3} \times 9\right\} \\
&= 20\{4 + 76 + 513\} \\
&= 11860.
\end{aligned}$$

3. Find the twelfth term of 4, 11, 28, 55, 92, ...

$$\begin{array}{lcccccccccccc}
\text{Series} & = & 4 & & 11 & & 28 & & 55 & & 92 & & \ldots\ldots \\
\text{1st diff.} & = & & 7 & & 17 & & 27 & & 37 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 10 & & 10 & & 10 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 0 & & 0 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 4, \quad a_1 = 7, \quad a_2 = 10, \quad a_3 = 0.$$

$$\begin{aligned}
\text{Twelfth term} &= 4 + 11 \times 7 + \frac{11 \times 10}{2} \times 10 \\
&= 4 + 77 + 550 \\
&= 631,
\end{aligned}$$

4. Find the sum of the series 43, 27, 14, 4, -3, ... to 12 terms.

$$\begin{array}{lcccccccccccc}
\text{Series} & = & 43 & & 27 & & 14 & & 4 & & -3 & & \ldots\ldots \\
\text{1st diff.} & = & & -16 & & -13 & & -10 & & -7 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 3 & & 3 & & 3 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 0 & & 0 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 43,\ a_1 = -16,\ a_2 = 3,\ a_3 = 0.$$

Sum to 12 terms

$$= 12\left\{43 - \tfrac{11}{2} \times 16 + \frac{11 \times 10}{1 \times 2 \times 3} \times 3\right\}$$
$$= 12\,(43 - 88 + 55)$$
$$= 120.$$

5. Find the seventh term of 1, 1.235, 1.471, 1.708, ...

$$\begin{array}{lcccccccccc}
\text{Series} & = & 1 & & 1.235 & & 1.471 & & 1.708 & & \ldots\ldots \\
\text{1st diff.} & = & & 0.235 & & 0.236 & & 0.237 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 0.001 & & 0.001 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 0 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 1,\ a_1 = 0.235,\ a_2 = 0.001,\ a_3 = 0.$$

$$\text{Seventh term} = 1 + 6 \times 0.235 + \frac{6 \times 5}{1 \times 2} \times 0.001$$
$$= 1 + 1.410 + 0.015$$
$$= 2.425.$$

6. Find the sum of the series 70, 66, 62.3, 58.9, ... to 15 terms.

$$\begin{array}{lcccccccccc}
\text{Series} & = & 70 & & 66 & & 62.3 & & 58.9 & & \ldots\ldots \\
\text{1st diff.} & = & & -4 & & -3.7 & & -3.4 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 0.3 & & 0.3 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 8 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 70,\ a_1 = -4,\ a_2 = 0.3,\ a_3 = 0.$$

Sum to 15 terms

$$= 15\left\{70 - \tfrac{14}{2} \times 4 + \frac{14 \times 13}{1 \times 2 \times 3} \times 0.3\right\}$$
$$= 15\{70 - 28 + 9.1\}$$
$$= 766.5.$$

7. Find the eleventh term of 343, 337, 326, 310, ...

Series	=	343	337	326	310	.....
1st diff.	=	-6	-11	-16	.....	
2d diff.	=	-5	-5	.....		
3d diff.	=	0	.....			

$$\therefore a = 343,\ a_1 = -6,\ a_2 = -5,\ a_3 = 0.$$

$$\text{Eleventh term} = 343 - 10 \times 6 - \frac{10 \times 9}{1 \times 2} \times 5$$
$$= 343 - 60 - 225$$
$$= 58.$$

8. Find the sum of the series 7×13, 6×11, 5×9, ... to 9 terms.

Series	=	91	66	45	28	.....
1st diff.	=	-25	-21	-17	.....	
2d diff.	=	4	4	.....		
3d diff.	=	0	.....			

$$\therefore a = 91,\ a_1 = -25,\ a_2 = 4,\ a_3 = 0.$$

$$\text{Sum to 9 terms} = 9\left\{91 - \tfrac{8}{2} \times 25 + \frac{8 \times 7}{1 \times 2 \times 3} \times 4\right\}$$
$$= 9\left(91 - 100 + \tfrac{112}{3}\right)$$
$$= 255.$$

9. Find the sum of n terms of the series 3×8, 6×11, 9×14, 12×17, ...

Series	=	24	66	126	204	.....
1st diff.	=	42	60	78	.....	
2d diff.	=	18	18	.....		
3d diff.	=	0	.....			

$$\therefore a = 24,\ a_1 = 42,\ a_2 = 18,\ a_3 = 0.$$

Sum to n terms

$$= n\left\{24 + \frac{n-1}{2} \times 42 + \frac{(n-1)(n-2)}{1 \times 2 \times 3} \times 18\right\}$$
$$= n\{24 + 21n - 21 + 3n^2 - 9n + 6\}$$
$$= 3n^3 + 12n^2 + 9n$$
$$= 3n(n+1)(n+3)$$

10. Find the sum to n terms of the series 1, 6, 15, 28, 45, ...

$$\begin{array}{llccccccccccc}
\text{Series} & = & 1 & & 6 & & 15 & & 28 & & 45 & & \ldots\ldots \\
\text{1st diff.} & = & & 5 & & 9 & & 13 & & 17 & & \ldots\ldots & \\
\text{2d diff.} & = & & & 4 & & 4 & & 4 & & \ldots\ldots & & \\
\text{3d diff.} & = & & & & 0 & & 0 & & \ldots\ldots & & &
\end{array}$$

$$\therefore a = 1,\quad a_1 = 5,\quad a_2 = 4,\quad a_3 = 0.$$

Sum to n terms

$$= n\left\{1 + \frac{n-1}{2} \times 5 + \frac{(n-1)(n-2)}{1 \times 2 \times 3} \times 4\right\}$$
$$= n\left(1 + \tfrac{5}{2}n - \tfrac{5}{2} + \tfrac{2}{3}n^2 - 2n + \tfrac{4}{3}\right)$$
$$= n\left(\tfrac{2}{3}n^2 + \tfrac{1}{2}n - \tfrac{1}{6}\right)$$
$$= \frac{n}{6}(4n^2 + 3n - 1)$$
$$= \frac{n}{6}(4n - 1)(n + 1).$$

Exercise CXXVIII.

1. Determine the number of shot in the side of the base of a triangular pile which contains 286 shot.

Let n denote the required number;

then $$\frac{n(n+1)(n+2)}{6} = 286,$$
$$n(n+1)(n+2) = 1716$$
$$= 11 \times 12 \times 13.$$
$$\therefore n = 11.$$

2. The number of shot in the upper course of a square pile is 169, and in the lowest course 1089. How many shot are there in the pile?

In the complete pile, $n = 33.$

Number of shot in complete pile,

$$\frac{33 \times 34 \times 67}{1 \times 2 \times 3} = 12{,}529.$$

In the part of the pile that is lacking,

$$n = 12.$$

Number of shot in part lacking,

$$\frac{12 \times 13 \times 25}{1 \times 2 \times 3} = 650,$$
$$12{,}529 - 650 = 11{,}879.$$

There are 11,879 shot in the pile.

3. Find the number of shot in a rectangular pile having 17 shot in one side of the base and 42 in the other.

The required number is

$$\begin{aligned}&\frac{n}{6}(n+1)(3n'-n+1)\\&=\tfrac{17}{6}(17+1)(126-17+1)\\&=5610.\end{aligned}$$

4. Find the number of shot in five courses of an incomplete triangular pile which has 15 in one side of the base.

Number in complete pile,

$$\frac{15\times 16\times 17}{1\times 2\times 3}=680.$$

Number in part lacking,

$$\frac{10\times 11\times 12}{1\times 2\times 3}=220.$$

$$680-220=460.$$

The number of shot in the pile is 460.

5. The number of shot in a triangular pile is to the number in a square pile, of the same number of courses, as 22 : 41. Find the number of shot in each pile.

The number of courses is in both cases the same as the number of shot in a side of the base. Hence

$$\frac{n(n+1)(n+2)}{1\times 2\times 3}:\frac{n(n+1)(2n+1)}{1\times 2\times 3}=22:41,$$

$$n+2:2n+1=22:41$$

$$n=20.$$

$$\therefore\ \frac{n(n+1)(n+2)}{1\times 2\times 3}=\frac{20\times 21\times 22}{1\times 2\times 3}=1540.$$

$$\frac{n(n+1)(2n+1)}{1\times 2\times 3}=\frac{20\times 21\times 41}{1\times 2\times 3}=2870.$$

The number of shot in the triangular pile is 1540; in the square pile, 2870.

6. Find the number of shot required to complete a rectangular pile having 15 and 6 shot, respectively, in the sides of the upper course.

In the first missing course $n = 5$ and $n' = 14$. Hence the number of shot required is
$$\tfrac{5}{6}(5+1)(42-5+1) = 190.$$

7. How many shot must there be in the lowest course of a triangular pile, so that 10 courses of the pile, beginning at the base, may contain 37,020 shot?

Number of shot in the complete pile is
$$\frac{n(n+1)(n+2)}{6}.$$
Number in lacking part is
$$\frac{(n-10)(n-9)(n-8)}{6}.$$
Hence
$$\frac{n(n+1)(n+2)}{6} - \frac{(n-10)(n-9)(n-8)}{6} = 37{,}020,$$
$$n(n+1)(n+2) - (n-10)(n-9)(n-8) = 222{,}120,$$
$$n^3 + 3n^2 + 2n - n^3 + 27n^2 - 242n + 720 = 222{,}120,$$
$$30n^2 - 240n = 221{,}400,$$
$$n^2 - 8n = 7380,$$
$$n = 90.$$
Therefore, the number of shot in a side of the lowest course is 90, and the number in the lowest course is
$$1 + 2 + 3 + \cdots\cdots + 90 = 45 \times 91 = 4095.$$

8. Find the number of shot in a complete rectangular pile of 15 courses, which has 20 shot in the longest side of its base.

Here, $n = 15$, $n' = 20$; and the required number is
$$\tfrac{15}{6} \times 16 \times (60 - 15 + 1) = 1840.$$

9. Find the number of shot in the bottom row of a square pile which contains 2600 more shot than a triangular pile of the same number of courses.
$$\frac{n(n+1)(2n+1)}{1 \times 2 \times 3} - \frac{n(n+1)(n+2)}{1 \times 2 \times 3} = 2600,$$
$$\frac{n(n+1)\{2n+1-(n+2)\}}{6} = 2600,$$
$$n(n+1)(n-1) = 15{,}600,$$
$$(n-1)n(n+1) = 24 \times 25 \times 26,$$
$$n = 25.$$
The required number is 25.

10. Find the number of shot in a complete square pile in which the number of shot in the base and the number in the fifth course above differ by 225.

$$n^2 - (n-5)^2 = 225,$$
$$10n - 25 = 225,$$
$$n = 25,$$
$$\frac{n(n+1)(2n+1)}{1 \times 2 \times 3} = \frac{25 \times 26 \times 51}{1 \times 2 \times 3}$$
$$= 5525.$$

The required number is 5525.

11. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.

The difference between the number of shot in the length of any course and the number in the width is the same for all courses.

Hence,
$$n' - n = 11 - 1$$
$$= 10,$$
$$nn' = 600.$$
$$\therefore n' = 30,$$
$$n = 20,$$
$$\frac{n}{6}(n+1)(3n' - n + 1) = \tfrac{20}{6} \times 21 \times 71$$
$$= 4970.$$

The required number is 4970.

Exercise CXXIX.

1. Sum to n terms, and to infinity, the series $\frac{1}{1 \times 4}$, $\frac{1}{2 \times 5}$, $\frac{1}{3 \times 6}$, ...

$$\frac{1}{1 \times 4} + \frac{1}{2 \times 5} + \frac{1}{3 \times 6} + \cdots\cdots + \frac{1}{n(n+3)}$$
$$= \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \cdots\cdots \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$
$$= \frac{1}{3}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$= \frac{1}{3}\left(\frac{11}{6} - \frac{3n^2 + 12n + 11}{(n+1)(n+2)(n+3)}\right)$$
$$= \frac{11n^3 + 48n^2 + 49n}{18(n+1)(n+2)(n+3)}$$
$$= \text{sum to } n \text{ terms}.$$

The sum to infinity
$$= \tfrac{1}{3}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}\right)$$
$$= \tfrac{11}{18}.$$

2. Sum to n terms, and to infinity, the series $\frac{1}{1 \times 3 \times 5}$, $\frac{1}{2 \times 4 \times 6}$, $\frac{1}{3 \times 5 \times 7}$, ...

The general term is $\frac{1}{n(n+2)(n+4)}$.

Let $$\frac{1}{n(n+2)(n+4)} = \frac{A}{n} + \frac{B}{n+2} + \frac{C}{n+4},$$

then
$$\begin{aligned} 1 &= A(n+2)(n+4) + Bn(n+4) \\ &\quad + Cn(n+2) \\ &= (A+B+C)n^2 + (6A+4B+2C)n \\ &\quad + 8A. \end{aligned}$$

$$\begin{aligned} \therefore A+B+C &= 0, & A &= \tfrac{1}{8}, \\ 6A+4B+2C &= 0, & B &= -\tfrac{1}{4}, \\ 8A &= 1, & C &= \tfrac{1}{8}. \end{aligned}$$

Hence
$$\begin{aligned} \frac{1}{n(n+2)(n+4)} &= \frac{1}{8n} - \frac{1}{4(n+2)} + \frac{1}{8(n+4)} \\ &= \frac{1}{8}\left(\frac{1}{n} - \frac{2}{n+2} + \frac{1}{n+4}\right). \end{aligned}$$

Hence
$$\begin{aligned} &\frac{1}{1 \times 3 \times 5} + \frac{1}{2 \times 4 \times 6} + \cdots + \frac{1}{n(n+2)(n+4)} \\ &= \frac{1}{8}\left[\left(\frac{1}{1} - \frac{2}{3} + \frac{1}{5}\right) + \left(\frac{1}{2} - \frac{2}{4} + \frac{1}{6}\right) + \cdots \right. \\ &\quad \left. + \left(\frac{1}{n} - \frac{2}{n+2} + \frac{1}{n+4}\right)\right] \\ &= \frac{1}{8}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \\ &\quad - \frac{1}{4}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2}\right) \\ &\quad + \frac{1}{8}\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{n+4}\right) \\ &= \frac{1}{8}\left(\frac{1}{1} + \frac{1}{2}\right) - \frac{1}{8}\left(\frac{1}{3} + \frac{1}{4}\right) - \frac{1}{8}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) \\ &\quad + \frac{1}{8}\left(\frac{1}{n+3} + \frac{1}{n+4}\right) \\ &= \frac{11}{96} - \frac{1}{8}\left(\frac{2n+3}{(n+1)(n+2)} - \frac{2n+7}{(n+3)(n+4)}\right) \\ &= \frac{11}{96} - \frac{2n^2+10n+11}{4(n+1)(n+2)(n+3)(n+4)} \\ &= \text{sum to } n \text{ terms.} \end{aligned}$$

Sum to infinity
$$\begin{aligned} &= \tfrac{1}{8}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) - \tfrac{1}{8}\left(\tfrac{1}{3} + \tfrac{1}{4}\right) \\ &= \tfrac{11}{96}. \end{aligned}$$

3. Sum to n terms, and to infinity, the series $\frac{1}{2\times4\times6}$, $\frac{1}{4\times6\times8}$, $\frac{1}{6\times8\times10}$, ...

The general term is $\frac{1}{2n(2n+2)(2n+4)}=\frac{1}{8n(n+1)(n+2)}$.

Let $$\frac{1}{n(n+1)(n+2)}=\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2},$$

then
$$\begin{aligned}1&=A(n+1)(n+2)+Bn(n+2)+Cn(n+1)\\&=(A+B+C)n^2+(3A+2B+C)n+2A.\end{aligned}$$

$$\begin{aligned}\therefore A+B+C&=0, & A&=\tfrac{1}{2},\\ 3A+2B+C&=0, & B&=-1,\\ 2A&=1, & C&=\tfrac{1}{2}.\end{aligned}$$

Hence $$\begin{aligned}\frac{1}{n(n+1)(n+2)}&=\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2(n+2)}\\&=\frac{1}{2}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right);\end{aligned}$$

and $$\frac{1}{8n(n+1)(n+2)}=\frac{1}{16}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right).$$

Hence $$\frac{1}{2\times4\times6}+\frac{1}{4\times6\times8}+\cdots\cdots+\frac{1}{2n(2n+2)(2n+4)}$$

$$\begin{aligned}&=\frac{1}{16}\left[\left(\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\cdots\cdots\right.\\&\qquad\left.+\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right)\right]\\
&=\frac{1}{16}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}\right)\\
&\quad-\frac{1}{8}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{n+1}\right)\\
&\quad+\frac{1}{16}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\cdots+\frac{1}{n+2}\right)\\
&=\frac{1}{16}\left(\frac{1}{1}\right)-\frac{1}{16}\left(\frac{1}{2}\right)-\frac{1}{16}\left(\frac{1}{n+1}\right)+\frac{1}{16}\left(\frac{1}{n+2}\right)\\
&=\frac{1}{16}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right)\\
&=\frac{n^2+3n}{32(n+1)(n+2)}\end{aligned}$$

= sum to n terms.

Sum to infinity $=\frac{1}{16}\left(\frac{1}{1}-\frac{1}{2}\right)$
$=\frac{1}{32}$.

4. Sum to n terms, and to infinity, the series $\frac{4}{2\times3\times4}$, $\frac{7}{3\times4\times5}$, $\frac{10}{4\times5\times6}$, ...

The general term is $\frac{3n+1}{(n+1)(n+2)(n+3)}$.

Let $$\frac{3n+1}{(n+1)(n+2)(n+3)} = \frac{A}{n+1} + \frac{B}{n+2} + \frac{C}{n+3},$$

then
$$\begin{aligned} 3n+1 &= A(n+2)(n+3) + B(n+1)(n+3) \\ &\quad + C(n+1)(n+2) \\ &= (A+B+C)n^2 + (5A+4B+3C)n \\ &\quad + 6A+3B+2C. \end{aligned}$$

$$\begin{aligned} \therefore A+B+C&=0, & A&=-1, \\ 5A+4B+3C&=3, & B&=5, \\ 6A+3B+2C&=1, & C&=-4. \end{aligned}$$

Hence $$\frac{3n+1}{(n+1)(n+2)(n+3)} = \frac{-1}{n+1} + \frac{5}{n+2} - \frac{4}{n+3},$$

and
$$\begin{aligned} &\frac{4}{2\times3\times4} + \frac{7}{3\times4\times5} + \cdots\cdots + \frac{3n+1}{(n+1)(n+2)(n+3)} \\ &= \left(-\frac{1}{2}+\frac{5}{3}-\frac{4}{4}\right) + \left(-\frac{1}{3}+\frac{5}{4}-\frac{4}{5}\right) + \cdots\cdots \\ &\quad + \left(-\frac{1}{n}+\frac{5}{n+1}-\frac{4}{n+2}\right) \\ &= -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{n+1}\right) \\ &\quad + 5\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\cdots+\frac{1}{n+2}\right) \\ &\quad - 4\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots\cdots+\frac{1}{n+3}\right) \\ &= -\frac{1}{2}+\frac{4}{3}+\frac{1}{n+2}-\frac{4}{n+3} \\ &= \frac{5}{6} - \frac{3n+5}{(n+2)(n+3)} \\ &= \frac{5n^2+7n}{6(n+2)(n+3)} \\ &= \frac{n(5n+7)}{6(n+2)(n+3)} \end{aligned}$$
= sum to n terms.

Sum to infinity $= \frac{5}{6}$.

5. Sum to n terms, and to infinity, the series $\frac{1}{1\times 2\times 3}$, $\frac{1}{2\times 3\times 4}$, $\frac{1}{3\times 4\times 5}$, ...

Each term of the series is 8 times the corresponding term in the series of Ex. 3. Hence the sum to n terms

$$=\frac{n^2+3n}{4(n+1)(n+2)},$$

and the sum to infinity $=\frac{1}{8}$.

Exercise CXXX.

1. If 6, 7, 8, 3, 2 are the digits of a number in the scale of r, beginning from the right, write the algebraical value of the number.

The algebraical value of the number is

$$2r^4+3r^3+8r^2+7r+6.$$

2. Find the product of 234 and 125 when r is the base of the scale.

$$\begin{aligned}234\times 125 &= (2r^2+3r+4)(r^2+2r+5)\\ &= 2r^4+7r^3+20r^2+23r+20.\end{aligned}$$

3. In what scale will the common number 756 be expressed by 530?

Let r be the base of the scale; then

$$5r^2+3r=756.$$
$$\therefore r=12.$$

4. In what scale will 540 be the square of 23?

Let r be the base of the scale; then

$$(2r+3)^2=5r^2+4r$$
$$-r^2+8r+9=0.$$
$$\therefore r=9.$$

5. Show that 1234321 will, in any scale, be a perfect square, and find its square root.

In any scale

$$\begin{aligned}1{,}234{,}321 &= r^6+2r^5+3r^4+4r^3+3r^2+2r+1\\ &= (r^3+r^2+r+1)^2\\ &= 1111^2 \text{ in the scale of } r.\end{aligned}$$

6. In what scale will 212, 1101, 1220 be in arithmetical progression?

Let r be the base of the scale; then

$$\begin{aligned} 2 \times 2 \times 1101 &= 212 + 1220, \\ 2(r^3 + r^2 + 1) &= (2r^2 + r + 2) + r^3 + 2r^2 + 2r, \\ r^3 - 2r^2 - 3r &= 0, \\ r(r+1)(r-3) &= 0. \\ \therefore r &= 3. \end{aligned}$$

7. Multiply 31.24 by 0.31 in the scale of 5.

$$\begin{array}{r} 31.24 \\ \underline{0.31} \\ 3124 \\ \underline{14432} \\ 20.2444 \end{array}$$

8. Find the least multiplier of 13,168 which will make the product a perfect cube.

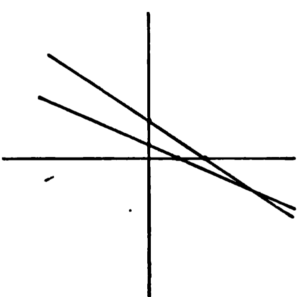
$$13{,}168 = 16 \times 823 = 2^4 \times 823.$$

Hence the required multiplier is $2^2 \times 823^2 = 1646^2 = 2{,}709{,}316$.

Exercise CXXXI.

1. Solve the following equations by constructing their loci:

$$\left.\begin{aligned} 2x + 3y &= 8 \\ 3x + 7y &= 7 \end{aligned}\right\}$$



(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = 2\frac{2}{3},\ 2,\ 1\frac{1}{3},\ \frac{2}{3},\ 0,\ -\frac{2}{3},\ 3\frac{1}{3},\ 4,\ 4\frac{2}{3},\ 5\frac{1}{3}.$

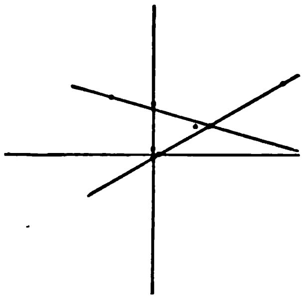
(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = 1,\ \frac{4}{7},\ \frac{1}{7},\ -\frac{2}{7},\ -\frac{5}{7},\ -1\frac{1}{7},\ 1\frac{3}{7},\ 1\frac{6}{7},\ 2\frac{2}{7},\ 2\frac{5}{7}.$

The loci are straight lines, as represented in the figure, and they meet at the point $x = 7$, $y = -2$.

2. Solve the following equations by constructing their loci:

$$\left.\begin{array}{l} 3x - 5y = 2 \\ 2x + 7y = 22 \end{array}\right\}$$



(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{2}{5},\ \frac{1}{5},\ \frac{4}{5},\ 1\frac{2}{5},\ 2,\ 2\frac{3}{5},\ -1,\ -1\frac{3}{5},\ -2\frac{1}{5},\ -2\frac{4}{5}.$

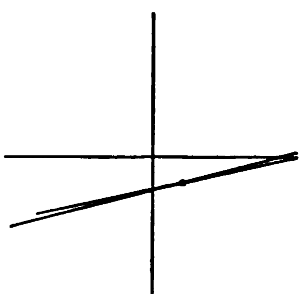
(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = 3\frac{1}{7},\ 2\frac{6}{7},\ 2\frac{4}{7},\ 2\frac{2}{7},\ 2,\ 1\frac{5}{7},\ 3\frac{3}{7},\ 3\frac{5}{7},\ 4,\ 4\frac{2}{7}.$

The loci are straight lines, as represented in the figure, and they meet at the point $x = 4$, $y = 2$.

3. Solve the following equations by constructing their loci:

$$\left.\begin{array}{l} 2x - 9y = 11 \\ 3x - 12y = 15 \end{array}\right\}$$



(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{11}{9},\ -1,\ -\frac{7}{9},\ -\frac{5}{9},\ -\frac{1}{3},\ -\frac{1}{9},\ -\frac{13}{9},\ -\frac{5}{3},\ -\frac{17}{9},\ -\frac{19}{9}.$

(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{5}{4},\ -1,\ -\frac{3}{4},\ -\frac{1}{2},\ -\frac{1}{4},\ -0,\ -\frac{3}{2},\ -\frac{7}{4},\ -2,\ -\frac{9}{4}.$

The loci are straight lines, as represented in the figure, and they meet at the point $x = 1$, $y = -1$.

4. Solve the following equations by constructing their loci:

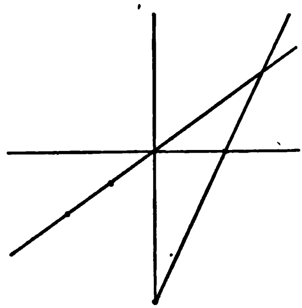
$$\left.\begin{array}{l}4x-2y=20\\6x=9y\end{array}\right\}$$

(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,$
$-2,\ -3,\ -4,$

$y = -10,\ -8,\ -6,\ -4,$
$-2,\ 0,\ -12,\ -14,$
$-16,\ -18.$

(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,$
$-2,\ -3,\ -4,$

$y = 0,\ \frac{2}{3},\ \frac{4}{3},\ 2,\ \frac{8}{3},\ \frac{10}{3},\ -\frac{2}{3},$
$-\frac{4}{3},\ -2,\ -\frac{8}{3}.$



The loci are straight lines, as represented in the figure, and they meet at the point $x = 7\frac{1}{2},\ y = 5$.

5. Solve the following equations by constructing their loci:

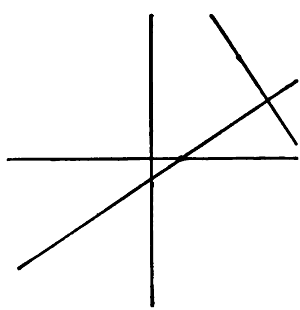
$$\left.\begin{array}{l}2x-3y=4\\3x+2y=32\end{array}\right\}$$

(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,$
$-2,\ -3,\ -4,$

$y = -\frac{4}{3},\ -\frac{2}{3},\ 0,\ \frac{2}{3},\ \frac{4}{3},\ 2,$
$-2,\ -\frac{8}{3},\ -\frac{10}{3},\ -4.$

(ii.) If $x = 0,\ 2,\ 4,\ 6,\ 8,\ 10,\ -2,$
$-4,\ -6,\ -8,$

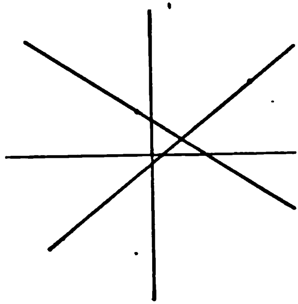
$y = 16,\ 13,\ 10,\ 7,\ 4,\ 1,$
$19,\ 22,\ 25,\ 28.$



The loci are straight lines, as represented in the figure, and they meet at the point $x = 8,\ y = 4$.

6. Solve the following equations by constructing their loci:

$$\left.\begin{aligned} 2x + 3y &= 7 \\ 4x - 5y &= 3 \end{aligned}\right\}$$



(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = \frac{7}{3},\ \frac{5}{3},\ 1,\ \frac{1}{3},\ -\frac{1}{3},\ -1,\ 3,\ \frac{11}{3},\ \frac{13}{3},\ 5.$

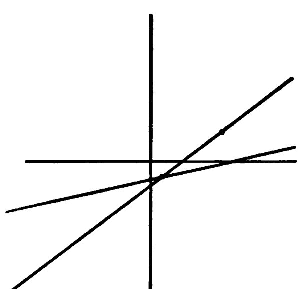
(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{3}{5},\ \frac{1}{5},\ 1,\ \frac{9}{5},\ \frac{13}{5},\ \frac{17}{5},\ -\frac{7}{5},\ -\frac{11}{5},\ -3,\ -\frac{19}{5}.$

The loci are straight lines, as represented in the figure, and they meet at the point $x = 2,\ y = 1$.

7. Solve the following equations by constructing their loci:

$$\left.\begin{aligned} 2x - 9y &= 11 \\ 3x - 4y &= 7 \end{aligned}\right\}$$



(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5 -1,\ -2,\ -3,\ -4,$

$y = -\frac{11}{9},\ -1,\ -\frac{7}{9},\ -\frac{5}{9},\ -\frac{1}{3},\ -\frac{1}{9},\ -\frac{13}{9},\ -\frac{5}{3},\ -\frac{17}{9},\ -\frac{19}{9}.$

(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{7}{4},\ -1,\ -\frac{1}{4},\ \frac{1}{2},\ \frac{5}{4},\ 2,\ -\frac{5}{2},\ -\frac{13}{4},\ -4,\ -\frac{19}{4}.$

The loci are straight lines, as represented in the figure, and they meet at the point $x = 1,\ y = -1$.

8. Solve the following equations by constructing their loci:

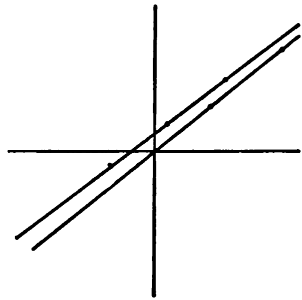
$$\left.\begin{array}{l}3x-4y=-5\\4x-5y=1\end{array}\right\}$$

(i.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = \frac{5}{4},\ 2,\ \frac{11}{4},\ \frac{7}{2},\ \frac{17}{4},\ 5,\ \frac{1}{2},\ -\frac{1}{4},\ -1,\ -\frac{7}{4}.$

(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

$y = -\frac{1}{5},\ \frac{3}{5},\ \frac{7}{5},\ \frac{11}{5},\ 3,\ \frac{19}{5},\ -1,\ -\frac{9}{5},\ -\frac{13}{5},\ -\frac{17}{5}.$



The loci are straight lines, as represented in the figure, and they meet at the point $x = 29,\ y = 23$.

9. Solve the following equations by constructing their loci:

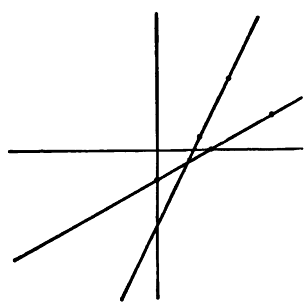
$$\left.\begin{array}{l}x-2y=4\\2x-y=5\end{array}\right\}$$

(i.) If $x = 0, 2, 4, 6, 8, -2, -4, -6,$

$y = -2, -1, 0, 1, 2, -3, -4, -5.$

(ii.) If $x = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ -1,\ -2,\ -3,\ -4,$

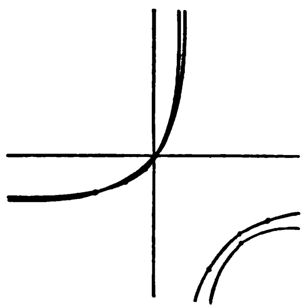
$y = -5,\ -3,\ -1,\ 1,\ 3,\ 5,\ -7,\ -9,\ -11,\ -13.$



The loci are straight lines, as represented in the figure, and they meet at the point $x = 2,\ y = -1$.

10. Solve the following equations by constructing their loci:

$$\left.\begin{aligned}\frac{3}{x}-\frac{4}{y}&=5\\ \frac{4}{x}-\frac{5}{y}&=6\end{aligned}\right\}$$



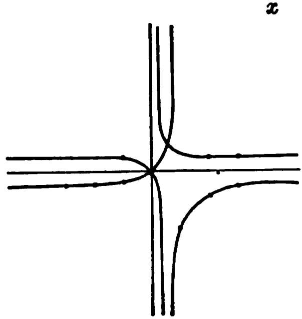
(i.) If $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1,$
$y = 0, \frac{4}{7}, 4, -4, -2, -\frac{4}{3}, -\frac{8}{7}, -\frac{4}{17}, -\frac{4}{11}, -\frac{4}{9}, -\frac{1}{2}.$

(ii.) If $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -1,$
$y = 0, \frac{1}{2}, \frac{5}{2}, -\frac{15}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{5}{4}, -\frac{5}{22}, -\frac{5}{14}, -\frac{15}{34}, -\frac{1}{2}.$

The loci are *hyperbolas*, consisting each of two infinite branches, as represented in the figure; and they intersect at the two points $x = 0$, $y = 0$, and $x = -1$, $y = -\frac{1}{2}$.

11. Solve the following equations by constructing their loci:

$$\left.\begin{aligned}\frac{1}{x}+\frac{2}{y}&=4\\ \frac{3}{x}-\frac{2}{y}&=4\end{aligned}\right\}$$



(i.) If $x = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, -1, -2, -3,$

$y = 0, \infty, 1, \frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{2}{5}, \frac{4}{9}, \frac{6}{13}.$

(ii.) If $x = 0, \frac{3}{4}, 1, 2, 3, -1, -2, -3, -4,$

$y = 0, \infty, -2, -\frac{4}{5}, -\frac{2}{3}, -\frac{2}{7}, -\frac{4}{11}, -\frac{2}{5}, -\frac{8}{19}.$

The loci are hyperbolas, as represented in the figures, and they intersect at the points $x = 0$, $y = 0$, and $x = \frac{1}{2}$, $y = 1$.

12. Solve the following equations by constructing their loci:

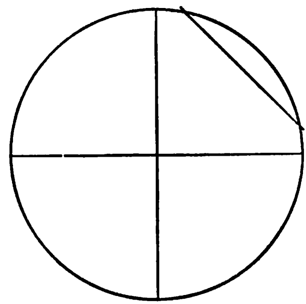
$$\left.\begin{array}{r} x^2+y^2=104 \\ x+y=12 \end{array}\right\}$$

(i.) If $x = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10,$

$y = \pm 10.2, \pm 10, \pm 9.4, \pm 8.2, \pm 6.3, \pm 2.$

(ii.) If $x = 0, 2, 4, 6, 8, 10, 12,$

$y = 12, 10, 8, 6, 4, 2, 0.$



The first locus is a circle, the second a straight line, as represented in the figure, and they intersect at the two points $x = 2$, $y = 10$, and $x = 10$, $y = 2$.

13. Solve the following equations by constructing their loci:

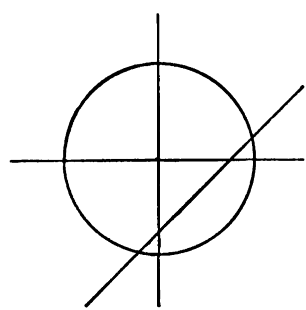
$$\left.\begin{array}{r} x-y=10 \\ x^2+y^2=178 \end{array}\right\}$$

(i.) If $x = 0, 2, 4, 6, 8, 10, 12, 13,$

$y = -10, -8, -6, -4, -2, 0, 2, 3.$

(ii.) If $x = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, \pm 13,$

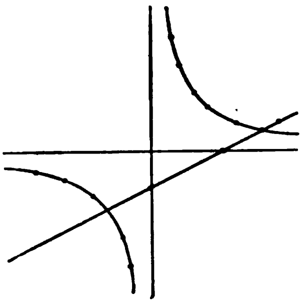
$y = \pm 13.3, \pm 13.2, \pm 12.7, \pm 11.9, \pm 10.7, \pm 8.8, \pm 5.5, \pm 3.$



The first locus is a straight line, the second a circle, as represented in the figure, and they intersect at the two points $x = 13$, $y = 3$, and $x = -3$, $y = -13$.

14. Solve the following equations by constructing their loci:

$$\left.\begin{aligned} xy - 12 &= 0 \\ x - 2y &= 5 \end{aligned}\right\}$$



(i.) If $x = 0, \ \pm 2, \ \pm 4, \ \pm 6, \ \pm 8, \ \infty,$

$y = \infty, \ \pm 6, \ \pm 3, \ \pm 2, \ \pm \frac{3}{2}, \ 0.$

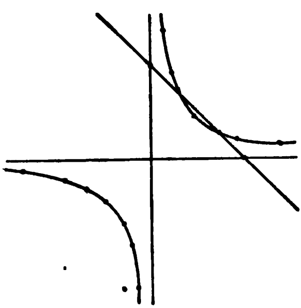
(ii.) If $x = 0, 1, 3, 5, 7, 9,$

$y = -\frac{5}{2}, -2, -1, 0, 1, 2.$

The first locus is a hyperbola, the second a straight line, as represented in the figure; and they intersect at the two points $x = 8$, $y = \frac{3}{2}$, and $x = -3$, $y = -4$.

15. Solve the following equations by constructing their loci:

$$\left.\begin{aligned} x + y &= 13 \\ xy &= 36 \end{aligned}\right\}$$



(i.) If $x = 0, 2, 4, 6, 8, 10, 12,$

$y = 13, 11, 9, 7, 5, 3, 1.$

(ii.) If $x = 0, \pm 2, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18,$

$y = \infty, \ \pm 18, \ \pm 9, \ \pm 6, \ \pm 4, \pm 3, \pm 2.$

The first locus is a straight line, the second a hyperbola, as represented in the figure; and they intersect at the two points $x = 4$, $y = 9$, and $x = 9$, $y = 4$.

16. Solve the following equations by constructing their loci:

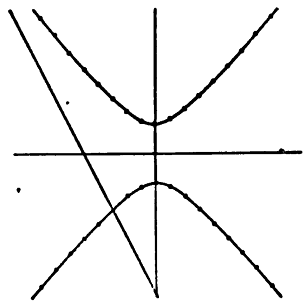
$$\left.\begin{aligned} 3y^2 - 4x^2 &= 12 \\ 2x + y &= -10 \end{aligned}\right\}$$

(i.) If $x = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10,$

$y = \pm 2, \pm 3.0, \pm 5.2, \pm 7.2, \pm 9.4, \pm 11.7.$

(ii.) If $x = 0, 4, -2, -4, -6, -8, -10,$

$y = -10, -18, -6, -2, 2, 6, 10.$



The first locus is a hyperbola, the second a straight line, as represented in the figure; and they intersect at the two points $x = -3$, $y = -4$, and $x = -12$, $y = 14$.

17. Solve the following equations by constructing their loci:

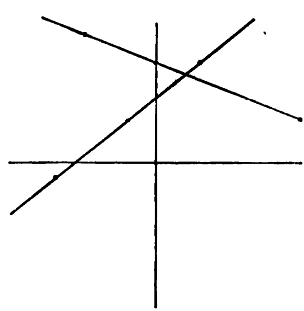
$$\left.\begin{aligned} \frac{4}{5+y} &= \frac{5}{12+x} \\ 2x + 5y &= 35 \end{aligned}\right\}$$

(i.) If $x = 0, 2, 4, 6, -2, -4, -6,$

$y = \frac{23}{5}, \frac{31}{5}, \frac{39}{5}, \frac{47}{5}, 3, \frac{7}{5}, -\frac{1}{5}.$

(ii.) If $x = 0, 2, 4, 6, -2, -4, -6,$

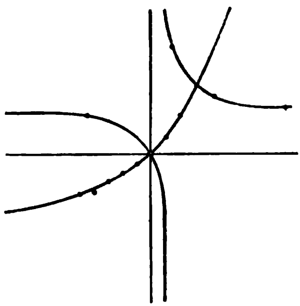
$y = 7, \frac{31}{5}, \frac{27}{5}, \frac{23}{5}, \frac{39}{5}, \frac{43}{5}, \frac{47}{5}.$



The loci are straight lines, as represented in the figure, and they intersect at the point $x = 2$, $y = \frac{31}{5}$.

18. Solve the following equations by constructing their loci:

$$\left.\begin{array}{l}\frac{2}{x}-\frac{5}{3y}=\frac{4}{27}\\ \frac{1}{4x}+\frac{1}{y}=\frac{11}{72}\end{array}\right\}$$

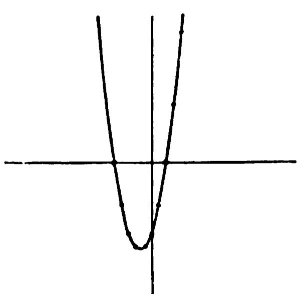


(i.) If $x = 0, 3, 6, 9, 13.5, 18, 20,$
$y = 0, 3.2, 9, 22.5, \infty, -45, 34.6.$

(ii.) If $x = 0, \frac{18}{11}, 3, 6, 9, -3, -6, -9,$
$y = 0, \infty, 14.4, 9, 8, 4.2, 5.1, 5.5.$

The loci are both hyperbolas, only one branch of the first being shown in the figure. They intersect at the two points $x = 0$, $y = 0$, and $x = 6$, $y = 9$.

Exercise CXXXII.



1. Construct the locus of the equation,

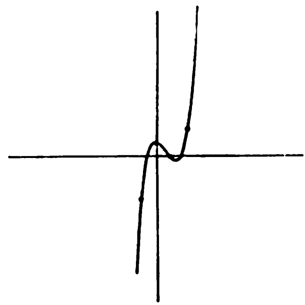
$$x^2 + 3x - 10 = 0.$$

Let $x^2 + 3x - 10 = y,$

then if $x = 0, 1, 2, 4, -2, -4, -5, -6,$
$y = -10, -6, 0, 18, -12, -6, 0, 8.$

The locus is represented in the figure. The roots are 2 and -5.

2. Construct the locus of the equation, $x^3 - 2x^2 + 1 = 0$.

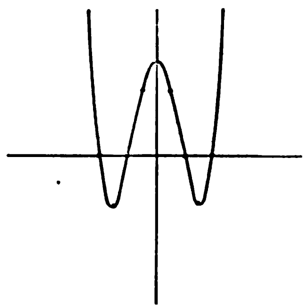


Let $x^3 - 2x^2 + 1 = y$,
then if $x = 0, 1, 2, 3, -1, -2$,
$y = 1, 0, 1, 10, -2, -15$.

The locus is represented in the figure. The roots are

$$1, \frac{1+\sqrt{5}}{2}, \text{ and } \frac{1-\sqrt{5}}{2}.$$

3. Construct the locus of the equation, $x^4 - 20x^2 + 64 = 0$.



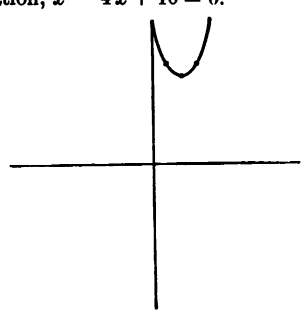
Let $x^4 - 20x^2 + 64 = y$,

then if $x = 0, \pm 2, \pm 3, \pm 4, \pm 5$,

$y = 64, 0, -35, 0, 189$.

The locus is represented in the figure, except that the ordinates are $\frac{1}{10}$ their computed length. The roots are ± 2 and ± 4.

4. Construct the locus of the equation, $x^2 - 4x + 10 = 0$.

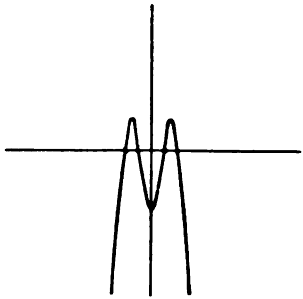


Let $x^2 - 4x + 10 = y$,

then if $x = 0, 1, 2, 3, 4, 5, -2$,

$y = 10, 7, 6, 7, 10, 15, 22$.

The locus is represented in the figure. The roots are imaginary.



5. Construct the locus of the equation,

$$x^4 - 5x^2 + 4 = 0.$$

Let $x^4 - 5x^2 + 4 = y$,

then if $x = 0, \pm 1, \pm 2, \pm 3$,

$$y = 4, 0, 0, 40.$$

The locus is represented in the figure. The roots are ± 1 and ± 2.

Exercise CXXXIII.

1. Determine whether -5 is a root of the equation

$$x^5 + 6x^4 - 10x^3 - 112x^2 - 207x - 110 = 0.$$

$$\begin{array}{l|l} 1 + 6 - 10 - 112 - 207 - 110 & \underline{-5} \\ \underline{\quad -5 - \ \ 5 + \ \ 75 + 185 + 110} & \\ \quad +1 - 15 - \ \ 37 - \ \ 22 + 0 & \end{array}$$

Hence -5 is a root.

2. Determine whether 1 is a root of the equation

$$x^5 - 8x^4 + 7x^3 + x^2 - 3x + 2 = 0.$$

$$\begin{array}{l|l} 1 - 8 + 7 + 1 - 3 + 2 & \underline{1} \\ \underline{\quad\ \ 1 - 7 + 0 + 1 - 2} & \\ \quad -7 + 0 + 1 - 2 + 0 & \end{array}$$

Hence 1 is a root.

3. Determine whether -7 is a root of the equation

$$x^4 + 21x + 7x^3 + 147 = 0.$$

$$\begin{array}{l|l} 1 + 7 + 0 + 21 + 147 & \underline{-7} \\ \underline{\quad -7 + 0 + \ \ 0 - 147} & \\ \quad\ \ 0 + 0 + 21 + 0 & \end{array}$$

Hence -7 is a root.

4. Determine whether -8 is a root of the equation

$$x^5 + 8x^4 - 7x^2 - 54x + 16 = 0.$$

$$\begin{array}{r|l}
1 + 8 - 7 - 54 + 16 & \underline{-8} \\
-8 + 0 + 56 - 16 & \\
\hline
0 - 7 + \ \ 2 + 0 &
\end{array}$$

Hence -8 is a root.

5. Determine whether 2 is a root of the equation

$$x^4 - 4x^3 - 3x^2 - 2x - 8 = 0.$$

$$\begin{array}{r|l}
1 - 4 - 3 - \ \ 2 - \ \ 8 & \underline{2} \\
+2 - 4 - 14 - 32 & \\
\hline
-2 - 7 - 16 - 40 &
\end{array}$$

Hence 2 is not a root.

6. Determine whether -7 is a root of the equation

$$x^3 + 14x^2 + 65x + 112 = 0.$$

$$\begin{array}{r|l}
1 + 14 + 65 + 112 & \underline{-7} \\
- \ \ 7 - 49 - 112 & \\
\hline
+ \ \ 7 + 16 + 0 &
\end{array}$$

Hence -7 is a root.

7. Determine whether 6 is a root of the equation

$$2x^4 - 4x^3 - 62x^2 + 114x - 180 = 0.$$

$$\begin{array}{r|l}
2 - \ \ 4 - 62 + 114 - 180 & \underline{6} \\
+12 + 48 - \ \ 84 + 180 & \\
\hline
+ \ \ 8 - 14 + \ \ 30 + 0 &
\end{array}$$

Hence 6 is a root.

8. Determine whether -5 is a root of the equation

$$x^4 - 7x - 2x^2 - 15 = 0.$$

$$\begin{array}{r|l}
1 + 0 - \ \ 2 - \ \ \ \ 7 - \ \ 15 & \underline{-5} \\
-5 + 25 - 115 + 610 & \\
\hline
-5 + 23 - 122 + 595 &
\end{array}$$

Hence -5 is not a root.

9. Determine whether -0.3 is a root of the equation

$$x^4+2.3x^3+3.6x^2+4.9x+1.2=0.$$

$$\begin{array}{rrrrr|l} 1 & +2.3 & +3.6 & +4.9 & +1.2 & \underline{-0.3} \\ & -0.3 & -0.6 & -0.9 & -1.2 & \\ \hline & +2 & +3 & +4 & +0 & \end{array}$$

Hence -0.3 is a root.

10. Determine whether $\frac{2}{3}$ is a root of the equation

$$x^3-\tfrac{1}{6}x^2-\tfrac{1}{12}x-\tfrac{1}{6}=0.$$

$$\begin{array}{rrrr|l} 1 & -\frac{1}{6} & -\frac{1}{12} & -\frac{1}{6} & \underline{\frac{2}{3}} \\ & \frac{2}{3} & +\frac{1}{3} & +\frac{1}{6} & \\ \hline & +\frac{1}{2} & +\frac{1}{4} & +0 & \end{array}$$

Hence $\frac{2}{3}$ is a root.

Exercise CXXXIV.

1. Find the equation whose roots are 2, 6, and -7.

The equation is $(x-2)(x-6)(x+7)=0$

or $x^3-x^2-44x+84=0.$

2. Find the equation of which the roots are 1, 4, -1, and -3.

The equation is

$$(x-1)(x-4)(x+1)(x+3)=0$$

or $(x^2-1)(x^2-x-12)=0$

or $x^4-x^3-13x^2+x+12=0.$

3. Find the equation of which the roots are 2, 3, -2, -3, and -6.

The equation is

$$(x-2)(x-3)(x+2)(x+3)(x+6)=0$$

or $(x^2-4)(x^2-9)(x+6)=0$

or $x^5+6x^4-13x^3-78x^2+36x+216=0.$

4. Find the equation of which the roots are 0.2, $\frac{1}{8}$, and -0.4.

The equation is

$$(x-0.2)(x-\tfrac{1}{8})(x+0.4)=0$$

or $(x-\frac{1}{5})(x-\frac{1}{8})(x+\frac{2}{5})=0$

or $(5x-1)(8x-1)(5x+2)=0$

or $200x^3+15x^2-21x+2=0.$

5. Find the equation of which the roots are

$$5,\ 3+\sqrt{-1},\ \text{and}\ 3-\sqrt{-1}.$$

The equation is

$$(x-5)(x-3-\sqrt{-1})(x-3+\sqrt{-1})=0$$

or $$(x-5)(x^2-6x+10)=0$$

or $$x^3-11x^2+40x-50=0.$$

Exercise CXXXV.

1. Form the equation whose roots are 2, 4, and -3.

$$2+4-3=3,$$
$$2\times 4+2\times(-3)+4(-3)=-10,$$
$$2\times 4\times(-3)=-24.$$

Hence the equation is

$$x^3-3x^2-10x+24=0.$$

2. Form the equation whose roots are 2, -1, and -7.

$$2-1-7=-6,$$
$$2\times(-1)+2\times(-7)+(-1)\times(-7)=-9,$$
$$2\times(-1)\times(-7)=14.$$

Hence the equation is

$$x^3+6x^2-9x-14=0.$$

3. Form the equation whose roots are 2, 0, and -2.

$$2+0-2=0,$$
$$2\times 0+2\times(-2)+0\times(-2)=-4,$$
$$2\times 0\times(-2)=0.$$

Hence the equation is

$$x^3-4x=0.$$

4. Form the equation whose roots are 6, 6, and 6.

$$6+6+6=18,$$
$$6\times 6+6\times 6+6\times 6=108,$$
$$6\times 6\times 6=216.$$

Hence the equation is

$$x^3-18x^2+108x-216=0.$$

5. Form the equation whose roots are 2, 1, -2, and -1.

$$2+1-2-1=0,$$
$$2\times 1+2\times(-2)+2\times(-1)+1\times(-2)+1\times(-1)$$
$$+(-2)\times(-1)=-5,$$
$$2\times 1\times(-2)+2\times 1\times(-1)+2\times(-2)\times(-1)$$
$$+1\times(-2)\times(-1)=0.$$
$$2\times 1\times(-2)\times(-1)=4.$$

Hence the equation is

$$x^4-5x^2+4=0.$$

6. Form the equation whose roots are 2, $\frac{1}{2}$, -2, $-\frac{1}{2}$.

$$2+\tfrac{1}{2}-2-\tfrac{1}{2}=0,$$
$$2\times\tfrac{1}{2}+2\times(-2)+2\times(-\tfrac{1}{2})+\tfrac{1}{2}\times(-2)+\tfrac{1}{2}\times(-\tfrac{1}{2})$$
$$+(-2)\times(-\tfrac{1}{2})=-\tfrac{17}{4},$$
$$2\times\tfrac{1}{2}\times(-2)+2\times\tfrac{1}{2}\times(-\tfrac{1}{2})+2\times(-2)\times(-\tfrac{1}{2})$$
$$+\tfrac{1}{2}\times(-2)\times(-\tfrac{1}{2})=0,$$
$$2\times\tfrac{1}{2}(-2)\times(-\tfrac{1}{2})=1.$$

Hence the equation is

$$x^4-\tfrac{17}{4}x^2+1=0.$$

Exercise CXXXVI.

1. Find the roots of the equation $x^2+11x+24=0$.

$$x^2+11x+24=(x+3)(x+8)=0.$$

Hence $x=-3$ or -8.

2. Find the roots of the equation $7x^2+161x+714=0$.

$$7x^2+161x+714=7(x^2+23x+102)$$
$$=7(x+17)(x+6).$$

Hence $x=-17$ or -6.

3. Find the roots of the equation $x^4-4a^2x^2+3a^4=0$.

$$x^4-4a^2x^2+3a^4=(x^2-a^2)(x^2-3a^2)$$
$$=(x+a)(x-a)(x+\sqrt{3}a)(x-\sqrt{3}a).$$

Hence $x=-a,\ +a,\ -\sqrt{3}a,$ or $+\sqrt{3}a$.

4. Find the roots of the equation $x^5 + 4x^3 + 8x^2 + 32 = 0$.

$$\begin{aligned} x^5 + 4x^3 + 8x^2 + 32 &= (x^3 + 8)(x^2 + 4) \\ &= (x + 2)(x^2 - 2x + 4)(x^2 + 4) \end{aligned}$$

Hence $\quad x = -2,\ 1 \pm \sqrt{-3},\ \pm 2\sqrt{-1}.$

5. Find the roots of the equation $12x^2 - 5x - 2 = 0$,

$$12x^2 - 5x - 2 = (3x - 2)(4x + 1).$$

Hence $\quad x = \frac{2}{3}$ or $-\frac{1}{4}$.

6. Find the roots of the equation $4x^4 - 9x^2 + 6x - 1 = 0$.

$$\begin{aligned} 4x^4 - 9x^2 + 6x - 1 &= (2x^2)^2 - (3x - 1)^2 \\ &= [2x^2 + (3x - 1)][2x^2 - (3x - 1)] \\ &= (2x^2 + 3x - 1)(2x^2 - 3x + 1) \\ &= (2x^2 + 3x - 1)(x - 1)(2x - 1). \end{aligned}$$

Hence $\quad x = 1,\ \frac{1}{2}$, or $\dfrac{-3 \pm \sqrt{17}}{4}$.

7. Find the roots of the equation $49x^2 - 112bx + 64b^2 = 0$.

$$49x^2 - 112bx + 64b^2 = (7x - 8b)^2.$$

Hence $\quad x = \dfrac{8b}{7},\ \dfrac{8b}{7}.$

8. Find the roots of the equation $x^6 - 64 = 0$.

$$\begin{aligned} x^6 - 64 &= (x^3 + 8)(x^3 - 8) \\ &= (x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4). \end{aligned}$$

Hence $\quad x = -2,\ 1 \pm \sqrt{-3},\ 2$, or $-1 \pm \sqrt{-3}$.

9. Find the roots of the equation $3x^3 - x^2 + 3x - 1 = 0$.

$$\begin{aligned} 3x^3 - x^2 + 3x - 1 &= 3(x^3 + x) - (x^2 + 1) \\ &= (x^2 + 1)(3x - 1). \end{aligned}$$

Hence $\quad x = \frac{1}{3}$ or $\pm\sqrt{-1}$.

10. Find the roots of the equation $x - 27x^4 = 0$.

$$\begin{aligned} x - 27x^4 &= x(1 - 27x^3) \\ &= x(1 - 3x)(1 + 3x + 9x^2). \end{aligned}$$

Hence $\quad x = 0,\ \frac{1}{3}$, or $\dfrac{-1 \pm \sqrt{-3}}{6}$.

Exercise CXXXVII.

1. Solve the equation $x^3+3x^2-25x-12=0$.

$$\begin{array}{rrrr} x^3 & +3x^2 & -25x & -12=0 \\ & -4 & +3 & \\ \hline & 7 & -28 & \end{array} \qquad -28\div 7=-4.$$

Hence $x^3+3x^2-25x-12=(x-4)(x^2+7x+3)$,

and $x=4$, or, $\dfrac{-7\pm\sqrt{37}}{2}$.

2. Solve the equation $x^3-4x^2-8x+8=0$.

$$\begin{array}{rrrr} x^3 & -4x^2 & -8x & +8=0 \\ & 2 & +4 & \\ \hline & -6 & -12 & \end{array} \qquad -12\div(-6)=2.$$

Hence $x^3-4x^2-8x+8=(x+2)(x^2-6x+4)$,

and $x=-2$, or $3\pm\sqrt{5}$.

3. Solve the equation $x^3-7x^2+19x-21=0$.

$$\begin{array}{rrrr} x^3 & -7x^2 & +19x & -21=0 \\ & -3 & +7 & \\ \hline & -4 & +12 & \end{array} \qquad 12\div(-4)=-3.$$

Hence $x^3-7x^2+19x-21=(x-3)(x^2-4x+7)$,

and $x=3$, or $2\pm\sqrt{-3}$.

4. Solve the equation $x^3-8x^2+21x-18=0$.

$$\begin{array}{rrrr} x^3 & -8x^2 & +21x & -18=0 \\ & -2 & +9 & \\ \hline & -6 & +12 & \end{array} \qquad 12\div 6=2.$$

Hence $x^3-8x^2+21x-18=(x-2)(x^2-6x+9)$
$=(x-2)(x-3)^2$

and $x=2, 3, 3$.

5. Solve the equation $x^3-26x-5=0$.

$$\begin{array}{rrrr} x^3 & +0x^2 & -26x & -5=0 \\ & 5 & -1 & \\ \hline & -5 & -25 & \end{array} \qquad -25\div(-5)=5.$$

Hence $x^3-26x-5=(x+5)(x^2-5x-1)$,

and $x=-5$, or $\dfrac{5\pm\sqrt{29}}{2}$.

6. Solve the equation $x^3 - 3x^2 - 54x - 104 = 0$.

$$\begin{array}{l}
x^3 - 3x^2 - 54x - 104 = 0 \\
\quad\ \underline{\ \ 4 \quad -26\ } \\
\quad\ -7 \quad -28 \qquad\qquad -28 \div (-7) = 4.
\end{array}$$

Hence $$x^3 - 3x^2 - 54x - 104 = (x + 4)(x^2 - 7x - 26),$$

and $$x = -4, \text{ or } \frac{7 \pm \sqrt{153}}{2}.$$

7. Solve the equation $x^3 + 9x^2 + 2x - 48 = 0$.

$$\begin{array}{l}
x^3 + 9x^2 + \ 2x - 48 = 0 \\
\quad\ \underline{\ -2 \quad +24\ } \\
\quad\ \ 11 \quad -22 \qquad\qquad -22 \div 11 = -2.
\end{array}$$

Hence $$\begin{aligned} x^3 + 9x^2 + 2x - 48 &= (x - 2)(x^2 + 11x + 24) \\ &= (x - 2)(x + 3)(x + 8), \end{aligned}$$

and $$x = 2, \ -3, \text{ or } -8.$$

8. Solve the equation $x^3 - 2x^2 - 25x + 50 = 0$.

$$\begin{array}{l}
x^3 - 2x^2 - 25x + 50 = 0 \\
\quad\ \underline{\ -2 \quad -25\ } \\
\quad\quad 0 \qquad 0
\end{array}$$

Hence $$\begin{aligned} x^3 - 2x^2 - 25x + 50 &= (x - 2)(x^2 - 25) \\ &= (x - 2)(x - 5)(x + 5), \end{aligned}$$

and $$x = 2, \ +5, \ -5.$$

9. Solve the equation $x^3 - 3x^2 - 61x + 63 = 0$.

$$\begin{array}{l}
x^3 - 3x^2 - 61x + 63 = 0 \\
\quad\ \underline{\ -1 \quad -63\ } \\
\quad\ -2 \quad +\ 2 \qquad\qquad 2 \div (-2) = -1.
\end{array}$$

Hence $$\begin{aligned} x^3 - 3x^2 - 61x + 63 &= (x - 1)(x^2 - 2x - 63) \\ &= (x - 1)(x + 7)(x - 9), \end{aligned}$$

and $$x = 1, \ -7, \text{ or } 9.$$

10. Solve the equation $x^3 - 37x - 84 = 0$.

$$\begin{array}{l}
x^3 + 0x^2 - 37x - 84 = 0 \\
\quad\ \underline{\ \ 4 \quad -21\ } \\
\quad\ -4 \quad -16 \qquad\qquad -16 \div (-4) = 4.
\end{array}$$

Hence $$\begin{aligned} x^3 - 37x - 84 &= (x + 4)(x^2 - 4x - 21) \\ &= (x + 4)(x + 3)(x - 7), \end{aligned}$$

and $$x = -4, \ -3, \text{ or } 7.$$

Exercise CXXXVIII.

1. Solve the equation $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$.

$$m + p = -2,$$
$$n + mp + q = -13,$$
$$np + mq = 38,$$
$$nq = -24.$$

By trial it is found that we may take

$$n = 2, \ q = -12, \ m = -3, \ p = 1.$$

Hence
$$x^4 - 2x^3 - 13x^2 + 38x - 24$$
$$= (x^2 - 3x + 2)(x^2 + x - 12)$$
$$= (x - 1)(x - 2)(x - 3)(x + 4),$$

and
$$x = 1, \ 2, \ 3, \text{ or } -4.$$

2. Solve the equation $x^4 - 5x^3 - 2x^2 + 12x + 8 = 0$.

$$x^4 - 5x^3 - 2x^2 + 12x + 8$$
$$= (x^2 - x - 2)(x^2 - 4x - 4)$$
$$= (x + 1)(x - 2)(x^2 - 4x - 4).$$

Hence
$$x = -1, \ 2, \text{ or } 2 \pm 2\sqrt{2}.$$

3. Solve the equation $x^4 - 4x^3 - 8x + 32 = 0$.

$$x^4 - 4x^3 - 8x + 32$$
$$= (x^2 - 6x + 8)(x^2 + 2x + 4)$$
$$= (x - 2)(x - 4)(x^2 + 2x + 4).$$

Hence
$$x = 2, \ 4, \text{ or } -1 \pm \sqrt{-3}.$$

4. Solve the equation $x^4 - 12x^3 + 50x^2 - 84x + 49 = 0$.

$$x^4 - 12x^3 + 50x^2 - 84x + 49$$
$$= (x^2 - 6x + 7)(x^2 - 6x + 7).$$

Hence
$$x = 3 \pm \sqrt{2}, \ 3 \pm \sqrt{2}.$$

5. Solve the equation $x^4 - 11x^2 + 18x - 8 = 0$.

$$x^4 - 11x^2 + 18x - 8$$
$$= x^2 - 2x + 1)(x^2 + 2x - 8)$$
$$= (x - 1)(x - 1)(x - 2)(x + 4).$$

Hence
$$x = 1, \ 1, \ 2, \text{ or } -4.$$

6. Solve the equation $x^4 - 10x^2 - 20x - 16 = 0$.

$$\begin{aligned} x^4 - 10x^2 - 20x - 16 \\ &= (x^2 - 2x - 8)(x^2 + 2x + 2) \\ &= (x+2)(x-4)(x^2+2x+2). \end{aligned}$$

Hence $\qquad x = -2,\ 4,\ \text{or}\ -1 \pm \sqrt{-1}.$

7. Solve the equation $x^4 - 7x^3 + 23x^2 - 47x + 42 = 0$.

$$\begin{aligned} x^4 - 7x^3 + 23x^2 - 47x + 42 \\ &= (x^2 - 5x + 6)(x^2 - 2x + 7) \\ &= (x-2)(x-3)(x^2-2x+7). \end{aligned}$$

Hence $\qquad x = 2,\ 3,\ \text{or}\ 1 \pm \sqrt{-6}.$

8. Solve the equation $x^4 + 2x^3 - 9x^2 - 8x + 20 = 0$.

$$\begin{aligned} x^4 + 2x^3 - 9x^2 - 8x + 20 \\ &= (x^2 - 4)(x^2 + 2x - 5) \\ &= (x-2)(x+2)(x^2+2x-5). \end{aligned}$$

Hence $\qquad x = 2,\ -2,\ \text{or}\ -1 \pm \sqrt{6}.$

9. Solve the equation $x^4 - 4x^3 - 102x^2 - 188x - 91 = 0$.

$$\begin{aligned} x^4 - 4x^3 - 102x^2 - 188x - 91 \\ &= (x^2 + 8x + 7)(x^2 - 12x - 13) \\ &= (x+1)(x+7)(x+1)(x-13) \end{aligned}$$

Hence $\qquad x = -1,\ -1,\ -7,\ \text{or}\ 13.$

10. Solve the equation $x^4 - 11x^3 + 46x^2 - 117x + 45 = 0$.

$$\begin{aligned} x^4 - 11x^3 + 46x^2 - 117x + 45 \\ &= (x^2 - 7x + 3)(x^2 - 7x + 15). \end{aligned}$$

Hence $\qquad x = \dfrac{7 \pm \sqrt{37}}{2},\ \text{or}\ \dfrac{2 \pm \sqrt{-11}}{2}.$

Exercise CXXXIX.

1. Determine the signs of the roots of the equation $x^4 + 4x^3 - 43x^2 - 58x + 240 = 0$, all the roots being real.

$$+ \quad + \quad - \quad - \quad +$$

No. of variations, 2.
No. of permanences, 2.

Hence there are two positive and two negative roots.

2. Determine the signs of the roots of the equation $x^3 - 22x^2 + 155x - 350 = 0$, all the roots being real.

$$+ \quad - \quad + \quad -$$

No. of variations, 3.
No. of permanences, 0.

Hence there are 3 positive roots.

3. Determine the signs of the roots of the equation $x^4 + 4x^3 - 35x^2 - 78x + 360 = 0$, all the roots being real.

$$+ \quad + \quad - \quad - \quad +$$

No. of variations, 2.
No. of permanences, 2.

Hence there are 2 positive and 2 negative roots.

4. Determine the signs of the roots of the equation $x^3 - 12x^2 - 43x - 30 = 0$, all the roots being real.

$$+ \quad - \quad - \quad -$$

No. of variations, 1.
No. of permanences, 2.

Hence there is 1 positive and 2 negative roots.

5. Determine the signs of the roots of the equation $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$, all the roots being real.

$$+ \quad - \quad - \quad + \quad + \quad -$$

No. of variations, 3.
No. of permanences, 2.

Hence there are 3 positive and 2 negative roots.

6. Determine the signs of the roots of the equation $x^3 + 12x^2 + 47x - 60 = 0$, all the roots being real.

$$+ \quad - \quad + \quad -$$

No. of variations, 3.
No. of permanences, 0.

Hence there are 3 positive roots.

7. Determine the signs of the roots of the equation $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$, all the roots being real.

$$+ \quad - \quad - \quad + \quad -$$

No. of variations, 3.
No. of permanences, 1.

Hence there are 3 positive and 1 negative roots.

8. Determine the signs of the roots of the equation $x^5 - x^4 - 187x^3 - 359x^3 + 186x + 330 = 0$, all the roots being real.

$$+ \quad - \quad - \quad - \quad + \quad +$$

No. of variations, 2.
No. of permanences, 3.

Hence there are 2 positive and 3 negative roots.

9. Determine the signs of the roots of the equation $x^6 - 10x^5 + 19x^4 + 110x^3 - 536x^2 + 800x - 384 = 0$, all the roots being real.

$$+ \quad - \quad + \quad + \quad - \quad + \quad -$$

No. of variations, 5.
No. of permanences, 1.

Hence there are 5 positive and 1 negative roots.

10. Determine the signs of the roots of the equation $x^7 - 10x^6 + 22x^5 + 32x^4 - 131x^3 + 50x^2 + 108x - 72 = 0$.

$$+ \quad - \quad + \quad + \quad - \quad + \quad + \quad -$$

No. of variations, 5.
No. of permanences, 2.

Hence there are 5 positive and 2 negative roots.

Exercise CXL.

1. Find the successive derivatives of the polynomial $x^2 + 2x + 3$.

$$F\ (x) = x^2 + 2x + 3,$$
$$F^{\text{I}}\ (x) = 2x + 2,$$
$$F^{\text{II}}\ (x) = 2,$$
$$F^{\text{III}}(x) = 0.$$

2. Find the successive derivatives of the polynomial $x^3 - 3x^2 + 7x + 25$.

$$F\ (x) = x^3 - 3x^2 + 7x + 25,$$
$$F^{\text{I}}\ (x) = 3x^2 - 6x + 7,$$
$$F^{\text{II}}\ (x) = 6x - 6,$$
$$F^{\text{III}}(x) = 6,$$
$$F^{\text{IV}}\ (x) = 0.$$

3. Find the successive derivatives of the polynomial $x^4 + 2x^3 - 5x^2 + 64$.

$$F\ (x) = x^4 + 2x^3 - 5x^2 + 64,$$
$$F^{\text{I}}\ (x) = 4x^3 + 6x^2 - 10x,$$
$$F^{\text{II}}\ (x) = 12x^2 + 12x,$$
$$F^{\text{III}}(x) = 24x + 12,$$
$$F^{\text{IV}}\ (x) = 24,$$
$$F^{\text{V}}\ (x) = 0.$$

4. Find the successive derivatives of the polynomial $x^5 + x^4 - 6x^3 + 3x^2 - 4x + 27$.

$$\begin{aligned}
F\ (x) &= x^5 + x^4 - 6x^3 + 3x^2 - 4x + 27,\\
F^{\text{I}}\ (x) &= 5x^4 + 4x^3 - 18x^2 + 6x - 4,\\
F^{\text{II}}\ (x) &= 20x^3 + 12x^2 - 36x + 6,\\
F^{\text{III}}(x) &= 60x^2 + 24x - 36,\\
F^{\text{IV}}\ (x) &= 120x + 24,\\
F^{\text{V}}\ (x) &= 120,\\
F^{\text{VI}}\ (x) &= 0.
\end{aligned}$$

5. Find the successive derivatives of the polynomial $x^4 - 3ax^3 + 6bx^2 - 9cx + mn$.

$$\begin{aligned}
F\ (x) &= x^4 - 3ax^3 + 6bx^2 - 9cx + mn,\\
F^{\text{I}}\ (x) &= 4x^3 - 9ax^2 + 12bx - 9c,\\
F^{\text{II}}\ (x) &= 12x^2 - 18ax + 12b,\\
F^{\text{III}}(x) &= 24x - 18a,\\
F^{\text{IV}}\ (x) &= 24,\\
F^{\text{V}}\ (x) &= 0.
\end{aligned}$$

Exercise CXLI.

1. Find all the roots of $x^3 - 8x^2 + 13x - 6 = 0$.

$$\begin{aligned}
F\ (x) &= x^3 - 8x^2 + 13x - 6,\\
F^{\text{I}}(x) &= 3x^2 - 16x + 13\\
&= (3x - 13)(x - 1).\\
\phi(x) &= x - 1.\\
\therefore F(x) &= (x - 1)^2(x - 6).
\end{aligned}$$

The roots are 1, 1, and 6.

2. Find all the roots of $x^3 - 7x^2 + 16x - 12 = 0$.

$$\begin{aligned}
F\ (x) &= x^3 - 7x^2 + 16x - 12,\\
F^{\text{I}}(x) &= 3x^2 - 14x + 16\\
&= (3x - 8)(x - 2).\\
\phi(x) &= x - 2.\\
\therefore F(x) &= (x - 2)^2(x - 3).
\end{aligned}$$

The roots are 2, 2, and 3.

3. Find all the roots of $x^4 - 6x^2 - 8x - 3 = 0$.

$$F(x) = x^4 - 6x^2 - 8x - 3,$$
$$F^1(x) = 4x^3 - 12x - 8.$$

$$\begin{array}{r|r|l}
4 + 0 - 12 - 8 & 1 + 0 - 6 - 8 - 3 & \\
4 + 8 + 4 & 4 & \\
-8 - 16 - 8 & 4 + 0 - 24 - 32 - 12 & x \\
-8 - 16 - 8 & 4 + 0 - 12 - 8 & \\
 & 12)\ -12 - 24 - 12 & \\
 & -1 - 2 - 1 & -4x - 8
\end{array}$$

$$\therefore \phi(x) = x^2 + 2x + 1$$
$$= (x + 1)^2.$$
$$\therefore F(x) = (x + 1)^3 (x - 3).$$

The roots are -1, -1, -1, and 3.

4. Find all the roots of $x^3 - 2x^2 - 15x + 36 = 0$.

$$F(x) = x^3 - 2x^2 - 15x + 36,$$
$$F^1(x) = 3x^2 - 4x - 15$$
$$= (3x + 5)(x - 3).$$
$$\phi(x) = x - 3.$$
$$\therefore F(x) = (x - 3)^2 (x + 4).$$

The roots are 3, 3, and -4.

5. Find all the roots of $x^4 - 7x^3 + 9x^2 + 27x - 54 = 0$.

$$F(x) = x^4 - 7x^3 + 9x^2 + 27x - 54,$$
$$F^1(x) = 4x^3 - 21x^2 + 18x + 27.$$

$$\begin{array}{r|r|l}
4 - 21 + 18 + 27 & 1 - 7 + 9 + 27 - 54 & \\
4 - 24 + 36 & 4 & \\
3 - 18 + 27 & 4 - 28 + 36 + 108 - 216 & x \\
3 - 18 + 27 & 4 - 21 + 18 + 27 & \\
 & -7 + 18 + 81 - 216 & \\
 & 4 & \\
 & -28 + 72 + 324 - 864 & -7 \\
 & -28 + 147 - 126 - 189 & \\
 & -75)\ -75 + 450 - 675 & \\
 & 1 - 6 + 9 & 4x + 3
\end{array}$$

$$\therefore \phi(x) = x^2 - 6x + 9$$
$$= (x - 3)^2.$$
$$\therefore F(x) = (x - 3)^3 (x + 2).$$

The roots are 3, 3, 3, and -2.

6. Find all the roots of $x^4 - 24x^2 + 64x - 48 = 0$.

$$F(x) = x^4 - 24x^2 + 64x - 48,$$
$$F^1(x) = 4x^3 - 48x + 64.$$

$$
\begin{array}{r|r|l}
4)\,4+0-48+64 & 1-0-14+64-48 & x \\
1+0-12-16 & 1+0-12+16 & \\
1-4+4 & -12)\,-12+48-48 & \\
4-16+16 & 1-\ \ 4+\ \ 4 & x+4 \\
4-16+16 & &
\end{array}
$$

$$\therefore \phi(x) = x^2 - 4x + 4$$
$$= (x-2)^2.$$
$$\therefore F(x) = (x-2)^3(x+6).$$

The roots are 2, 2, 2, and -6.

7. Find all the roots of $x^4 - 10x^3 + 24x^2 + 10x - 25 = 0$.

$$F(x) = x^4 - 10x^3 + 24x^2 + 10x - 25,$$
$$F^1(x) = 4x^3 - 30x^2 + 48x + 10.$$

$$
\begin{array}{r|r|l}
4-30+48+10 & 1-10+24+10-25 & \\
9 & 4 & \\
36-270+432+90 & 4-40+96+40-100 & x \\
36-200+100 & 4-30+48+10 & \\
-70+332+90 & -10+48+30-100 & \\
9 & 2 & \\
-630+2988+810 & -20+96+60-200 & -5 \\
-630+3500-1750 & -20+150-240-50 & \\
-512)\,-512+2560 & -6)\,-54+300-150 & \\
1-5 & 9-50+25 & 4x-70
\end{array}
$$

$$\therefore \phi(x) = x - 5,$$
$$F(x) = (x-5)^2(x^2-1).$$

The roots are 5, 5, 1, and -1.

8. Find all the roots of $x^5 - 11x^4 + 19x^3 + 115x^2 - 200x - 500 = 0$.

$$F\ (x) = x^5 - 11x^4 + 19x^3 + 115x^2 - 200x - 500,$$

$$F^1(x) = 5x^4 - 44x^3 + 57x^2 + 230x - 200.$$

$$\begin{array}{r|r|l}
5 - 44 + 57 + 230 - 200 & 1 - 11 + 19 + 115 - 200 - 500 & \\
5 - 40 + 25 + 250 & 5 \qquad\qquad\qquad\qquad\qquad\qquad & \\
\hline
- 4 + 32 - 20 - 200 & 5 - 55 + 95 + 575 - 1000 - 2500 & x \\
- 4 + 32 - 20 - 200 & 5 - 44 + 57 + 230 - 200 \qquad & \\
\hline
 & - 11 + 38 + 345 - 800 - 2500 & \\
 & 5 \qquad\qquad\qquad\qquad\qquad & \\
 & - 55 + 190 + 1725 - 4000 - 12500 & - 11 \\
 & - 55 + 484 - 627 - 2530 + 2200 & \\
 & -294) - 294 + 2352 - 1470 - 14700 & \\
 & 1 - 8 + 5 + 50 & 5x - 4
\end{array}$$

$$\therefore \phi\ (x) = x^3 - 8x^2 + 5x + 50,$$

$$\phi'(x) = 3x^2 - 16x + 5.$$

$$\begin{array}{r|r|l}
3 - 16 + 5 & 1 - 8 + 5 + 50 & \\
3 - 15 & 3 \qquad\qquad\qquad & \\
\hline
- 1 + 5 & 3 - 24 + 15 + 150 & x \\
- 1 + 5 & 3 - 16 + 5 \qquad & \\
\hline
 & 2 - 8 + 10 + 150 & \\
 & 3 \qquad\qquad\quad & \\
 & - 24 + 30 + 450 & - 8 \\
 & - 24 + 128 - 40 & \\
 & - 48) - 98 - 490 & \\
 & 1 - 5 & 3x - 1
\end{array}$$

$\therefore$ H.C.F. of $\phi\ (x)$ and $\phi'(x) = x - 5$.

$$\phi\ (x) = (x - 5)^2(x + 2),$$

$$\therefore F\ (x) = (x - 5)^3\ (x + 2)^2.$$

The roots are 5, 5, 5, -2, and -2.

9. Find all the roots of $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0$.

$$F\ (x) = x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3,$$
$$F^{\text{I}}(x) = 5x^4 - 8x^3 + 9x^2 - 14x + 8,$$
$$F^{\text{II}}(x) = 20x^3 - 24x^2 + 18x - 14.$$

Find H.C.F. of $F^{\text{I}}(x)$ and $F^{\text{II}}(x)$.

$2)20 - 24 + 18 - 14$	$5 - 8 + 9 - 14 + 8$	
$10 - 12 + 9 - 7$	2	
7	$10 - 16 + 18 - 28 + 16$	x
$70 - 84 + 63 - 49$	$10 - 12 + 9 - 7$	
$70 - 280 + 220$	$-4 + 9 - 21 + 16$	
$206 - 157 - 49$	5	
7	$-20 + 45 - 105 + 80$	-2
$1442 - 1099 - 343$	$-20 + 24 - 18 + 14$	
$1442 - 5974 + 4532$	$3)21 - 87 + 66$	
$4875)4875 - 4875$	$7 - 29 + 22$	$10x - 206$
$1 - 1$	$7 - 7$	$7x - 22$
	$-22 + 22$	
	$-22 + 22$	

$\therefore$ H.C.F. of $F^{\text{I}}(x)$ and $F^{\text{II}}(x) = x - 1$.
Hence $F^{\text{I}}(x)$ contains $(x-1)^2$ as a factor.
But 1 is a root of $F(x) = 0$.

$$\therefore F(x) = (x-1)^3(x^2 + x + 3).$$

The roots are 1, 1, 1, and $\dfrac{-1 \pm \sqrt{-11}}{2}$.

10. Find all the roots of $x^4 + 6x^3 + x^2 - 24x + 16 = 0$.

$$F\ (x) = x^4 + 6x^3 + x^2 - 24x + 16.$$
$$F^{\text{I}}(x) = 4x^3 + 18x^2 + 2x - 24.$$

$2)4 + 18 + 2 - 24$	$1 + 6 + 1 - 24 + 16$	
$2 + 9 + 1 - 12$	2	
$2 + 6 - 8$	$2 + 12 + 2 - 48 + 32$	x
$3 + 9 - 12$	$2 + 9 + 1 - 12$	
$3 + 9 - 12$	$3 + 1 - 36 + 32$	
	2	
	$6 + 2 - 72 + 64$	3
	$6 + 27 + 3 - 36$	
	$-25)-25 - 75 + 100$	
	$1 + 3 - 4$	$2x + 3$

$$\therefore \phi(x) = x^2 + 3x - 4$$
$$= (x-1)(x+4).$$
$$\therefore F(x) = (x-1)^2(x+4)^2.$$

The roots are 1, 1, -4 and -4.

Exercise CXLII.

1. Put the equation $2x^3 + \frac{2}{3}x^2 - x + \frac{1}{6} = 0$ in the form $f(x) = 0$.

Put $\frac{x}{6}$ for x;

then the equation becomes

$$2x^3 + 4x^2 - 36x + 36 = 0,$$

or

$$x^3 + 2x^2 - 18x + 18 = 0.$$

2. Put the equation $3x^3 + 5x^2 - \frac{7}{2}x - 8 = 0$ in the form $f(x) = 0$.

Put $\frac{x}{6}$ for x;

then the equation becomes

$$3x^3 + 30x^2 - 126x - 1728 = 0,$$

or

$$x^3 + 10x^2 - 42x - 576 = 0.$$

3. Put the equation $5x^4 - x^3 - \frac{15}{2}x^2 - \frac{10}{3}x + 1 = 0$ in the form $f(x) = 0$.

Put $\frac{x}{30}$ for x;

then the equation becomes

$$5x^4 - 30x^3 - 6750x^2 - 90000x + 30^4 = 0,$$

or

$$x^4 - 6x^3 - 1350x^2 - 18000x + 162000 = 0.$$

4. Put the equation $x^5 + \frac{1}{2}x^4 + \frac{2}{3}x^3 - \frac{1}{3}x^2 + x - 3 = 0$ in the form $f(x) = 0$.

Put $\frac{x}{6}$ for x;

then the equation becomes

$$x^5 + 3x^4 + 24x^3 - 72x^2 + 1296x - 23{,}328 = 0.$$

5. Put the equation $x^4 - 2x^2 + \frac{1}{2}x - 14 = 0$ in the form $f(x) = 0$.

Put $\frac{x}{2}$ for x;

then the equation becomes

$$x^4 - 8x^2 + 4x - 224 = 0.$$

EXERCISE CXLIII.

1. Diminish the roots of the equation $x^3 - 11x^2 + 31x - 12 = 0$ by 1.

$$\begin{array}{l}
1 - 11 + 31 - 12 \,\lfloor 1 \\
\quad + 1 - 10 + 21 \\
\hline
1 - 10 + 21 + 9 \\
\quad + 1 - 9 \\
\hline
1 - 9 + 12 \\
\quad + 1 \\
\hline
1 - 8
\end{array}$$

The required equation is
$$y^3 - 8y^2 + 12y + 9 = 0.$$

2. Diminish the roots of the equation $x^4 - 6x^3 + 4x^2 + 18x - 5 = 0$ by 2.

$$\begin{array}{l}
1 - 6 + 4 + 18 - 5 \,\lfloor 2 \\
\quad + 2 - 8 - 8 + 20 \\
\hline
1 - 4 - 4 + 10 + 15 \\
\quad + 2 - 4 - 16 \\
\hline
1 - 2 - 8 - 6 \\
\quad + 2 + 0 \\
\hline
1 + 0 - 8 \\
\quad + 2 \\
\hline
1 + 2
\end{array}$$

The required equation is
$$y^4 + 2y^3 - 8y^2 - 6y + 15 = 0.$$

3. Diminish the roots of the equation $x^3 + 10x^2 + 13x - 24 = 0$ by -2.

$$\begin{array}{l}
1 + 10 + 13 - 24 \,\lfloor -2 \\
\quad - 2 - 16 + 6 \\
\hline
1 + 8 - 3 - 18 \\
\quad - 2 - 12 \\
\hline
1 + 6 - 15 \\
\quad - 2 \\
\hline
1 + 4
\end{array}$$

The required equation is
$$y^3 + 4y^2 - 15y - 18 = 0.$$

4. Diminish the roots of the equation $x^3 - 9x^2 + 22x - 12 = 0$ by 3.

$$\begin{array}{l}
1 - 9 + 22 - 12 \,\lfloor 3 \\
\quad + 3 - 18 + 12 \\
\hline
1 - 6 + 4 + 0 \\
\quad + 3 - 9 \\
\hline
1 - 3 - 5 \\
\quad + 3 \\
\hline
1 + 0
\end{array}$$

The required equation is
$$y^3 - 5y = 0.$$

5. Diminish the roots of the equation $x^4 + x^3 - 16x^2 - 4x + 48 = 0$ by 4.

$$\begin{array}{l}
1 + 1 - 16 - 4 + 48 \,\lfloor 4 \\
\quad + 4 + 20 + 16 + 48 \\
\hline
1 + 5 + 4 + 12 + 96 \\
\quad + 4 + 36 + 160 \\
\hline
1 + 9 + 40 + 172 \\
\quad + 4 + 52 \\
\hline
1 + 13 + 92 \\
\quad + 4 \\
\hline
1 + 17
\end{array}$$

The required equation is $y^4 + 17y^3 + 92y^2 + 172y + 96 = 0$.

6. Diminish the roots of the equation $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0$ by 0.7.

$$\begin{array}{llllll}
1 & +\ 2 & -\ 25 & -\ 26 & +\ 120 & \underline{|0.7} \\
 & +\ 0.7 & +\ 1.89 & -\ 16.177 & -\ 29.5239 & \\ \hline
1 & +\ 2.7 & -\ 23.11 & -\ 42.177 & +\ 90.4761 & \\
 & +\ 0.7 & +\ 2.38 & -\ 14.511 & & \\ \hline
1 & +\ 3.4 & -\ 20.73 & -\ 56.688 & & \\
 & +\ 0.7 & +\ 2.87 & & & \\ \hline
1 & +\ 4.1 & -\ 17.86 & & & \\
 & +\ 0.7 & & & & \\ \hline
1 & +\ 4.8 & & & &
\end{array}$$

The required equation is

$$y^4 + 4.8y^3 - 17.86y^2 - 56.688y + 90.4761 = 0.$$

7. Diminish the roots of the equation $x^4 - x^2 - 3x + 4 = 0$ by 0.3.

$$\begin{array}{llllll}
1 & +\ 0 & +\ 1 & -\ 3 & +\ 4 & \underline{|0.3} \\
 & +\ 0.3 & +\ 0.09 & +\ 0.327 & -\ 0.8019 & \\ \hline
1 & +\ 0.3 & +\ 1.09 & -\ 2.673 & +\ 3.1981 & \\
 & +\ 0.3 & +\ 0.18 & +\ 0.381 & & \\ \hline
1 & +\ 0.6 & +\ 1.27 & -\ 2.292 & & \\
 & +\ 0.3 & +\ 0.27 & & & \\ \hline
1 & +\ 0.9 & +\ 1.54 & & & \\
 & +\ 0.3 & & & & \\ \hline
1 & +\ 1.2 & & & &
\end{array}$$

The required equation is

$$y^4 + 1.2y^3 + 1.54y^2 - 2.292y + 3.1981 = 0.$$

8. Diminish the roots of the equation $x^5 + x^4 + 3x^2 - 2x - 16 = 0$ by 0.5.

$$\begin{array}{lllllll}
1 & +\ 1 & +\ 0 & +\ 3 & -\ 2 & -\ 16 & \underline{|0.5} \\
 & +\ 0.5 & +\ 0.75 & +\ 0.375 & +\ 1.6875 & -\ 0.15625 & \\ \hline
1 & +\ 1.5 & +\ 0.75 & +\ 3.375 & -\ 0.3125 & -\ 16.15625 & \\
 & 0.5 & +\ 1 & +\ 0.875 & +\ 2.125 & & \\ \hline
1 & +\ 2 & +\ 1.75 & +\ 4.25 & +\ 1.8125 & & \\
 & +\ 0.5 & +\ 1.25 & +\ 1.50 & & & \\ \hline
1 & +\ 2.5 & +\ 3 & +\ 5.75 & & & \\
 & +\ 0.5 & +\ 1.5 & & & & \\ \hline
1 & +\ 3 & +\ 4.5 & & & & \\
 & +\ 0.5 & & & & & \\ \hline
1 & +\ 3.5 & & & & &
\end{array}$$

The required equation is

$$y^5 + 3.5y^4 + 4.5y^3 + 5.75y^2 + 1.8125y - 16.15625 = 0.$$

9. Diminish the roots of the equation

$$x^5 - 3x^4 - 2x^3 + 3x^2 - 7x + 12 = 0 \text{ by } -1.$$

$$\begin{array}{rrrrrrl}
1 & -3 & -2 & +3 & -7 & +12 & \underline{|-1} \\
 & -1 & +4 & -2 & -1 & +8 & \\ \cline{1-6}
1 & -4 & +2 & +1 & -8 & +20 & \\
 & -1 & +5 & -7 & +6 & & \\ \cline{1-5}
1 & -5 & +7 & -6 & -2 & & \\
 & -1 & +6 & -13 & & & \\ \cline{1-4}
1 & -6 & +13 & -19 & & & \\
 & -1 & +7 & & & & \\ \cline{1-3}
1 & -7 & +20 & & & & \\
 & -1 & & & & & \\ \cline{1-2}
1 & -8 & & & & &
\end{array}$$

The required equation is $y^5 - 8y^4 + 20y^3 - 19y^2 - 2y + 20 = 0.$

10. Diminish the roots of the equation

$$x^6 - x^5 + 2x^4 - 3x^3 + 4x^2 - 5x + 6 = 0 \text{ by } 0.2.$$

$$\begin{array}{rrrrrrrl}
1 & -1 & +2 & -3 & +4 & -5 & +6 & \underline{|0.2} \\
 & +0.2 & -0.16 & +0.368 & -0.5264 & +0.69472 & -0.861056 & \\ \cline{1-7}
1 & -0.8 & +1.84 & -2.632 & +3.4736 & -4.30528 & +5.138944 & \\
 & +0.2 & -0.12 & +0.344 & -0.4576 & +0.60320 & & \\ \cline{1-6}
1 & -0.6 & +1.72 & -2.288 & +3.0160 & -3.70208 & & \\
 & +0.2 & -0.08 & +0.328 & -0.3920 & & & \\ \cline{1-5}
1 & -0.4 & +1.64 & -1.960 & +2.624 & & & \\
 & +0.2 & -0.04 & +0.320 & & & & \\ \cline{1-4}
1 & -0.2 & +1.6 & -1.64 & & & & \\
 & +0.2 & +0 & & & & & \\ \cline{1-3}
1 & +0 & +1.6 & & & & & \\
 & +0.2 & & & & & & \\ \cline{1-2}
1 & +0.2 & & & & & &
\end{array}$$

The required equation is

$$y^6 + 0.2y^5 + 1.6y^4 - 1.64y^3 + 2.624y^2 - 3.70208y + 5.138944 = 0.$$

Exercise CXLIV.

1. Find the two commensurable roots of the equation
$$x^4 - 4x^3 - 8x + 32 = 0.$$

Try 4; then 2.

$$\begin{array}{l}
1 - 4 + 0 - 8 + 32 \\
\;\;\; + 4 + 0 + 0 - 32 \\
\hline
1 + 0 + 0 - 8 \quad 0 \\
\;\;\; + 2 + 4 + 8 \\
\hline
1 + 2 + 4 \quad 0
\end{array}$$

The commensurable roots are 2 and 4.

2. Find the one commensurable root of the equation
$$x^3 - 6x^2 + 10x - 8 = 0.$$

Try 4.

$$\begin{array}{l}
1 - 6 + 10 - 8 \\
\;\;\; + 4 - \;\; 8 + 8 \\
\hline
1 - 2 + \;\; 2 \quad 0
\end{array}$$

The commensurable root is 4.

3. Find the four commensurable roots of the equation
$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$$

Try 1; then 2; then -2; then -3.

$$\begin{array}{l}
1 + 2 - \;\; 7 - \;\; 8 + 12 \\
\;\;\; + 1 + \;\; 3 - \;\; 4 - 12 \\
\hline
1 + 3 - \;\; 4 - 12 \quad 0 \\
\;\;\; + 2 + 10 + 12 \\
\hline
1 + 5 + \;\; 6 \quad 0 \\
\;\;\; - 2 - \;\; 6 \\
\hline
1 + 3 \quad 0 \\
\;\;\; - 3 \\
\hline
1 \quad 0
\end{array}$$

The roots are 1, 2, -2, and -3,

4. Find the one commensurable root of the equation
$$x^3 + 3x^2 - 30x + 36 = 0.$$

Try 3.

$$\begin{array}{l}
1 + 3 - 30 + 36 \\
\;\;\; + 3 + 12 - 36 \\
\hline
1 + 6 - 18 \quad 0
\end{array}$$

The commensurable root is 3.

5. Find the two commensurable roots of the equation
$$x^4 - 12x^3 + 32x^2 + 27x - 18 = 0.$$

Try 6; then -1.

$$\begin{array}{l}
1 - 12 + 32 + 27 - 18 \\
\;\;\; + \;\; 6 - 36 - 24 + 18 \\
\hline
1 - \;\; 6 - \;\; 4 + \;\; 3 \quad 0 \\
\;\;\; - \;\; 1 + \;\; 7 - \;\; 3 \\
\hline
1 - \;\; 7 + \;\; 3 \quad 0
\end{array}$$

The commensurable roots are 6 and -1.

6. Find the two commensurable roots of the equation
$$x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$$

Try 4; then 5.

$$\begin{array}{l}
1 - 9 + 17 + 27 - 60 \\
\;\;\; + 4 - 20 - 12 + 60 \\
\hline
1 - 5 - \;\; 3 + 15 \quad 0 \\
\;\;\; + 5 + \;\; 0 - 15 \\
\hline
1 + 0 - \;\; 3 \quad 0
\end{array}$$

The commensurable roots are 4 and 5.

7. Find the five commensurable roots of the equation

$$x^5 - 5x^4 + 3x^3 + 17x^2 - 28x + 12 = 0.$$

Try 1; then 1; then 2; then 3; then -2.

$$\begin{array}{l}
1 - 5 + 3 + 17 - 28 + 12 \\
\quad + 1 - 4 - 1 + 16 - 12 \\
\hline
1 - 4 - 1 + 16 - 12 \quad 0 \\
\quad + 1 - 3 - 4 + 12 \\
\hline
1 - 3 - 4 + 12 \quad 0 \\
\quad + 2 - 2 - 12 \\
\hline
1 - 1 - 6 \quad 0 \\
\quad + 3 + 6 \\
\hline
1 + 2 \quad 0 \\
\quad - 2 \\
\hline
1 \quad 0
\end{array}$$

The roots are 1, 1, 2, 3, and -2.

8. Find the four commensurable roots of the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

Try 1; then 2; then 3; then 4.

$$\begin{array}{l}
1 - 10 + 35 - 50 + 24 \\
\quad + 1 - 9 + 26 - 24 \\
\hline
1 - 9 + 26 - 24 \quad 0 \\
\quad + 2 - 14 + 24 \\
\hline
1 - 7 + 12 \quad 0 \\
\quad + 3 - 12 \\
\hline
1 - 4 \quad 0 \\
\quad + 4 \\
\hline
1 \quad 0
\end{array}$$

The roots are 1, 2, 3, and 4.

9. Find the three commensurable roots of the equation

$$x^5 - 8x^4 + 11x^3 + 29x^2 - 36x - 45 = 0.$$

Try 3; then 5; then -1.

$$\begin{array}{l}
1 - 8 + 11 + 29 - 36 - 45 \\
\quad + 3 - 15 - 12 + 51 + 45 \\
\hline
1 - 5 - 4 + 17 + 15 \quad 0 \\
\quad + 5 + 0 - 20 - 15 \\
\hline
1 + 0 - 4 - 3 \quad 0 \\
\quad - 1 + 1 + 3 \\
\hline
1 - 1 - 3 \quad 0
\end{array}$$

The commensurable roots are 3, 5, and -1.

10. Find the one commensurable root of the equation

$$x^5 - x^4 - 6x^3 + 9x^2 + x - 4.$$

Try 1.
$$\begin{array}{l} 1-1-6+9+1-4 \\ \underline{\quad +1+0-6+3+4} \\ 1+0-6+3+4 \quad 0 \end{array}$$

The commensurable root is 1.

Exercise CXLV.

1. Compute the value of $x^4 - 5x^3 + 26x^2 - 4x + 7$ when $x = 5$.

$$\begin{array}{l} -5+26-\quad 4+\quad 7\,\underline{|5} \\ \underline{+5+\ \ 0+130+630} \\ +0+26+126+637 \end{array}$$

The required value is 637.

2. Compute the value of $x^3 - 4x^2 + 5x - 22$ when $x = -7$.

$$\begin{array}{l} -\quad 4+\quad 5-\quad 22\,\underline{|-7} \\ \underline{-\quad 7+77-574} \\ -11+82-596 \end{array}$$

The required value is -596.

3. Compute the value of $x^5 - 2x^4 + 3x^3 + x^2 - 28$ when $x = 2$.

$$\begin{array}{l} -2+3+1+\ \ 0-28\,\underline{|2} \\ \underline{+2+0+6+14+28} \\ \quad 0+3+7+14+0 \end{array}$$

The required value is 0.

4. Compute the value of $x^5 + 7x^3 - 2x^2 - 49$ when $x = -3$.

$$\begin{array}{l} \quad 0+\quad 7-\quad 2+\quad 0-\quad 49\,\underline{|-3} \\ \underline{-3+\ \ 9-48+150-450} \\ -3+16-50+150-499 \end{array}$$

The required value is -499.

5. Compute the value of $x^5 - 14x^3 + 473$ when $x = 6$.

$$\begin{array}{l} \quad 0-14+\quad 0+\quad 0+\ 473\,\underline{|6} \\ \underline{+6+36+132+792+4752} \\ +6+22+132+792+5225 \end{array}$$

The required value is 5225.

6.. Compute the value of $x^6 - 2x^5 + 3x^4 + 2x^3 + x^2 - 7x - 96$ when $x = -2$.

$$\begin{array}{l|l}
-2 + \ 3 + \ 2 + \ 1 - \ 7 - \ 96 & \underline{-2} \\
-2 + \ 8 - 22 + 40 - 82 + 178 & \\
\hline
-4 + 11 - 20 + 41 - 89 + \ 82 &
\end{array}$$

The required value is 82.

7. Compute the value of $x^6 - x^5 - 2x^4 + x^3 - 6x + 14$ when $x = 3$.

$$\begin{array}{l|l}
-1 - 2 + \ 1 + \ 0 - \ 6 + \ 14 & \underline{3} \\
+3 + 6 + 12 + 39 + 117 + 333 & \\
\hline
+2 + 4 + 13 + 39 + 111 + 347 &
\end{array}$$

The required value is 347.

8. Compute the value of $x^5 - 4x^4 + 2x^2 - 7x + 16$ when $x = 10$.

$$\begin{array}{l|l}
- \ 4 + \ 0 + \ 2 - \ 7 + \ 16 & \underline{10} \\
+10 + 60 + 600 + 6020 + 60130 & \\
\hline
+ \ 6 + 60 + 602 + 6013 + 60146 &
\end{array}$$

The required value is 60,146.

9. Compute the value of $x^7 - x^6 - 2x^5 - 3x^4 + 2x^3 + x^2 - x + 4$ when $x = -2$.

$$\begin{array}{l|l}
-1 - 2 - \ 3 + \ 2 + \ 1 - \ 1 + \ 4 & \underline{-2} \\
-2 + 6 - \ 8 + 22 - 48 + 94 - 186 & \\
\hline
-3 + 4 - 11 + 24 - 47 + 93 - 182 &
\end{array}$$

The required value is -182.

10. Compute the value of $x^7 - 5x^5 + 6x^3 + 3x - 1$ when $x = 4$.

$$\begin{array}{l|l}
0 - \ 5 + \ 0 + \ 6 + \ 0 + \ 3 - \ 1 & \underline{4} \\
+4 + 16 + 44 + 176 + 728 + 2912 + 11660 & \\
\hline
+4 + 11 + 44 + 182 + 728 + 2915 + 11659 &
\end{array}$$

The required value is 11,659.

Exercise CXLVI.

1. Determine the first significant figure of each root of the equation

$$x^3 - x^2 - 2x + 1 = 0.$$

If $\quad x = -2, -1, 0, +0.4, +0.5, +1, +2,$

$f(x) = -7, +1, +1, +0.104, -0.125, -1, +1.$

Hence the roots are $-1.+$, $0.4+$, and $1.+$.

2. Determine the first significant figure of each root of the equation
$$x^3 - 5x - 3 = 0.$$

If
$$x = -2, -1 -0.7, -0.6, +0, +1, +2, +3,$$
$$f(x) = -1, +1, +0.157, -0.216, -3, -7, -5, +9.$$

Hence the roots are $-1.+$, $-0.6+$, and $2.+$.

3. Determine the first significant figure of each root of the equation
$$x^3 - 5x^2 + 7 = 0.$$

If
$$x = -2, -1, 0, +1, +2, +3, +4, +5,$$
$$f(x) = -21, +1, +7, +3, -5, -11, -9, +7.$$

Hence the roots are $-1.+$, $1.+$, and $4.+$.

4. Determine the first significant figure of each root of the equation
$$x^3 - 7x + 7 = 0.$$

If
$$x = -4, -3, +1, +1.3, +1.4, +1.6, +1.7,$$
$$f(x) = -29, +1, +1, +0.097, -0.056, -0.104, +0.013.$$

Hence the roots are $-3.+$, $1.3+$, and $1.6+$.

5. Determine the first significant figure of each root of the equation
$$x^3 + 2x^2 - 30x + 39 = 0.$$

If
$$x = -8, -7, +1, +2, +3, +4,$$
$$f(x) = -89, +4, +12, -5, -6, +15.$$

Hence the roots are $-7.+$, $1.+$, and $3.+$.

6. Determine the first significant figure of each root of the equation
$$x^3 - 6x^2 + 3x + 5 = 0.$$

If
$$x = -1, -0.7, -0.6, +1, +2, +5, +6,$$
$$f(x) = -5, -0.383, +0.824, +3, -5, -5, +23.$$

Hence the roots are $-0.6+$, $1.+$, and $5.+$.

7. Determine the first significant figure of each root of the equation
$$x^3 + 9x^2 + 21x + 17 = 0.$$

If
$$x = -5, -4, -3, -2, -1,$$
$$f(x) = -3, +1, -1, -3, +1.$$

Hence the roots are $-4.+$, $-3.+$, and $-1.+$.

8. Determine the first significant figure of each root of the equation $$x^3 - 15x^2 + 63x - 50 = 0.$$

If $$x = 0, +1, +2, +6, +7, +8,$$
$$f(x) = -50, -1, +24, +4, -1, +6.$$

Hence the roots are 1.+, 6.+, and 7.+.

9. Determine the first significant figure of each root of the equation $$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

If $$x = -1, -0.8, -0.7, +0.7, +0.8, +2, +3, +5, +6,$$
$$f(x) = +11, +2.2+, -0.9+, -0.8+, +0.4+, +8, -5, -13, +88.$$

Hence the roots are $-0.7+$, $0.7+$, 2.+, and 5.+.

10. Determine the first significant figure of each root of the equation $$x^4 - 12x^2 + 12x - 3.$$

If $$x = -4, -3, 0, +0.4, +0.5, +0.6, +0.7, +2, +3,$$
$$f(x) = +13, -66, -3, -0.09+, +0.06+, 0.0+, -0.2, -11, +6.$$

Hence the roots are $-3.+$, $0.4+$, $0.6+$, and 2.+.

EXERCISE CXLVII.

1. Compute to six decimal places the root of the equation $x^3 + 10x^2 + 6x - 120$ which lies between 2 and 3.

+ 10	+ 6	− 120 | 2.833066+
+ 2	+ 24	+ 60
+ 12	+ 30	− 60
+ 2	+ 28	+ 57.152
+ 14	+ 58	− 2.848
+ 2	+ 13.44	+ 2.582187
+ 16	+ 71.44	− 0.265813
+ 0.8	+ 14.08	+ 0.260046
+ 16.8	+ 85.52	− 0.005767
+ 0.8	+ 0.5529	+ 0.005208
+ 17.6	+ 86.0729	− 0.000559
+ 0.8	+ 0.5538	+ 0.000522
+ 18.4	+ 86.6267	− 0.000037
+ 0.03	+ 86.627	
+ 18.43	+ 0.055	
+ 0.03	+ 86.682	
+ 18.46	+ 0.055	
+ 0.03	+ 86.837	
+ 18.49	+ 86.84	
18	+ 86.8	
	+ 87	

2. Compute to six decimal places the root of the equation $x^3 + x^2 + x - 100 = 0$ which lies between 4 and 5.

+ 1	+ 1	− 100 | 4.264429 +
+ 4	+ 20	+ 84
+ 5	+ 21	− 16
+ 4	+ 36	+ 11.928
+ 9	+ 57	− 4.072
+ 4	+ 2.64	+ 3.788376
+ 13	+ 59.64	− 0.283624
+ 0.2	+ 2.68	+ 0.256076
+ 13.2	+ 62.32	− 0.027548
+ 0.2	+ 0.8196	+ 0.025632
+ 13.4	+ 63.1396	− 0.001916
+ 0.2	+ 0.8232	+ 0.001282
+ 13.6	+ 63.9628	− 0.000634
+ 0.06	+ 63.963	+ 0.000576
+ 13.66	+ 0.056	− 0.000058
+ 0.06	+ 64.019	
+ 13.72	+ 0.056	
+ 0.06	+ 64.075	
+ 13.78	64.08	
+ 14	64.1	
	64	

3. Compute to six decimal places the root of the equation $x^4 - 2x^3 + 21x - 23 = 0$ which lies between 1 and 2.

-2	$+0$	$+21$	-23 | $1.157450+$
$+1$	-1	-1	$+20$
-1	-1	$+20$	-3
$+1$	$+0$	-1	$+1.9021$
0	-1	$+19$	-1.0979
$+1$	$+1$	$+0.021$	$+0.95515625$
$+1$	0	$+19.021$	-0.14274375
$+1$	$+0.21$	$+0.043$	$+0.13411146$
$+2$	$+0.21$	$+19.064$	-0.00863229
$+0.1$	$+0.22$	$+0.039125$	$+0.00766660$
$+2.1$	$+0.43$	$+19.103125$	-0.00096569
$+0.1$	$+0.23$	$+0.045375$	$+0.00095835$
$+2.2$	$+0.66$	$+19.151500$	-0.00000734
$+0.1$	$+0.1225$	$+19.15150$	
$+2.3$	$+0.7825$	$+0.00728$	
$+0.1$	$+0.1250$	$+19.15878$	
$+2.4$	$+0.9075$	$+0.00728$	
$+0.05$	$+0.1275$	$+19.16606$	
$+2.45$	$+1.0350$	$+19.1661$	
$+0.05$	$+1.04$	$+0.0004$	
$+2.50$	$+1$	$+19.1665$	
$+0.05$		$+0.0004$	
$+2.55$		$+19.1669$	
$+0.05$		$+19.167$	
$+2.60$			

4. Compute to six decimal places the root of the equation $x^4 - 5x^3 + 3x^2 + 35x - 70 = 0$ which lies between 2 and 3.

− 5	+ 3	+ 35	− 70	2.645751+
+ 2	− 6	− 6	+ 58	
− 3	− 3	+ 29	− 12	
+ 2	− 2	− 10	+ 11.0976	
− 1	− 5	+ 19	− 0.9024	
+ 2	+ 2	− 0.504	+ 0.78780416	
+ 1	− 3	+ 18.496	− 0.11459584	
+ 2	+ 2.16	+ 1.008	+ 0.09960665	
+ 3	− 0.84	+ 19.504	− 0.01498919	
+ 0.6	+ 2.52	+ 0.191104	+ 0.01396598	
+ 3.6	+ 1.68	+ 19.695104	− 0.00102321	
+ 0.6	+ 2.88	+ 0.199872	+ 0.00099775	
+ 4.2	+ 4.56	+ 19.894976	− 0.00002546	
+ 0.6	+ 0.2176	+ 19.89498	+ 0.00001995	
+ 4.8	+ 4.7776	+ 0.02635	− 0.00000551	
+ 0.6	+ 0.2192	+ 19.92133		
+ 5.4	+ 4.9968	+ 0.02660		
+ 0.04	+ 0.2208	+ 19.94793		
+ 5.44	+ 5.2176	+ 19.9479		
+ 0.04	+ 5.22	+ 0.0035		
+ 5.48	+ 0.05	+ 19.9514		
+ 0.04	+ 5.27	+ 0.0035		
+ 5.52	+ 0.05	+ 19.9549		
+ 0.04	+ 5.32	+ 19.955		
+ 5.56	+ 0.05	+ 19.95		
+ 10	+ 5.37			
	+ 5			

5. Compute to six decimal places the root of the equation $x^4 - 12x^2 + 12x - 3$ which lies between -3 and -4.

0	− 12	− 12	− 3	3.907378 +
+ 3	+ 9	− 9	− 63	
+ 3	− 3	− 21	− 66	
+ 3	+ 18	+ 45	+ 65.0241	
+ 6	+ 15	+ 24	− 0.9759	
+ 3	+ 27	+ 48.249	− 0.97590000	
+ 9	+ 42	+ 72.249	+ 0.92562260	
+ 3	+ 11.61	+ 59.427	− 0.05027740	
+ 12	+ 53.61	+ 131.676	+ 0.03984378	
+ 0.9	+ 12.42	+ 131.67600	− 0.01043362	
+ 12.9	+ 66.03	+ 0.55580	+ 0.00929908	
+ 0.9	+ 13.23	+ 132.23180	− 0.00113454	
+ 13.8	+ 79.26	+ 0.55678	+ 0.00106280	
+ 0.9	+ 79.26	+ 132.78858	− 0.00007174	
+ 14.7	+ 0.14	+ 132.7886		
+ 0.9	+ 79.40	+ 0.0240		
+ 15.6	+ 0.14	+ 132.8126		
+ 15.60	+ 79.54	+ 0.0240		
20	+ 0.14	+ 132.8366		
	+ 79.68	+ 132.837		
	+ 80	+ 0.007		
	+ 100	+ 132.844		
		+ 0.007		
		132.851		
		132.85		

6. Compute to six places of decimals the root of the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321 = 0$ which lies between 8 and 9.

+ 2	+ 3	+ 4	+ 5	−54321	8.414454 +
+ 8	+ 80	+ 664	+ 5344	+42792	
+10	+ 83	+ 668	+ 5349	−11529	
+ 8	+144	+1816	+19872	+11088.97344	
+18	+227	+2484	+25221	− 440.02656	
+ 8	+208	+3480	+ 2501.4336	+ 304.11052	
+26	+435	+5964	+27722.4336	− 135.91604	
+ 8	+272	+ 289.584	+ 2620.0064	+ 122.02904	
+34	+707	+6253.584	+30342.4400	− 13.88700	
+ 8	+ 16.96	+ 296.432	+30342.440	+ 12.21504	
+42	+723.96	+6550.016	+ 68.612	− 1.67196	
+ 0.4	+ 17.12	+ 303.344	+30411.052	+ 1.52700	
+42.4	+741.08	+6853.360	+ 68.690	− 0.14496	
+ 0.4	+ 17.28	+6853.4	+30479.742	+ 0.12216	
+42.8	+758.36	+ 7.8	+30479.74	− 0.02280	
+ 0.4	+ 17.44	+6861.2	+ 27.52		
+43.2	+775.80	+ 7.8	+30507.26		
+ 0.4	+780	+6869.0	+ 27.52		
+43.6		+ 7.8	+30534.78		
+ 0.4		+6876.8	+30534.8		
+44.0		+6880	+ 2.8		
		+7000	+30537.6		
			+ 2.8		
			+30540.4		
			+30540		

7. Compute to six places of decimals the root of the equation $x^4 - 59x^2 + 840 = 0$, which lies between 4 and 5.

0	− 59	0	+ 840 | 4.898989+
+ 4	+ 16	− 172	− 688
+ 4	− 43	− 172	+ 152
+ 4	+ 32	− 44	− 140.5184
+ 8	− 11	− 216	+ 11.4816
+ 4	+ 48	+ 40.352	− 10.50697359
+ 12	+ 37	− 175.648	+ 0.97462641
+ 4	+ 13.44	+ 51.616	− 0.86897824
+ 16	+ 50.44	− 124.032	+ 0.10564817
+ 0.8	+ 14.08	+ 7.287849	− 0.09708075
+ 16.8	+ 64.52	− 116.744151	+ 0.00856742
+ 0.8	+ 14.72	+ 7.444827	− 0.00754481
+ 17.6	+ 79.24	− 109.299324	+ 0.00102261
+ 0.8	+ 1.7361	− 109.29932	− 0.00097002
+ 18.4	+ 80.9761	+ 0.67704	+ 0.00005259
+ 0.8	+ 1.7442	− 108.62228	
+ 19.2	+ 82.7203	+ 0.67832	
+ 0.09	+ 1.7523	− 107.94396	
+ 19.29	+ 84.4726	− 107.9440	
+ 0.09	+ 84.47	+ 0.0765	
+ 19.38	+ 0.16	− 107.8675	
+ 0.09	+ 84.63	+ 0.0765	
+ 19.47	+ 0.16	− 107.7910	
+ 0.09	+ 84.79	− 107.791	
+ 19.56	+ 0.16	+ 0.008	
+ 20	+ 84.95	− 107.783	
	+ 85.	+ 0.008	
	+ 100.	− 107.775	
		− 107.78	

8. Compute to six places of decimals the real root of the equation $x^3 - 35499 = 0$.

0	0	− 35499 | 32.865378+
+ 30	+ 900	+ 27000
+ 30	+ 900	− 8499
+ 30	+ 1800	+ 5768
+ 60	+ 2700	− 2731
+ 30	+ 184	+ 2519.552
+ 90	+ 2884	− 211.448
+ 2	+ 188	+ 194.005656
+ 92	+ 3072	− 17.442344
+ 2	+ 77.44	+ 16.199170
+ 94	+ 3149.44	− 1.243174
+ 2	+ 78.08	
+ 96	+ 3227.52	+ 0.972108
+ 0.8	+ 5.9076	− 0.271066
+ 96.8	+ 3233.4276	+ 0.240828
+ 0.8	+ 5.9112	− 0.030238
+ 97.6	+ 2339.3388	+ 0.027520
+ 0.8		− 0.002718
	+ 3239.339	
+ 98.4	+ 0.495	
+ 0.06	+ 3239.834	
+ 98.46	+ 0.495	
+ 0.06	+ 3240.329	
+ 98.52	+ 3240.33	
+ 0.06	+ 0.03	
+ 98.58	+ 3240.36	
+ 99	+ 0.03	
+ 100	+ 3440.39	
	+ 3440.4	
	+ 3440	

9. Compute to six decimal places the real root of the equation $x^3 - 242970624 = 0$.

0	0	− 242970624	624
+ 600	+ 360000	+ 216000000	
+ 600	+ 360000	− 26970624	
+ 600	+ 720000	+ 22328000	
+ 1200	+ 1080000	− 4642624	
+ 600	+ 36400	+ 4642624	
+ 1800	+ 1116400	0	
+ 20	+ 36800		
+ 1820	+ 1153200		
+ 20	+ 7456		
+ 1840	+ 1160656		
+ 20			
+ 1860			
+ 4			
+ 1864			

10. Compute to six decimal places the positive real root of the equation $x^4 - 707281 = 0$.

0	0	0	− 707281	29
+ 20	+ 400	+ 8000	+ 160000	
+ 20	+ 400	+ 8000	− 547281	
+ 20	+ 800	+ 24000	− 547281	
+ 40	+ 1200	+ 32000	0	
+ 20	+ 1200	+ 28809		
+ 60	+ 2400	+ 60809		
+ 20	+ 801			
+ 80	+ 3201			
+ 9				
+ 89				

11. Compute to six places of decimals the real root of the equation $x^5 - 147008443 = 0$.

0	0	0	0	− 147008443	43
+ 40	+ 1600	+ 64000	+ 2560000	+ 102400000	
+ 40	+ 1600	+ 64000	+ 2560000	− 44608443	
+ 40	+ 3200	+ 192000	+ 10240000	+ 44608443	
+ 80	+ 4800	+ 256000	+ 12800000	0	
+ 40	+ 4800	+ 384000	+ 2069481		
+ 120	+ 9600	+ 640000	+ 14869481		
+ 40	+ 6400	+ 49827			
+ 160	+ 16000	+ 689827			
+ 40	+ 609				
+ 200	+ 16609				
+ 3					
+ 203					

12. Compute to six places of decimals the positive root of the equation $x^2 - 551791 = 0$.

0	− 551791 | 742.826359 +
+ 700	+ 490000
+ 700	− 61791
+ 700	
+ 1400	+ 57600
	− 4191
+ 40	+ 2964
+ 1440	− 1227
+ 40	
+ 1480	+ 1187.84
	− 39.16
+ 2	+ 29.7124
+ 1482	− 9.4476
+ 2	
+ 1484	+ 8.913876
	− 0.533724
+ 0.8	+ 0.445695
+ 1484.8	− 0.088029
+ 0.8	+ 0.074285
+ 1485.6	− 0.013744
+ 0.02	+ 0.013374
+ 1485.62	− 0.000370
+ 0.02	
+ 1485.64	
+ 0.006	
+ 1485.646	
+ 0.006	
+ 1485.652	
+ 1485.65	
+ 1495.7	
+ 1486	

13. Compute to six places of decimals the root of the equation $x^2 - 17x + 70.3 = 0$ which lies between 7 and 8.

-17	$+70.3$	$7.103575+$
$+7$	$-70.$	
-10	$+0.3$	
$+7$	-0.29	
-3	$+0.01$	
$+0.1$	-0.008391	
-2.9	$+0.001609$	
$+0.1$	-0.00139675	
-2.8	$+0.00021225$	
$+0.003$	-0.00019551	
-2.797	$+0.00001674$	
$+0.003$	-0.00001395	
-2.794	$+0.00000279$	
$+0.0005$		
-2.7935		
$+0.0005$		
-2.7930		
-2.793		
-2.79		

14. Compute to six places of decimals the root of the equation $x^3 + 9x^2 + 24x + 17 = 0$ which lies between -4 and -5.

-9	$+24$	-17	$4.532088+$
$+4$	-20	$+16$	
-5	$+4$	-1	
$+4$	-4	$+0.875$	
-1	0	-0.125	
$+4$	$+1.75$	$+0.116577$	
$+3$	$+1.75$	-0.008423	
$+0.5$	$+2.00$	$+0.008063768$	
$+3.5$	$+3.75$	-0.000359232	
$+0.5$	$+0.1359$	$+0.000323288$	
$+4.0$	$+3.8859$	-0.000035944	
$+0.5$	$+0.1368$	$+0.000032328$	
$+4.5$	$+4.0227$	-0.000003616	
$+0.03$	$+0.009184$		
$+4.53$	$+4.031884$		
$+0.03$	$+0.009188$		
$+4.56$	$+4.041072$		
$+0.03$	$+4.0411$		
$+4.59$	$+4.041$		
$+0.002$			
$+4.592$			
$+0.002$			
$+4.594$			
$+0.002$			
$+4.596$			

15. Compute to six places of decimals the root of the equation $x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$ which lies between 0 and -1.

+ 8	+ 14	− 4	− 8 | 0.7320508+
+ 0.7	+ 6.09	+ 14.063	+ 7.0441
+ 8.7	+ 20.09	+ 10.063	− 0.9559
+ 0.7	+ 6.58	+ 18.669	+ 0.89261841
+ 9.4	+ 26.67	+ 28.732	− 0.06328159
+ 0.7	+ 7.07	+ 1.021947	+ 0.06171030
+ 10.1	+ 33.74	+ 29.753947	− 0.00157129
+ 0.7	+ 0.3249	+ 1.031721	+ 0.00154625
+ 10.8	+ 34.0649	+ 30.785668	− 0.00002504
+ 0.03	+ 0.3258	+ 30.78567	+ 0.00002472
+ 10.83	+ 34.3907	+ 0.06948	− 0.00000032
+ 0.03	+ 0.3267	+ 30.85515	
+ 10.86	+ 34.7174	+ 0.06952	
+ 0.03	+ 34.72	+ 30.92467	
+ 10.89	+ 0.02	+ 20.925	
+ 0.03	+ 34.74	+ 30.9	
+ 10.92	+ 0.02		
+ 10	+ 34.76		
	+ 0.02		
	+ 34.78		

EXERCISE CXLVIII.

1. Solve $x^4 + 7x^3 - 7x - 1 = 0$.

$$x^4 + 7x^3 - 7x - 1 = 0,$$
$$(x^2 - 1)(x^2 + 7x + 1) = 0.$$
$$\therefore x = \pm 1, \text{ or } \frac{-7 \pm 3\sqrt{5}}{2}.$$

2. Solve $x^4 + x^3 + x^2 + x + 1 = 0$.

$$x^4 + x^3 + x^2 + x + 1 = 0,$$
$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0,$$
$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0,$$
$$\left(x + \frac{1}{x}\right)^2 - 2 + \left(x + \frac{1}{x}\right) + 1 = 0,$$
$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = 0.$$
$$\therefore x + \frac{1}{x} = \frac{-1 \pm \sqrt{5}}{2}.$$

(1) $$x^2 + \frac{1 - \sqrt{5}}{2}x + 1 = 0,$$
$$x = \frac{\sqrt{5} - 1}{4} \pm \tfrac{1}{4}\sqrt{-2\sqrt{5} - 10}.$$

(2) $$x^2 + \frac{1 + \sqrt{5}}{2}x + 1 = 0,$$
$$x = \frac{-\sqrt{5} - 1}{4} \pm \tfrac{1}{4}\sqrt{2\sqrt{5} - 10}.$$

3. Solve $x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0$.

$$x^6 - 3x^5 + 5x^4 - 5x^2 + 3x - 1 = 0,$$
$$(x^2 - 1)(x^4 - 3x^3 + 6x^2 - 3x + 1) = 0.$$
$$\therefore x = \pm 1,$$

or $$x^4 - 3x^3 + 6x^2 - 3x + 1 = 0,$$
$$x^2 - 3x + 6 - \frac{3}{x} + \frac{1}{x^2} = 0,$$

$$\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)+6=0,$$

$$\left(x+\frac{1}{x}\right)^2-2-3\left(x+\frac{1}{x}\right)+6=0,$$

$$\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+4=0.$$

$$\therefore x+\frac{1}{x}=\frac{3\pm\sqrt{-7}}{2}.$$

(1) $$x^2-\frac{3+\sqrt{-7}}{2}x+1=0,$$

$$x=\frac{3+\sqrt{-7}}{4}\pm\tfrac{1}{4}\sqrt{6\sqrt{-7}-14}.$$

(2) $$x^2-\frac{3-\sqrt{-7}}{2}x+1=0,$$

$$x=\frac{3-\sqrt{-7}}{4}\pm\tfrac{1}{4}\sqrt{-6\sqrt{-7}-14}.$$

4. Solve $x^4-5x^3+6x^2-5x+1=0$.

$$x^4-5x^3+6x^2-5x+1=0,$$

$$x^2-5x+6-\frac{5}{x}+\frac{1}{x^2}=0,$$

$$\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0,$$

$$\left(x+\frac{1}{x}\right)^2-2-5\left(x+\frac{1}{x}\right)+6=0,$$

$$\left(x+\frac{1}{x}\right)^2-5\left(x+\frac{1}{x}\right)+4=0.$$

$$\therefore x+\frac{1}{x}=1, \text{ or } 4.$$

(1) $$x^2-x+1=0,$$

$$x=\frac{1\pm\sqrt{-3}}{2}.$$

(2) $$x^2-4x+1=0,$$

$$x=2\pm\sqrt{3}.$$

5. Solve $2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0.$

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0,$$

$$2x^2 - 5x + 6 - \frac{5}{x} + \frac{2}{x^2} = 0,$$

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0,$$

$$2\left(x + \frac{1}{x}\right)^2 - 4 - 5\left(x + \frac{1}{x}\right) + 6 = 0,$$

$$2\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 2 = 0.$$

$$\therefore x + \frac{1}{x} = 2, \text{ or } \tfrac{1}{2}.$$

(1) $$x^2 - 2x + 1 = 0,$$
$$x = 1.$$

(2) $$x^2 - \tfrac{1}{2}x + 1 = 0,$$
$$x = \frac{1 \pm \sqrt{-15}}{4}.$$

6. Solve $x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0.$

$$x^5 - 4x^4 + x^3 + x^2 - 4x + 1 = 0,$$

$$(x + 1)(x^4 - 5x^3 + 6x^2 - 5x + 1) = 0.$$

$$\therefore x = -1,$$

or $$x^4 - 5x^3 + 6x^2 - 5x + 1 = 0,$$

$$x^2 - 5x + 6 - \frac{5}{x} + \frac{1}{x^2} = 0,$$

$$x^2 + \frac{1}{x^2} - 5\left(x + \frac{1}{x}\right) + 6 = 0,$$

$$\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) + 4 = 0.$$

$$\therefore x + \frac{1}{x} = 1, \text{ or } 4.$$

(1) $$x^2 - x + 1 = 0,$$
$$x = \frac{1 \pm \sqrt{-3}}{2}.$$

(2) $$x^2 - 4x + 1 = 0,$$
$$x = 2 \pm \sqrt{3}.$$

7. Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

$$x^4 - 10x^3 + 26x^2 - 10x + 1 = 0,$$

$$x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2} = 0,$$

$$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0,$$

$$\left(x + \frac{1}{x}\right)^2 - 10\left(x + \frac{1}{x}\right) + 24 = 0.$$

$$\therefore x + \frac{1}{x} = 4 \text{ or } 6.$$

(1) $$x^2 - 4x + 1 = 0,$$
$$x = 2 \pm \sqrt{3}.$$

(2) $$x^2 - 6x + 1 = 0,$$
$$x = 3 \pm 2\sqrt{2}.$$

8. Solve $x^3 + mx^2 + mx + 1 = 0$.

$$x^3 + mx^2 + mx + 1 = 0,$$

$$(x + 1)\{x^2 + (m - 1)x + 1\} = 0.$$

$$\therefore x = -1,$$

or $$x^2 + (m - 1)x + 1 = 0,$$

$$x = \frac{1 - m \pm \sqrt{m^2 - 2m - 3}}{2}.$$

9. Solve $x^5 + 1 = 0$.

$$x^5 + 1 = 0,$$

$$(x + 1)(x^4 - x^3 + x^2 - x + 1) = 0.$$

$$\therefore x = -1,$$

or $$x^4 - x^3 + x^2 - x + 1 = 0,$$

$$x^2 - x + 1 - \frac{1}{x} + \frac{1}{x^2} = 0,$$

$$\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 1 = 0,$$

$$\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) - 1 = 0.$$

$$\therefore x + \frac{1}{x} = \frac{1 \pm \sqrt{5}}{2}.$$

(1) $$x^2 - \frac{1 + \sqrt{5}}{2}x + 1 = 0,$$

$$x = \frac{1+\sqrt{5}}{4} \pm \tfrac{1}{4}\sqrt{2\sqrt{5}-10}.$$

(2) $$x^2 - \frac{1-\sqrt{5}}{2}x + 1 = 0,$$

$$x = \frac{1-\sqrt{5}}{4} \pm \tfrac{1}{4}\sqrt{-2\sqrt{5}-10}.$$

10. Solve $3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0.$

$$3x^5 - 2x^4 + 5x^3 - 5x^2 + 2x - 3 = 0,$$
$$(x-1)(3x^4 + x^3 + 6x^2 + x + 3) = 0.$$
$$\therefore x = 1,$$

or $$3x^4 + x^3 + 6x^2 + x + 3 = 0,$$
$$3x^2 + x + 6 + \frac{1}{x} + \frac{3}{x^2} = 0,$$
$$3\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 6 = 0,$$
$$3\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 0.$$
$$\therefore x + \frac{1}{x} = 0 \text{ or } -\tfrac{1}{3}.$$

(1) $$x^2 + 1 = 0,$$
$$x = \pm\sqrt{-1}.$$

(2) $$x^2 + \tfrac{1}{3}x + 1 = 0,$$
$$x = \frac{-1 \pm \sqrt{-35}}{6}.$$

Exercise CXLIX.

1. Solve $11^x = 346.$

$$11^x = 346,$$
$$x \log 11 = \log 346,$$
$$x = \frac{\log 346}{\log 11}$$
$$= \frac{2.5391}{1.0414}$$
$$= 2.438+.$$

2. Solve $3^x = 10.$

$$3^x = 10,$$
$$x \log 3 = \log 10,$$
$$x = \frac{\log 10}{\log 3}$$
$$= \frac{1.0000}{0.4771}$$
$$= 2.096+.$$

3. Solve $10^x = 745$.

$$10^x = 745,$$
$$x \log 10 = \log 745,$$
$$x = \frac{\log 745}{\log 10} = \frac{2.8722}{1.0000} = 2.8722+.$$

4. Solve $7^x = 324$.

$$7^x = 324,$$
$$x \log 7 = \log 324,$$
$$x = \frac{\log 324}{\log 7} = \frac{2.5105}{0.8451} = 2.970+.$$

5. Solve $4^x = 3.74$.

$$4^x = 3.74,$$
$$x \log 4 = \log 3.74,$$
$$x = \frac{\log 3.74}{\log 4} = \frac{0.5729}{0.6021} = 0.951+.$$

6. Solve $146^x = 12984$.

$$146^x = 12984,$$
$$x \log 146 = \log 12984,$$
$$x = \frac{\log 12984}{\log 146} = \frac{4.1134}{2.1644} = 1.900+.$$

7. Solve $0.2^x = 0.4$

$$0.2^x = 0.4,$$
$$x \log 0.2 = \log 0.4,$$
$$x = \frac{\log 0.4}{\log 0.2} = \frac{9.6021-10}{9.3010-10} = \frac{0.3979}{0.6990} = 0.569+.$$

8. Solve $14.74^x = 8.64$.

$$14.74^x = 8.64,$$
$$x \log 14.74 = \log 8.64,$$
$$x = \frac{\log 8.64}{\log 14.74} = \frac{0.9365}{1.1685} = 0.801+.$$

9. Solve $x^x = 2.767$.

$$x^x = 2.767,$$
$$x \log x = \log 2.767 = 0.4420.$$
$$1 \log 1 = 1 \times 0 = 0,$$
$$1.7 \log 1.7 = 1.7 \times 0.2304 = 0.3917,$$
$$1.77 \log 1.77 = 1.77 \times 0.2480 = 0.4390,$$
$$1.774 \log 1.774 = 1.774 \times 0.2490 = 0.4417.$$
$$\therefore x = 1.774+.$$

10. Solve $x^x = 23.10$.

$$x^x = 23.10,$$
$$x \log x = \log 23.10$$
$$= 1.3636.$$
$$2 \log 2 = 2 \times 0.3010 = 0.6020,$$
$$2.9 \log 2.9 = 2.9 \times 0.4624 = 1.3410,$$
$$2.92 \log 2.92 = 2.92 \times 0.4654 = 1.3590,$$
$$2.925 \log 2.925 = 2.925 \times 0.4661 = 1.3635.$$
$$\therefore x = 2.925 +.$$

11. Given $P = 750$, $A = 1797.42$, $r = 6\%$; find t.

$$t = \frac{\log A - \log P}{\log (1+r)}$$
$$= \frac{3.2547 - 2.8751}{0.0253}$$
$$= \frac{0.3796}{0.0253}$$
$$= 15.$$

12. Given $P = 780$, $A = 1559.22$, $r = 8\%$; find t.

$$t = \frac{\log A - \log P}{\log (1+r)}$$
$$= \frac{3.1926 - 2.8921}{0.0334}$$
$$= \frac{0.3005}{0.0334}$$
$$= 9.$$

13. Given $P = 5630.75$, $A = 21789.22$, $r = 7\%$; find t.

$$t = \frac{\log A - \log P}{\log (1+r)}$$
$$= \frac{4.3383 - 3.7506}{0.0294}$$
$$= \frac{0.5877}{0.0294}$$
$$= 20.$$

14. Given $P = 300$, $A = 515.46$, $r = 7\%$; find t.

$$t = \frac{\log A - \log P}{\log (1+r)}$$
$$= \frac{2.7122 - 2.4771}{0.0294}$$
$$= \frac{0.2351}{0.0294}$$
$$= 8.$$

15. Given $P = 84.65$, $A = 289.47$ $r = 7\frac{1}{2}\%$; find t.

$$t = \frac{\log A - \log P}{\log (1+r)}$$
$$= \frac{2.4616 - 1.9276}{0.0314}$$
$$= \frac{0.5340}{0.0314}$$
$$= 17.$$

EXERCISE CL.

1. $x^3 + 12x^2 + 45x + 50 = 0.$

Let $x = y - 4.$ (Cf. § 600.)

$$\begin{array}{rrrr|l} 1 & +12 & +45 & +50 & \underline{-4} \\ & -4 & -32 & -52 & \\ \hline 1 & +8 & +13 & -2 & \\ & -4 & -16 & & \\ \hline 1 & +4 & -3 & & \\ & -4 & & & \\ \hline 1 & +0 & & & \end{array}$$

$\therefore y^3 - 3y - 2 = 0.$

$p = -3,\ q = -2.$

$$z = \sqrt[3]{-\tfrac{1}{2}q \pm \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

$$= \sqrt[3]{1 \pm \sqrt{-1 + 1}}$$

$$= 1.$$

$$x = z - \frac{p}{3z} - \frac{m}{3}$$

$$= 1 + 1 - 4$$

$$= -2.$$

Divide the given equation by $x + 2.$

$$\begin{array}{rrrr|l} 1 & +12 & +45 & +50 & \underline{-2} \\ & -2 & -20 & & \\ \hline 1 & +10 & +25 & & \end{array}$$

$x^2 + 10x + 25 = 0,$

$(x + 5)^2 = 0.$

$\therefore x = -2,\ -5,\ -5.$

2. $x^3 - 21x^2 + 159x - 490 = 0.$

Let $x = y + 7.$

$$\begin{array}{rrrr|l} 1 & -21 & +159 & -490 & \underline{+7} \\ & +7 & -98 & +427 & \\ \hline 1 & -14 & +61 & -63 & \\ & +7 & -49 & & \\ \hline 1 & -7 & +12 & & \\ & +7 & & & \\ \hline 1 & +0 & & & \end{array}$$

$\therefore y^3 + 12y - 63 = 0.$

$p = 12,\ q = -63.$

$$z = \sqrt[3]{\tfrac{63}{2} \pm \sqrt{64 + \tfrac{3969}{4}}}$$

$$= \sqrt[3]{\tfrac{63}{2} \pm \tfrac{65}{2}}$$

$$= -1 \text{ or } 4,$$

$$x = z - \frac{p}{3z} - \frac{m}{3}$$

$$= -1 + 4 + 7 \text{ or } 4 - 1 + 7$$

$$= 10.$$

$$\begin{array}{rrrr|l} 1 & -21 & +159 & -490 & \underline{10} \\ & +10 & -110 & & \\ \hline 1 & -11 & +49 & & \end{array}$$

$x^2 - 11x + 49 = 0.$

$\therefore x = 10,$ or $\dfrac{11 \pm 5\sqrt{-3}}{2}.$

3. $x^3 - 6x^2 + 13x - 10.$

Let $x = y + 2.$

$$\begin{array}{rrrr|l} 1 & -6 & +13 & -10 & \underline{2} \\ & +2 & -8 & +10 & \\ \hline 1 & -4 & +5 & +0 & \\ & +2 & -4 & & \\ \hline 1 & -2 & +1 & & \\ & +2 & & & \\ \hline 1 & +0 & & & \end{array}$$

$\therefore y^3 + y = 0.$

$y(y^2 + 1) = 0,$

$y = 0,$ or $\pm\sqrt{-1},$

$x = y + 2$

$= 2,$ or $2 \pm \sqrt{-1}.$

4. $x^3+3x^2+9x-13=0.$

Let $x=y-1.$

$$\begin{array}{l} 1+3+9-13\,\underline{|-1} \\ \underline{\quad -1-2-\ \ 7} \\ 1+2+7-20 \\ \underline{\quad -1-1} \\ 1+1+6 \\ \underline{\quad -1} \\ 1+0 \end{array}$$

$\therefore y^3+6y-20=0.$

$p=6,\ q=-20.$

$$\begin{aligned} z &= \sqrt[3]{10 \pm \sqrt{8+100}} \\ &= \sqrt[3]{10 \pm 6\sqrt{3}} \\ &= 1 \pm \sqrt{3}, \end{aligned}$$

$$\begin{aligned} x &= z - \frac{p}{3z} - \frac{m}{3} \\ &= 1+\sqrt{3} - \frac{2}{1+\sqrt{3}} - 1 \\ &= 1+\sqrt{3}+1-\sqrt{3}-1 \\ &= 1. \end{aligned}$$

$$\begin{array}{l} 1+3+\ \ 9-13\,\underline{|1} \\ \underline{\quad +1+\ \ 4+13} \\ 1+4+13 \qquad 0 \end{array}$$

$x^2+4x+13=0.$

$\therefore x=1,$ or $-2\pm 3\sqrt{-1}.$

5. $y^3+48y+504=0.$

$p=48,\ q=504.$

$$\begin{aligned} z &= \sqrt[3]{-252 \pm \sqrt{4096+63504}} \\ &= \sqrt[3]{-252 \pm 260} \\ &= 2, \end{aligned}$$

$$\begin{aligned} y &= z - \frac{p}{3z} \\ &= 2-8 \\ &= -6. \end{aligned}$$

$$\begin{array}{l} 1+0+48+504\,\underline{|-6} \\ \underline{\quad -6+36-504} \\ 1-6+84 \end{array}$$

$y^2-6y+84=0.$

$\therefore y=-6,$ or $3\pm 5\sqrt{-3}.$

6. $y^3-21y-344=0.$

$p=-21,\ q=-344.$

$$\begin{aligned} z &= \sqrt[3]{172 \pm \sqrt{343+29584}} \\ &= \sqrt[3]{172 \pm 173} \\ &= 1, \end{aligned}$$

$$\begin{aligned} y &= z - \frac{p}{3z} \\ &= 1+7 \\ &= 8. \end{aligned}$$

$$\begin{array}{l} 1+0-21-344\,\underline{|8} \\ \underline{\quad +8+64+344} \\ 1+8+43 \end{array}$$

$y^2+8y+43=0.$

$\therefore y=8,$ or $-4\pm 3\sqrt{-3}.$

7. $y^3-3y+2=0.$

$p=-3,\ q=2.$

$$\begin{aligned} z &= \sqrt[3]{-1 \pm \sqrt{-1+1}} \\ &= -1, \end{aligned}$$

$$\begin{aligned} y &= z - \frac{p}{3z} \\ &= -1-1 \\ &= -2. \end{aligned}$$

$$\begin{array}{l} 1+0-3+2\,\underline{|-2} \\ \underline{\quad -2+4} \\ 1-2+1 \end{array}$$

$y^2-2y+1-0.$

$\therefore y=-2,\ 1,\ 1.$

8. $y^3 - 60y + 671 = 0.$

$p = -60,\ q = 671.$

$$z = \sqrt[3]{-\tfrac{671}{2} \pm \sqrt{-8000 + \tfrac{450241}{4}}}$$

$$= \sqrt[3]{\frac{-671 \pm 11\sqrt{41}}{2}}$$

$$= \frac{-11 \pm \sqrt{41}}{2},$$

$$y = z - \frac{p}{3z}$$

$$= \frac{-11 + \sqrt{41}}{2} + \frac{40}{-11 + \sqrt{41}}$$

$$= \frac{-11 + \sqrt{41}}{2} - \frac{11 + \sqrt{41}}{2}$$

$$= -11.$$

$$\begin{array}{rrrr|l} 1 + & 0 - & 60 + & 671 & -11 \\ & -11 + & 121 - & 671 & \\ \hline 1 - & 11 + & 61 & & \end{array}$$

$$y^2 - 11y + 61 = 0.$$

$$\therefore y = -11, \text{ or } \frac{11 \pm \sqrt{-123}}{2}.$$

Exercise CLI.

1. $x^3 + 3x - 5 = 0.$

$p = 3,\ q = -5.$

$$\tan\theta = \frac{2p}{2q}\sqrt{\frac{p}{3}}$$

$$= -\frac{6}{15}\sqrt{1}$$

$$= -0.4000.$$

$\log\tan\ \theta = 9.60206\ (n),$

$\theta = 158°\ 11'\ 55'',$

$\tfrac{1}{2}\theta = 79°\ 5'\ 58''.$

$\log\tan\tfrac{1}{2}\theta = 10.71539,$

$\log\tan\tfrac{1}{2}\phi = \tfrac{1}{3}\log\tan\tfrac{1}{2}\theta$

$= 10.23846,$

$\tfrac{1}{2}\phi = 59°\ 59'\ 39'',$

$\phi = 119°\ 51'\ 18'',$

$\cot\phi = -0.57709,$

$\csc\phi = 1.1546,$

$$x_1 = -2\sqrt{\frac{p}{3}}\cot\phi$$

$$= -1.15418.$$

$$x_2 = \sqrt{\frac{p}{3}}\cot\phi + \sqrt{-p}\csc\phi$$

$$= -0.57709 + 1.9998\sqrt{-1}.$$

$$x_3 = \sqrt{\frac{p}{3}}\cot\phi - \sqrt{-p}\csc\phi$$

$$= -0.57709 - 1.9998\sqrt{-1}.$$

2. $x^3 + 7x + 3.$

$p = 7,\ q = 3.$

$$\tan\theta = \frac{2p}{3q}\sqrt{\frac{p}{3}}$$

$$= \frac{14}{9}\sqrt{\frac{7}{3}}.$$

$\log\ 14 = 1.14613$

$\log\sqrt{7} = 0.42255$

$\text{colog}\ 9 = 9.04576 - 10$

$\text{colog}\sqrt{3} = 9.76144 - 10$

$\log\tan\ \theta = 10.37588$

$$\theta = 67° 10' 36'',$$
$$\frac{\theta}{2} = 33° 35' 18'',$$
$$\log \tan \frac{\theta}{2} = 9.82223,$$
$$\log \tan \frac{\phi}{2} = \tfrac{1}{3} \log \tan \tfrac{1}{2}\theta$$
$$= 6.94074,$$
$$\frac{\phi}{2} = 41° 6' 12'',$$
$$\phi = 82° 12' 24'',$$
$$\cot \phi = 0.13686,$$
$$\csc \phi = 1.0093,$$
$$x_1 = -2\sqrt{\frac{p}{3}} \cot \phi.$$

$$\begin{aligned} \log\ 2 &= 0.30103 \\ \log \sqrt{p} &= 0.42255 \\ \text{colog} \sqrt{3} &= 9.76144 - 10 \\ \log \cot \phi &= \underline{9.13629} \\ \log x_1 &= 9.62131 - 10 \end{aligned}$$

$$x_1 = -0.41813.$$

$$x_2 = \sqrt{\frac{p}{3}} \cot \phi + \sqrt{-p} \csc \phi.$$

$$\begin{aligned} \log \sqrt{p} &= 0.42255 \\ \log \csc \phi &= \underline{0.00403} \\ &\ 0.42658 \end{aligned}$$

$$x_2 = 0.20906 + 2.6704\sqrt{-1},$$
$$x_3 = 0.20906 - 2.6704\sqrt{-1}.$$

3. $x^3 - 7x + 11 = 0.$

$$p = 7, q = 11, \left(\frac{p^3}{27} < \frac{q^2}{4}\right).$$

$$\sin \theta = \frac{2p}{3q}\sqrt{\frac{p}{3}}$$
$$= \frac{14}{33}\sqrt{\frac{7}{3}}.$$

$$\begin{aligned} \log\ 14 &= 1.14613 \\ \log \sqrt{7} &= 0.42255 \\ \text{colog}\ 33 &= 8.48149 - 10 \\ \text{colog} \sqrt{3} &= \underline{9.76144 - 10} \\ \log \sin \theta &= 9.81161 \end{aligned}$$

$$\theta = 40° 23' 40'',$$
$$\frac{\theta}{2} = 20° 11' 50'',$$
$$\log \tan \frac{\theta}{2} = 9.56570,$$
$$\log \tan \frac{\phi}{2} = \tfrac{1}{3} \log \tan \frac{\theta}{2}$$
$$= 9.85523,$$
$$\frac{\phi}{2} = 35° 37' 21'',$$
$$\phi = 71° 14' 42''.$$
$$\log \cot \phi = 9.53091,$$
$$\log \csc \phi = 0.02369.$$

$$x_1 = -2\sqrt{\frac{p}{3}} \csc \phi.$$

$$\begin{aligned} \log\ 2 &= 0.30103 \\ \log \sqrt{p} &= 0.42255 \\ \text{colog} \sqrt{3} &= 9.76144 - 10 \\ \log \csc \phi &= \underline{0.02369} \\ \log x_1 &= 0.50871 \end{aligned}$$

$$x_1 = -3.2264.$$

$$x_2 = \sqrt{\frac{p}{3}} \csc \phi + \sqrt{-p} \cot \phi.$$

$$\begin{aligned} \log \sqrt{p} &= 0.42255 \\ \log \cot \phi &= \underline{9.53091} \\ &\ 9.95346 \end{aligned}$$

$$x_2 = 1.6132 + 0.89838\, i,$$
$$x_3 = 1.6132 - 0.89838\, i.$$

4. $x^3 - 4x - 5 = 0.$

$$p = 4,\ q = -5,\ \left(\frac{p^3}{27} < \frac{q^2}{4}\right).$$

$$\sin\theta = \frac{2p}{3q}\sqrt{\frac{p}{3}}$$

$$= -\frac{16}{15\sqrt{3}}.$$

$$\begin{aligned} \log 16 &= 1.20412 \\ \text{colog } 15 &= 8.82391 \\ \text{colog}\sqrt{3} &= 9.76144 - 10 \\ \hline \log\sin\theta &= 9.78947 - 10 \end{aligned}$$

$$\theta = 38°\ 0'\ 48'',$$

$$\frac{\theta}{2} = 19°\ 0'\ 24''.$$

$$\log\tan\frac{\theta}{2} = 9.53713,$$

$$\log\tan\frac{\phi}{2} = \frac{1}{3}\log\tan\frac{\theta}{2}$$

$$= 9.84571,$$

$$\frac{\phi}{2} = 35°\ 1'\ 48'',$$

$$\phi = 70°\ 3'\ 36''.$$

$$\log\cot\phi = 9.55965,$$

$$\log\csc\phi = 0.02685.$$

$$x_1 = -2\sqrt{\frac{p}{3}}\csc\phi = -\frac{4}{\sqrt{3}}\csc\phi.$$

$$\begin{aligned} \log 4 &= 0.60206 \\ \text{colog}\sqrt{3} &= 9.76144 - 10 \\ \log\csc\phi &= 0.02685 \\ \hline \log x_1 &= 0.39035 \end{aligned}$$

$$x_1 = -2.4567.$$

$$x_2 = \sqrt{\frac{p}{3}}\csc\phi + \sqrt{-p}\cot\phi.$$

$$\begin{aligned} \log\sqrt{p} &= 0.30103 \\ \log\cot\phi &= 9.55965 \\ \hline & 9.86068 \end{aligned}$$

$$x_2 = 1.2283 + 0.72557\sqrt{-1},$$

$$x_3 = 1.2283 - 0.72557\sqrt{-1}.$$

5. $x^3 - 5x + 4 = 0.$

$$p = 5,\quad q = 4,\quad \left(\frac{p^3}{27} > \frac{q^2}{4}\right).$$

$$\sin\theta = \frac{3q}{2p}\sqrt{\frac{3}{p}}$$

$$= \frac{6}{5}\sqrt{\frac{3}{5}}.$$

$$\begin{aligned} \log\ \ 6 &= 0.77815 \\ \log\sqrt{3} &= 0.23856 \\ \text{colog}\ \ 5 &= 9.30103 \\ \text{colog}\sqrt{5} &= 9.65051 \\ \hline \log\sin\theta &= 9.96825 \end{aligned}$$

$$\theta = 68°\ 21'\ 24'',$$

$$\frac{\theta}{3} = 22°\ 47'\ \ 8'',$$

$$60° - \frac{\theta}{3} = 37°\ 12'\ 52'',$$

$$60° + \frac{\theta}{3} = 82°\ 47'\ \ 8''.$$

$$x_1 = 2\sqrt{\frac{p}{3}}\sin\frac{\theta}{3}.$$

$$\begin{aligned} \log\ \ 2 &= 0.30103 \\ \log\sqrt{5} &= 0.34949 \\ \text{colog}\sqrt{3} &= 9.76144 \\ \hline \log 2\sqrt{\frac{p}{3}} &= 0.41196 \\ \log\sin\frac{\theta}{3} &= 9.58803 \\ \hline \log x_1 &= 9.99999 \end{aligned}$$

$$x_1 = 1.$$

$$x_2 = 2\sqrt{\frac{p}{3}}\sin\left(60° - \frac{\theta}{3}\right).$$

$$\begin{aligned}
\log 2\sqrt{\frac{p}{3}} &= 0.41196 \\
\log\sin\left(60° - \frac{\theta}{3}\right) &= 9.78161 \\
\log x_2 &= \overline{0.19357}
\end{aligned}$$

$$x_2 = 1.5616.$$

$$x_3 = -2\sqrt{\frac{p}{3}}\sin\left(60° + \frac{\theta}{3}\right).$$

$$\begin{aligned}
\log 2\sqrt{\frac{p}{3}} &= 0.41196 \\
\log\sin\left(60° + \frac{\theta}{3}\right) &= 9.99655 \\
\log x_3 &= \overline{0.40851}
\end{aligned}$$

$$x_3 = -2.5616.$$